MONODROMY OF $p$-ADIC SOLUTIONS OF PICARD-FUCHS EQUATIONS
(Special Differential Equations)

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MONODROMY OF $p$-ADIC SOLUTIONS OF PICARD-FUCHS EQUATIONS *

by JAN STIENSTRA

Picard-Fuchs equations are differential equations coming from (algebraic) geometry. Classically their solutions can be written as period integrals for families of varieties. In this note we want to look at $p$-adic solutions of the same differential equations. In $p$-adic analysis we can not use period integrals to describe these solutions.

**Katz-Oda construction of the Gauss-Manin connection**

First recall the purely algebraic construction of the differential equations due to Katz and Oda. Let $S = \text{Spec} A$ an affine scheme which is smooth over an open part of $\text{Spec} \mathbb{Z}$. Let $f : X \to S$ be a projective smooth morphism. The Koszul filtration on the absolute De Rham complex $\Omega^\cdot_X$ is defined by

$$K^{i\cdot} := \text{image}(f^*\Omega^i_S \otimes \Omega^{*-i}_X \to \Omega^\cdot_X).$$

Then

$$K^{0\cdot}/K^{1\cdot} \simeq \Omega^{\cdot}_{X/S}, \quad K^{1\cdot}/K^{2\cdot} \simeq f^*\Omega^1_S \otimes \Omega^{*-1}_{X/S}.$$ 

The Gauss-Manin connection

$$\nabla : \mathcal{H}^m(X, \Omega^\cdot_{X/S}) \to \Omega^1_S \otimes \mathcal{H}^m(X, \Omega^\cdot_{X/S})$$

is the boundary map in the hypercohomology sequence associated with the exact sequence of complexes

$$0 \to K^{1\cdot}/K^{2\cdot} \to K^{0\cdot}/K^{2\cdot} \to K^{0\cdot}/K^{1\cdot} \to 0$$

*details for this note are presented in


From this we see in particular
\[
\text{image}(I^{m}(X, \Omega_{X}) \to I^{m}(X, \Omega_{X/S})) \subset \ker \nabla
\]
Let $\text{Diff}_{S}$ denote the algebra of differential operators on $A$ relative to $Z$ and let $\text{Diff}'_{S}$ be the subalgebra generated by the derivations of $A$. Then the Gauss-Manin connection defines a Lie algebra homomorphism
\[
\nabla : \text{Der}A \to \text{End}_{\mathbb{Z}}(I^{\ast}(X, \Omega_{X/S}))
\]
\[
\nabla(D) = (D \otimes 1) \circ \nabla
\]
which extends to an algebra homomorphism
\[
\nabla : \text{Diff}'_{S} \to \text{End}_{\mathbb{Z}}(I^{\ast}(X, \Omega_{X/S}))
\]
In other words: the Gauss-Manin connection makes $\text{End}_{\mathbb{Z}}(I^{\ast}(X, \Omega_{X/S}))$ a module over $\text{Diff}'_{S}$. Linear relations in this module are Picard-Fuchs differential equations.

For our treatment of $p$-adic solutions of we use the generalized De Rham-Witt complex $\mathcal{W}_{X}$. This complex can be constructed for every scheme $X$ on which 2 is invertible. It is a Zariski sheaf of anticommutative differential graded algebras with the following structures and properties:

- all degrees $\geq 0$.
  \[
  \mathcal{W}_{X}^{0} = \mathcal{W}_{X} \]
  is the sheaf of generalized Witt vectors on $X$

- For all $N \geq 1$ there is a graded algebra endomorphism $F_{N}$ on $\mathcal{W}_{X}^{\ast}$ (F for Frobenius). These satisfy
  \[
  F_{N}F_{M} = F_{NM} \quad \forall N, M
  \]
  \[
  dF_{N} = NF_{N}d \quad \forall N
  \]
  where $d$ = differential of $\mathcal{W}_{X}^{\ast}$

