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MONODROMY OF \( p \)-ADIC SOLUTIONS OF PICARD-FUCHS EQUATIONS *

by JAN STIENSTRA

Picard-Fuchs equations are differential equations coming from (algebraic) geometry. Classically their solutions can be written as period integrals for families of varieties. In this note we want to look at \( p \)-adic solutions of the same differential equations. In \( p \)-adic analysis we can not use period integrals to describe these solutions.

Katz-Oda construction of the Gauss-Manin connection

First recall the purely algebraic construction of the differential equations due to Katz and Oda. Let \( S = \text{Spec}A \) an affine scheme which is smooth over an open part of \( \text{Spec} \mathbb{Z} \). Let \( f : X \rightarrow S \) be a projective smooth morphism. The Koszul filtration on the absolute De Rham complex \( \Omega_X^* \) is defined by

\[
K^{i*} := \text{image}(f^*\Omega_S^i \otimes \Omega_X^{*-i} \rightarrow \Omega_X^*).
\]

Then

\[
K^{0*}/K^{1*} \simeq \Omega_{X/S}^1, \quad K^{1*}/K^{2*} \simeq f^*\Omega_S^1 \otimes \Omega_{X/S}^{*-1}.
\]

The Gauss-Manin connection

\[
\nabla : \mathcal{H}^m(X, \Omega_{X/S}^*) \rightarrow \Omega_S^1 \otimes \mathcal{H}^m(X, \Omega_{X/S}^*)
\]

is the boundary map in the hypercohomology sequence associated with the exact sequence of complexes

\[
0 \rightarrow K^{1*}/K^{2*} \rightarrow K^{0*}/K^{2*} \rightarrow K^{0*}/K^{1*} \rightarrow 0
\]

*details for this note are presented in


From this we see in particular
\[
\text{image}(\mathcal{H}^m(X, \Omega_X^\bullet) \to \mathcal{H}^m(X, \Omega_{X/S}^\bullet)) \subset \ker \nabla
\]
Let \(\text{Diff}_S\) denote the algebra of differential operators on \(A\) relative to \(Z\) and let \(\text{Diff}'_S\) be the subalgebra generated by the derivations of \(A\). Then the Gauss-Manin connection defines a Lie algebra homomorphism
\[
\nabla : \text{Der} A \to \text{End}_Z (\mathcal{H}^*(X, \Omega_{X/S}^\bullet))
\]
\[
\nabla(D) = (D \otimes 1) \circ \nabla
\]
which extends to an algebra homomorphism
\[
\nabla : \text{Diff}'_S \to \text{End}_Z (\mathcal{H}^*(X, \Omega_{X/S}^\bullet))
\]
In other words: the Gauss-Manin connection makes \(\text{End}_Z (\mathcal{H}^*(X, \Omega_{X/S}^\bullet))\) a module over \(\text{Diff}'_S\). Linear relations in this module are Picard-Fuchs differential equations.

For our treatment of \(p\)-adic solutions of we use the generalized De Rham-Witt complex \(\mathcal{W}\Omega_X^\bullet\). This complex can be constructed for every scheme \(X\) on which 2 is invertible. It is a Zariski sheaf of anti-commutative differential graded algebras with the following structures and properties:

- all degrees \(\geq 0\).
  \(\mathcal{W}\Omega_X^0 = \mathcal{W}\mathcal{O}_X\) is the sheaf of generalized Witt vectors on \(X\)

- For all \(N \geq 1\) there is a graded algebra endomorphism \(F_N\) on \(\mathcal{W}\Omega_X^\bullet\) (F for Frobenius). These satisfy
  \[
  F_N F_M = F_{NM} \quad \forall N, M
  \]
  \[
  dF_N = NF_N d \quad \forall N
  \]
  where \(d\) = differential of \(\mathcal{W}\Omega_X^\bullet\)

- Let \(\overline{\Omega}_X^\bullet := \oplus_{i \geq 0} \Omega_X^i/(i!\text{-torsion in } \Omega_X^i)\) where \(\Omega_X^\bullet\) is the De Rham complex on \(X\) rel. \(Z\). Then there exists a homomorphism of sheaves of differential graded algebras
  \[
  \pi : \mathcal{W}\Omega_X^\bullet \to \overline{\Omega}_X^\bullet;
  \]
  such that \(\pi : \mathcal{W}\mathcal{O}_X \to \mathcal{O}_X\) gives the first Witt vector coordinate.
• \( \forall a \in \mathcal{O}_X \ \exists \underline{a} \in \mathcal{W}\mathcal{O}_X \) s.t. \( \pi \underline{a} = a \)

\[
F_N \underline{a} = \underline{a}^N \ \forall N, \quad a \cdot b = ab \ \forall a, b
\]