- Let $\mathcal{W}_{X}^{\ast} := \bigoplus_{i \geq 0} \Omega_{X}^{i}/(i!\text{-torsion in } \Omega_{X}^{i})$ where $\Omega_{X}^{\ast}$ is the De Rham complex on $X$ rel. $Z$. Then there exists a homomorphism of sheaves of differential graded algebras
  \[
  \pi : \mathcal{W}_{X}^{\ast} \to \mathcal{W}_{X}^{\ast};
  \]
  such that $\pi : \mathcal{W}_{X} \to \mathcal{O}_{X}$ gives the first Witt vector coordinate.
\[ \forall a \in \mathcal{O}_X \quad \exists a \in \mathcal{WO}_X \text{ s.t. } \pi a = a \]

\[ F_N a = a^N \quad \forall N, \quad a \cdot b = ab \quad \forall a, b \]

Because of \( dF_N = N F_N d \) we have a homomorphism of differential graded algebras

\[ F_N : \bigoplus_i \mathcal{WO}_X^i[-i] \to \mathcal{WO}_X^*/N \]

equal to \( F_N \) in each degree. This fits into the following commutative diagrams

\[
\begin{array}{ccc}
\bigoplus_i H^{m-i}(X, \mathcal{WO}_X^i) & \xrightarrow{F_N} & H^m(X, \mathcal{WO}_X^*/N) \\
\downarrow \tau_N & & \downarrow \pi \\
H^m(X, \Omega_{X/S}/\mathcal{N}) & \xrightarrow{\nabla} & \Omega_S^1 \otimes H^m(X, \Omega_{X/S}/\mathcal{N})
\end{array}
\]

\[
\begin{array}{ccc}
H^m(X, \mathcal{WO}_X) & \xrightarrow{F_N} & H^m(X, \mathcal{WO}_X^*/N) \\
\downarrow F_N & & \downarrow \\
H^m(X, \mathcal{WO}_X) & \xrightarrow{\nabla} & \Omega_S^1 \otimes H^m(X, \Omega_{X/S}/\mathcal{N})
\end{array}
\]

Assume:

\[ \text{Assume:} \quad S = \text{Spec}A \text{ smooth over open part of Spec}Z[\frac{1}{2}] \]

\[ f : X \to S \text{ projective smooth morphism, relative dimension } r \]

all \( H^i(X, \Omega_{X/S}^i) \) are free \( A \)-modules, \( H^r(X, \Omega_{X/S}^r) \cong A \).

Then \( \pi : H^m(X, \mathcal{WO}_X) \to H^m(X, \mathcal{O}_X) \) is surjective. Choose:
\{\omega_1, \ldots, \omega_h\} \text{ basis of } H^m(X, \mathcal{O}_X)\\
\{\check{\omega}_1, \ldots, \check{\omega}_h\} \text{ dual basis of } H^{r-m}(X, \Omega_{X/S}^r)\\
\tilde{\omega}_1, \ldots, \tilde{\omega}_h \in H^m(X, \mathcal{W}\mathcal{O}_X) \text{ s.t. } \pi \tilde{\omega}_i = \omega_i

Define for \( N \in \mathbb{N} \) the \( h \times h \)-matrix \( B_N \) over \( A \) by

\[ \pi F_N \underline{\tilde{\omega}} = B_N \underline{\omega} \]

where \( \underline{\omega} \) = column vector with components \( \omega_1, \ldots, \omega_h \); similarly for \( \underline{\tilde{\omega}} \).

\( B_p \mod p \) for prime \( p \) is known as the Hasse-Witt matrix of ...

**Theorem.** Suppose \( P_1, \ldots, P_h \in \text{Diff}_S \) are such that

\[ \nabla(P_1)\check{\omega}_1 + \cdots + \nabla(P_h)\check{\omega}_h = 0 \]

in \( H^{2r-m}(X, \Omega_{X/S}^r) \)

Then one has the following congruence differential equation

\[ P_i B_{N,i1} + \cdots + P_h B_{N,ih} \equiv 0 \mod N \]

for all \( N \in \mathbb{N} \), for \( i = 1, \ldots, h \).

Idea of proof: for every derivation \( D \) on \( A \)

\[ \langle \tau_N \tilde{\omega}_i, \check{\omega}_j \rangle \equiv B_{N,ij} \mod N \]

\[ \nabla(D)(\tau_N \tilde{\omega}_i) = 0 \]

\[ D(\tau_N \tilde{\omega}_i, \check{\omega}_j) = \langle \tau_N \tilde{\omega}_i, \nabla(D)(\check{\omega}_j) \rangle. \]

**Hypergeometric curves**

Let \( 0 < a, b, c < n \) be integers with \( \gcd(n, a, b, c) = 1 \). Let \( X = X_{n;a,b,c} \) be the smooth projective model, over \( A := \mathbb{Z}[\mu_n][\lambda, (n\lambda(1-\lambda))^{-1}] \), of

\[ y^n = x^a(x-1)^b(x-\lambda)^c. \]

The cohomology \( H^1(X, \mathcal{O}_X) \) can be calculated as \( \check{\text{Cech}} \) cohomology with respect to covering of \( X \) \( X_1 = \{x \neq \infty\}, X_2 = \{x \neq 0\} \). For a detailed description we need:

\[ \alpha = a/n, \quad \beta = b/n, \quad \gamma = c/n, \]
\[ \langle l \rangle = -\langle l\alpha \rangle - \langle l\beta \rangle - \langle l\gamma \rangle \in \{0, 1, 2, 3\} \]
\[ \mathcal{J} := \{(l, j) \in (\mathbb{Z}/n\mathbb{Z}) \times \mathbb{Z} \mid 0 < j < \langle l \rangle \}; \]

\[ \text{[ ] and } \langle \cdot \rangle \text{ are the usual integral and fractional part functions.} \]

For \( (l, j) \in \mathcal{J} \) define
\[ v_l = y^\tilde{l} x^{-\langle l\alpha \rangle} (x - 1)^{-\langle l\beta \rangle} (x - \lambda)^{-\langle l\gamma \rangle} \]
\[ \omega_{(l,j)} = \text{coho class of Čech 1-cocycle } x^{-j} v_l \]
\[ \tilde{\omega}_{(l,j)} = n^{-1} x^{j-1} v_{\tilde{l}}^{-1} dx \]
\[ = n^{-1} x^{j-1-\langle l\alpha \rangle}(x - 1)^{-\langle l\beta \rangle}(x - \lambda)^{-\langle l\gamma \rangle} dx \]

with \( \tilde{l} \in \mathbb{N}, l \equiv \tilde{l} \mod n \). Then
\[ \{\omega_{(l,j)}\}_{(l,j) \in \mathcal{J}} = \text{basis of } H^1(X, \mathcal{O}_X) \]
\[ \{\tilde{\omega}_{(l,j)}\}_{(l,j) \in \mathcal{J}} = \text{dual basis for } H^0(X, \Omega^1_{X/S}) \]

Lift \( \omega_{(l,j)} \) to \( \tilde{\omega}_{(l,j)} \) in \( H^1(X, \mathcal{W}\Omega_X) \) as follows. \( x^{-j} v_l \) is section of \( \mathcal{W}\Omega_X \) over \( X_1 \cap X_2 \). The Čech cocycle condition is trivially satisfied! Take
\[ \tilde{\omega}_{(l,j)} = \text{cohomology class of the Čech 1-cocycle } x^{-j} v_l. \]

Then
\[ \pi F_N \tilde{\omega}_{(l,j)} = \text{cohomology class of the Čech 1-cocycle } (x^{-j} v_l)^N \]

Recall the definition \( \pi F_N \tilde{\omega} = B_N \tilde{\omega}. \) Thus, indexing the rows and columns of \( B_N \) with elements of \( \mathcal{J} \), one finds
\[ B_{N,(l,j),(l',j')} = 0 \]
if \( l' \neq lN \), whereas for \( l' = lN \)
\[ B_{N,(l,j),(l',j')} = (-1)^L \sum_k \binom{[N <l\beta>]}{L - k} \binom{[N <l\gamma>]}{k} \lambda^k \]

here \( L = j' - jN + [N <l\alpha>] + [N <l\beta>] + [N <l\gamma>]. \)