Because of \( dF_N = NF_N d \) we have a homomorphism of differential graded algebras

\[
F_N : \bigoplus_i \mathcal{W}\Omega^i_X[-i] \to \mathcal{W}\Omega^*_X/N
\]
equal to \( F_N \) in each degree. This fits into the following commutative diagrams

\[
\begin{array}{c}
\oplus_i H^{m-i}(X, \mathcal{W}\Omega^i_X) \xrightarrow{F_N} \bigoplus_i H^m(X, \mathcal{W}\Omega^i_X)/N \\
\downarrow \tau_N \downarrow \\
\bigoplus_i H^m(X, \Omega^*_{X/S}/N) \xrightarrow{\bigoplus_i H^m(X, \Omega^*_{X/S}/N) \otimes} \Omega^1_S \otimes \bigoplus_i H^m(X, \Omega^*_{X/S}/N)
\end{array}
\]

Assume:

\( S = \text{Spec}A \) smooth over open part of \( \text{Spec} Z[\frac{1}{2}] \)

\( f : X \to S \) projective smooth morphism, relative dimension \( r \)

all \( H^i(X, \Omega^i_{X/S}) \) are free \( A \)-modules, \( H^r(X, \Omega^*_{X/S}) \simeq A \).

Then \( \pi : H^m(X, \mathcal{W}\mathcal{O}_X) \to H^m(X, \mathcal{O}_X) \) is surjective. Choose:
\{\omega_1, \ldots, \omega_h\} \text{ basis of } H^m(X, \mathcal{O}_X) \\
\{\check{\omega}_1, \ldots, \check{\omega}_h\} \text{ dual basis of } H^{r-m}(X, \Omega^r_{X/S}) \\
\tilde{\omega}_1, \ldots, \tilde{\omega}_h \in H^m(X, \mathcal{W}\mathcal{O}_X) \text{ s.t. } \pi \tilde{\omega}_i = \omega_i \\
Define \text{ for } N \in \mathbb{N} \text{ the } h \times h\text{-matrix } B_N \text{ over } A \text{ by } \\
\pi F_N \tilde{\omega} = B_N \omega \\
where \omega = \text{ column vector with components } \omega_1, \ldots, \omega_h; \text{ similarly for } \tilde{\omega}. \\
B_p \mod p \text{ for prime } p \text{ is known as the Hasse-Witt matrix of } . . . \\

**Theorem.** Suppose \( P_1, \ldots, P_h \in \text{Diff}_S \) are such that \\
\( \nabla(P_1) \check{\omega}_1 + \cdots + \nabla(P_h) \check{\omega}_h = 0 \) \text{ in } H^{2r-m}(X, \Omega^r_{X/S}) \\
Then one has the following congruence differential equation \\
\( P_1 B_{N,i1} + \cdots + P_h B_{N,ih} \equiv 0 \mod N \) \\
for all \( N \in \mathbb{N} \), for \( i = 1, \ldots, h \). \\
Idea of proof: for every derivation \( D \) on \( A \) \\
\( \langle \tau_N \check{\omega}_i, \check{\omega}_j \rangle \equiv B_{N,ij} \mod N \) \\
\( \nabla(D)(\tau_N \check{\omega}_i) = 0 \) \\
\( D(\tau_N \check{\omega}_i, \check{\omega}_j) = \langle \tau_N \check{\omega}_i, \nabla(D)(\check{\omega}_j) \rangle \).

**Hypergeometric curves** \\
Let \( 0 < a, b, c < n \) be integers with \( \gcd(n, a, b, c) = 1 \). Let \( X = X_{n;a,b,c} \) be the smooth projective model, over \( A := \mathbb{Z}[\mu_n][\lambda, (n\lambda(1-\lambda))^{-1}] \), of \\
y^n = x^a(x-1)^b(x-\lambda)^c. \\
The cohomology \( H^1(X, \mathcal{O}_X) \) can be calculated as Čech cohomology with respect to covering of \( X \) \( X_1 = \{x \neq \infty\} \), \( X_2 = \{x \neq 0\} \). For a detailed description we need: \\
\( \alpha = a/n, \ \beta = b/n, \ \gamma = c/n, \)
\[ \langle l \rangle = [- <l\alpha> - <l\beta> - <l\gamma> \in \{0, 1, 2, 3\} \]

\[ \mathcal{J} := \{(l, j) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z} | 0 < j < \langle l \rangle \}; \]

[\cdot] and \(<\cdot> are the usual integral and fractional part functions.