Then one easily checks the following congruence differential equation
\[ \nabla(P_{(l',j')}) B_{N,(l,j),(l',j')} \equiv 0 \mod NA \]
where $P_{(l',j')}$ is the hypergeometric differential operator, with $\Theta = \lambda \frac{d}{d\lambda}$,

\[
\Theta(\Theta - j' + \langle l'\alpha \rangle + \langle l'\gamma \rangle) - \\
- \lambda(\Theta + \langle l'\gamma \rangle)(\Theta - j' + \langle l'\alpha \rangle + \langle l'\beta \rangle + \langle l'\gamma \rangle))
\]

We now turn to $p$-adic solutions, $p$ prime $> 2$. Our method is based on the commutativity of the diagram

\[
\begin{array}{ccc}
H^m(X, \mathcal{W}\mathcal{O}_X) & \xrightarrow{F_p} & H^m(X, \mathcal{W}\mathcal{O}_X) \\
\downarrow & & \downarrow \\
\downarrow F_{p^{r+1}} & & \downarrow F_{p^r} \\
\downarrow H^m(X, \mathcal{W}\Omega^\bullet_x/p^{r+1}) & \rightarrow & H^m(X, \mathcal{W}\Omega^\bullet_x/p^r) \\
\downarrow & & \downarrow \\
\downarrow H^m(X, \Omega^\bullet_x)/p^{r+1} & \rightarrow & H^m(X, \Omega^\bullet_x)/p^r \\
\downarrow & & \downarrow \\
\downarrow H^m(X, \Omega^\bullet_{x/S})/p^{r+1} & \rightarrow & H^m(X, \Omega^\bullet_{x/S})/p^r \\
\downarrow \nabla & & \downarrow \nabla \\
\Omega^1_S \otimes H^m(X, \Omega^\bullet_{x/S}/p^{r+1}) & \rightarrow & \Omega^1_S \otimes H^m(X, \Omega^\bullet_{x/S}/p^r)
\end{array}
\]

In the limit for $r \to \infty$ it gives

\[
\lim_{r \to \infty} H^m(X, \mathcal{W}\mathcal{O}_X) \to (H^m(X, \Omega^\bullet_{x/S}) \otimes \mathbb{Z}_p)^\nabla
\]

and thus we try to find $p$-adic solutions of Picard-Fuchs equations by "lifting against Frobenius". This amounts to solving algebraic equations!

**Vectors fixed by Frobenius**

Assume $\det B_p \not\in pA$. Let

\[
A^0 = A[\det B_p]^{-1}, \quad A_0 = A^0/pA^0, \quad A^\wedge = \lim_{n} A^0/p^n A^0.
\]

$A_0$ is a direct product of domains. Fix one such component and let $R$ be its inverse image in $A^\wedge$. Then $R$ is complete and separated in the $p$-adic topology and $\det B_p$ is invertible in $R$.

Let $P$ be the set of primes $\neq p$. For every scheme $Y$ such that every $l \in P$ is invertible in $\mathcal{O}_Y$ one can use the idempotent operator $E_p :=$
\[ \prod_{l \in P} (1 - l^{-1} V_l F_l) \] on \( \mathcal{W}O_Y \) to split off the sheaf of \emph{p-typical Witt vectors} on \( Y \).

\[ \mathcal{W}O_Y = E_p \mathcal{W}O_Y \]

There exists a \( \mathcal{I}_p \)-algebra endomorphism \( \sigma \) of \( R \) such that

\[ \sigma(x) \equiv x^p \mod pR \quad \forall x \in R \]

There are many such \( \sigma \). Given a choice for \( \sigma \) there is a unique homomorphism of rings

\[ \lambda : R \to \mathcal{W}(R) \]

such that \( \pi F^n \lambda = \sigma^n \quad \forall n \in \mathbb{N} \); here \( \mathcal{W}(R) \) is the ring of \( p \)-typical Witt vectors over \( R \) and \( \pi : \mathcal{W}(R) \to R \) is the projection onto first coordinate.