For \((l, j) \in \mathcal{J}\) define

\[ v_l = y^l x^{-[l\alpha]} (x - 1)^{-[l\beta]} (x - \lambda)^{-[l\gamma]} \]

\[ \omega_{(l,j)} = \text{coho class of Čech 1-cocycle } x^{-j} v_l \]

\[ \tilde{\omega}_{(l,j)} = n^{-1} x^{j-1} v_{\tilde{l}}^{-1} dx \]

with \(\tilde{l} \in \mathbb{N}, \ l \equiv \tilde{l} \mod n\). Then

\[ \{\omega_{(l,j)}\} \in \mathcal{J} = \text{basis of } H^1(X, \mathcal{O}_X) \]

\[ \{\tilde{\omega}_{(l,j)}\} \in \mathcal{J} = \text{dual basis for } H^0(X, \Omega_{X/S}) \]

Lift \(\omega_{(l,j)}\) to \(\tilde{\omega}_{(l,j)}\) in \(H^1(X, \mathcal{W}\mathcal{O}_X)\) as follows. \(x^{-j} v_l\) is section of \(\mathcal{W}\mathcal{O}_X\) over \(X_1 \cap X_2\). The Čech cocycle condition is trivially satisfied! Take

\[ \tilde{\omega}_{(l,j)} = \text{cohomology class of the Čech 1-cocycle } x^{-j} v_l \]

Then

\[ \pi F_N \tilde{\omega}_{(l,j)} = \text{cohomology class of the Čech 1-cocycle } (x^{-j} v_l)^N \]

Recall the definition \(\pi F_N \tilde{\omega} = B_N \omega\). Thus, indexing the rows and columns of \(B_N\) with elements of \(\mathcal{J}\), one finds

\[ B_{N,(l,j),(l',j')} = 0 \]

if \(l' \neq lN\), whereas for \(l' = lN\)

\[ B_{N,(l,j),(l',j')} = (-1)^L \sum_k \left( \begin{array}{c} [N <l\beta>] \cr L - k \end{array} \right) \left( \begin{array}{c} [N <l\gamma>] \cr k \end{array} \right) \lambda^k \]

here \(L = j' - jN + [N <l\alpha>] + [N <l\beta>] + [N <l\gamma>]\).

Then one easily checks the following congruence differential equation

\[ \nabla(P_{(l',j')}) B_{N,(l,j),(l',j')} \equiv 0 \mod NA \]
where $P_{(l',j')}$ is the hypergeometric differential operator, with $\Theta = \lambda \frac{d}{d\lambda}$,

$$
\Theta(\Theta-j'+<l'\alpha>+<l'\gamma>)-
-\lambda(\Theta+<l'\gamma>)(\Theta-j'+<l'\alpha>+<l'\beta>+<l'\gamma>))
$$

We now turn to \textbf{p-adic solutions}, $p$ prime $> 2$. Our method is based on the commutativity of the diagram

\[
\begin{array}{ccc}
H^m(X, \mathcal{W}\mathcal{O}_X) & \xrightarrow{F_p} & H^m(X, \mathcal{W}\mathcal{O}_X) \\
\downarrow & & \downarrow \\
F_{p^{r+1}} & & F_{p^r} \\
\downarrow & & \downarrow \\
\mathcal{H}^m(X, \mathcal{W}\Omega_X/p^{r+1}) & \rightarrow & \mathcal{H}^m(X, \mathcal{W}\Omega_X/p^r) \\
\downarrow & & \downarrow \\
\mathcal{H}^m(X, \Omega^*_X)/p^{r+1} & \rightarrow & \mathcal{H}^m(X, \Omega^*_X)/p^r \\
\downarrow & & \downarrow \\
\mathcal{H}^m(X, \Omega^*_{X/S})/p^{r+1} & \rightarrow & \mathcal{H}^m(X, \Omega^*_{X/S})/p^r \\
\downarrow \nabla & & \downarrow \nabla \\
\Omega^1_S \otimes \mathcal{H}^m(X, \Omega^*_{X/S}/p^{r+1}) & \rightarrow & \Omega^1_S \otimes \mathcal{H}^m(X, \Omega^*_{X/S}/p^r)
\end{array}
\]

In the limit for $r \rightarrow \infty$ it gives

$$
\lim_{\rightarrow F_p} H^m(X, \mathcal{W}\mathcal{O}_X) \rightarrow (\mathcal{H}^m(X, \Omega^*_{X/S}) \otimes \mathbb{Z}_p)^\nabla
$$

and thus we try to find \textbf{p-adic solutions} of Picard-Fuchs equations by "lifting against Frobenius". This amounts to solving algebraic equations!