Notations:

\[ \sigma(x) = x^\sigma, \quad F = F_p; \]

for a matrix \( M = (m_{ij}) \)

\[ M^{(p^r)} = (m_{ij}^{p^r}), \quad M^{\sigma^r} = (m_{ij}^{\sigma^r}), \quad \lambda(M) = (\lambda(m_{ij})), \quad \underline{M} = (m_{ij}); \]

for \( A \)-algebra \( A' \)

\[ X \otimes A = X \times_{\mathcal{S}} \text{Spec} A'. \]

\textbf{Theorem}

\[ \exists H \in GL_h(R) \ s.t. \ B_{p^{r+1}} \equiv B_p^\sigma H \mod p^{r+1} \quad \forall r \geq 0. \]

\[ \exists \hat{\omega}_1, \ldots, \hat{\omega}_h \in H^m(X \otimes R, \mathcal{W}O_{X \otimes R}) \ s.t. \ F\hat{\omega} = \lambda(H)\hat{\omega} \text{ and } \pi \hat{\omega}_i = \omega_i, \]

\[ \hat{\omega} = \text{column vector } (\hat{\omega}_1, \ldots, \hat{\omega}_h)^t. \]

Fix an algebraically closed field \( \Omega \supset R/pR \) and define

\[ (R/pR)^{\acute{e}t} := \lim_{\overset{\longrightarrow}{B \in \mathcal{B}}} B. \]

where \( \mathcal{B} \) is the set of finite étale extensions of \( R/pR \) in \( \Omega \). For every \( B \in \mathcal{B} \) there is a unique finite étale \( \tilde{B} \) over \( R \) such that \( B = \tilde{B}/p\tilde{B} \). We define

\[ R^{\acute{e}t} := \text{the } p\text{-adic completion of } \lim_{\overset{\longrightarrow}{B \in \mathcal{B}}} \tilde{B}. \]
$(R/pR)^{\text{et}}$ is an infinite étale extension of $R/pR$ and $R^{\text{et}}/pR^{\text{et}} = (R/pR)^{\text{et}}$. The algebraic fundamental group $\pi_1(\text{Spec}(R/pR), \Omega)$ is by definition the Galois group of $(R/pR)^{\text{et}}/(R/pR)$. It acts on $R^{\text{et}}$. $\sigma$ induces an endomorphism $\sigma$ of $R^{\text{et}}$.

$$(R^{\text{et}})^{\sigma} = \mathbb{Z}_p, \quad (R^{\text{et}})^{\pi_1} = R.$$  

**Proposition**  \( \exists C \in GL_h(R^{\text{et}}) \) s.t. \( C^{\sigma} H = C \).

**idea of proof:** The system of equations

\[
\begin{align*}
C_0^{(p)} H - C_0 &= 0, \\
\delta \cdot \det C_0 - 1 &= 0, \\
C_{i+1}^{(p)} H - C_{i+1} + p^{-1}[C_i^{(p)} - C_i] H &= 0 \quad (i \geq 0)
\end{align*}
\]

can inductively be solved with \( h \times h \)-matrices $C_i$ over $R^{\text{et}}$. Then $C := \Sigma_i p^i C_i$ is a solution.

$R \hookrightarrow R^{\text{et}}$ induces $H^m(X \otimes R, \mathcal{W}\mathcal{O}_{X \otimes R}) \hookrightarrow H^m(X \otimes R^{\text{et}}, \mathcal{W}\mathcal{O}_{X \otimes R^{\text{et}}})$.

Define

$$\xi_1, \ldots, \xi_h \in H^m(X \otimes R^{\text{et}}, \mathcal{W}\mathcal{O}_{X \otimes R^{\text{et}}})$$

by

$$\underline{\xi} = \lambda(C) \underline{\hat{\omega}}.$$  

Then

$$F \underline{\xi} = \underline{\xi}, \quad \pi \underline{\xi} = C \underline{\hat{\omega}}.$$  

**Proposition**

$H^m(X \otimes R^{\text{et}}, \mathcal{W}\mathcal{O}_{X \otimes R^{\text{et}}})$ is a free $\mathcal{W}(R^{\text{et}})$-module with bases \( \{\xi_1, \ldots, \xi_h\} \) and \( \{\hat{\omega}_1, \ldots, \hat{\omega}_h\} \).

$H^m(X \otimes R, \mathcal{W}\mathcal{O}_{X \otimes R})$ is a free $\mathcal{W}(R)$-module with basis \( \{\hat{\omega}_1, \ldots, \hat{\omega}_h\} \).

\( \pi : H^m(X \otimes R^{\text{et}}, \mathcal{W}\mathcal{O}_{X \otimes R^{\text{et}}}) \rightarrow H^m(X \otimes R^{\text{et}}, \mathcal{O}_{X \otimes R^{\text{et}}}) \) restricts to an isomorphism \( \pi : \Lambda \simeq \pi \Lambda \) on

$$\Lambda := \ker(F - 1 \text{ on } H^m(X \otimes R^{\text{et}}, \mathcal{W}\mathcal{O}_{X \otimes R^{\text{et}}})).$$
Write $\Lambda$ resp. $\xi$ instead of $\pi\Lambda$ resp. $\pi\xi$.

**Theorem.** $\Lambda$ is a free $\mathbb{Z}_p$-module with basis $\{\xi_1, \ldots, \xi_h\}$.

$$H^m(X, \mathcal{O}_X) \otimes^\Lambda R^{et} = \Lambda \otimes_{\mathbb{Z}_p} R^{et}$$

$$\xi = C\omega, \quad \nabla\xi = 0$$

Thus the rows of $C$ satisfy the same differential equations as $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_h\}$.

$\pi_1 := \pi_1(\text{Spec}(R/pR), \Omega)$ acts on $R^{et}$. By functoriality this induces an action of $\pi_1$ on $H^m(X, \mathcal{O}_X) \otimes_A R^{et}$ and on $H^m(X \otimes R^{et}, \mathcal{W}\mathcal{O}_{X\otimes R^n})$. Since $F$ and $\pi$ are $\pi_1$ equivariant we obtain the **$p$-adic monodromy representation**:

$$\mathcal{M}: \pi_1(\text{Spec}(R/pR), \Omega) \rightarrow \text{Aut}_{\mathbb{Z}_p}(\Lambda)$$

$$\mathcal{M}(\tau)\xi = C^\tau C^{-1}\xi \quad \text{for} \quad \tau \in \pi_1.$$

$\xi = \text{column vector } (\xi_1, \ldots, \xi_h)^t$

The **$p$-adic monodromy group** $\mathcal{M}(\pi_1)$ for the hypergeometric curve $y^5 = x(x-1)^2(x-\lambda)^3$.

is computed in J. Stienstra, M. van der Put, B. van der Marel, *On p-adic monodromy*. It turns out to be conjugate to:

case $p \equiv \pm 1 \text{mod } 5$

$$\begin{cases} 
\begin{pmatrix} \eta a & \eta^2 b & 0 \\ \eta^2 b & \eta^{-2} b & 0 \\ 0 & \eta^{-1} a & \end{pmatrix} 
\end{cases} \begin{array}{c} a, b \in \mathbb{Z}_p^*, \\
\eta \in \mu_5 \end{array}.$$ 


case $p \equiv \pm 2 \text{mod } 5$

$$\begin{cases} 
\begin{pmatrix} \eta a & 0 \\ \eta^2 a^\sigma & 0 \\ 0 & \eta^{-2} a^\sigma & \eta^{-1} a \end{pmatrix} 
\end{cases} \begin{array}{c} a \in \mathcal{W}(\mathbb{F}_{p^2})^*, \\
\eta \in \mu_5 \end{array}.$$