\textbf{Vectors fixed by Frobenius}

Assume $\det B_p \notin pA$. Let

$$
A^0 = A[(\det B_p)^{-1}], \quad A_0 = A^0 / pA^0, \quad A^\wedge = \lim_{\rightarrow n} A^0 / p^n A^0.
$$

$A_0$ is a direct product of domains. Fix one such component and let $R$ be its inverse image in $A^\wedge$. Then $R$ is complete and separated in the $p$-adic topology and $\det B_p$ is invertible in $R$.

Let $P$ be the set of primes $\neq p$. For every scheme $Y$ such that every $l \in P$ is invertible in $\mathcal{O}_Y^*$ one can use the idempotent operator $E_p :=$
\[ \prod_{l \in P}(1 - l^{-1}V_l F_l) \] on \( \mathcal{W}O_Y \) to split off the sheaf of \textit{p-typical Witt vectors} on \( Y \).

\[ \mathcal{W}O_Y = E_p \mathcal{W}O_Y \]

There exists a \( \mathcal{I}_p \)-algebra endomorphism \( \sigma \) of \( R \) such that

\[ \sigma(x) \equiv x^p \mod pR \quad \forall x \in R \]

There are many such \( \sigma \). Given a choice for \( \sigma \) there is a unique homomorphism of rings

\[ \lambda : R \to \mathcal{W}(R) \]

such that \( \pi F^n \lambda = \sigma^n \quad \forall n \in \mathbb{N} \); here \( \mathcal{W}(R) \) is the ring of \( p \)-typical Witt vectors over \( R \) and \( \pi : \mathcal{W}(R) \to R \) is the projection onto first coordinate

Notations:

\[ \sigma(x) = x^\sigma, \quad F = F_p; \]

for a matrix \( M = (m_{ij}) \)

\[ M^{(\sigma)} = (m_{ij}^p), \quad M^{\sigma} = (m_{ij}^\sigma), \quad \lambda(M) = (\lambda(m_{ij})), \quad \underline{M} = (m_{ij}); \]

for \( A \)-algebra \( A' \)

\[ X \otimes A = X \times_S \text{Spec}A'. \]

\[ \exists H \in GL_h(R) \text{ s.t. } B_{p^{r+1}} \equiv B_{p^r}^\sigma H \mod p^{r+1} \quad \forall r \geq 0. \]

\[ \exists \tilde{\omega}_1, \ldots, \tilde{\omega}_h \in H^m(X \otimes R, \mathcal{W}O_{X \otimes R}) \text{ s.t. } F \tilde{\omega} = \lambda(H) \tilde{\omega} \text{ and } \pi \tilde{\omega}_i = \omega_i, \]

\[ \tilde{\omega} = \text{column vector } (\tilde{\omega}_1, \ldots, \tilde{\omega}_h)^t. \]

Fix an algebraically closed field \( \Omega \supset R/pR \) and define

\[ (R/pR)^{\text{ét}} := \lim_{B \in \mathcal{B}} B. \]

where \( \mathcal{B} \) is the set of finite étale extensions of \( R/pR \) in \( \Omega \). For every \( B \in \mathcal{B} \) there is a unique finite étale \( \tilde{B} \) over \( R \) such that \( B = \tilde{B}/p\tilde{B} \). We define

\[ R^{\text{ét}} := \text{the } p\text{-adic completion of } \lim_{B \in \mathcal{B}} \tilde{B}. \]
$(R/pR)^{\text{ét}}$ is an infinite étale extension of $R/pR$ and $R^{\text{ét}}/pR^{\text{ét}} = (R/pR)^{\text{ét}}$. The algebraic fundamental group $\pi_1(\text{Spec}(R/pR), \Omega)$ is by definition the Galois group of $(R/pR)^{\text{ét}}/(R/pR)$. It acts on $R^{\text{ét}}$. $\sigma$ induces an endomorphism $\sigma$ of $R^{\text{ét}}$.

$$(R^{\text{ét}})^{\sigma} = Z_p, \quad (R^{\text{ét}})^{\pi_1} = R.$$ 

**Proposition**  \[ \exists C \in GL_h(R^{\text{ét}}) \text{ s.t. } C^\sigma H = C. \]

**idea of proof:** The system of equations

$$C_0^{(p)} H - C_0 = 0, \quad \delta \cdot \det C_0 - 1 = 0,$$

$$C_{i+1}^{(p)} H - C_{i+1} + p^{-1}[C_i^\sigma - C_i^{(p)}] H = 0 \quad (i \geq 0)$$

can inductively be solved with $h \times h$-matrices $C_i$ over $R^{\text{ét}}$. Then $C := \Sigma_i p^i C_i$ is a solution.

$R \hookrightarrow R^{\text{ét}}$ induces $H^m(X \otimes R, \mathcal{W}\mathcal{O}_{X \otimes R}) \hookrightarrow H^m(X \otimes R^{\text{ét}}, \mathcal{W}\mathcal{O}_{X \otimes R^{\text{ét}}})$.

Define

$$\xi_1, \ldots, \xi_h \in H^m(X \otimes R^{\text{ét}}, \mathcal{W}\mathcal{O}_{X \otimes R^{\text{ét}}})$$

by

$$\xi = \lambda(C) \hat{\omega}.$$ 

Then

$$F \xi = \xi, \quad \pi \xi = C \omega.$$ 

**Proposition** 

$H^m(X \otimes R^{\text{ét}}, \mathcal{W}\mathcal{O}_{X \otimes R^{\text{ét}}})$ is a free $\mathcal{W}(R)$-module with basis $\{\xi_1, \ldots, \xi_h\}$

$H^m(X \otimes R, \mathcal{W}\mathcal{O}_{X \otimes R})$ is a free $\mathcal{W}(R)$-module with basis $\{\hat{\omega}_1, \ldots, \hat{\omega}_h\}$.

$\pi: H^m(X \otimes R^{\text{ét}}, \mathcal{W}\mathcal{O}_{X \otimes R^{\text{ét}}}) \rightarrow H^m(X \otimes R^{\text{ét}}, \mathcal{O}_{X \otimes R^{\text{ét}}})$ restricts to an isomorphism $\pi: \Lambda \simeq \pi \Lambda$ on

$$\Lambda := \ker(F - 1 \text{ on } H^m(X \otimes R^{\text{ét}}, \mathcal{W}\mathcal{O}_{X \otimes R^{\text{ét}}})).$$
Write $\Lambda$ resp. $\xi$ instead of $\pi\Lambda$ resp. $\pi\xi$.

Theorem. $\Lambda$ is a free $\mathbb{Z}_p$-module with basis $\{\xi_1, \ldots, \xi_h\}$.

\[ H^m(X, \mathcal{O}_X) \otimes_A R^{et} = \Lambda \otimes_{\mathbb{Z}_p} R^{et} \]

\[ \xi = C\omega, \quad \nabla\xi = 0 \]

Thus the rows of $C$ satisfy the same differential equations as $\{\check{\omega}_1, \ldots, \check{\omega}_h\}$.

$\pi_1 := \pi_1(\text{Spec}(R/pR), \Omega)$ acts on $R^{et}$. By functoriality this induces an action of $\pi_1$ on $H^m(X, \mathcal{O}_X) \otimes_A R^{et}$ and on $H^m(X \otimes R^{et}, \mathcal{W}\mathcal{O}_{X \otimes R^{et}})$. Since $F$ and $\pi$ are $\pi_1$ equivariant we obtain the $p$-adic monodromy representation:

\[ \mathcal{M} : \pi_1(\text{Spec}(R/pR), \Omega) \to \text{Aut}_{\mathbb{Z}_p}(\Lambda) \]

\[ \mathcal{M}(\tau)\xi = C^\tau C^{-1}\xi \quad \text{for} \quad \tau \in \pi_1. \]

$\xi = \text{column vector} (\xi_1, \ldots, \xi_h)^t$

The $p$-adic monodromy group $\mathcal{M}(\pi_1)$ for the hypergeometric curve

\[ y^5 = x(x - 1)^2(x - \lambda)^3. \]

is computed in J. Stienstra, M. van der Put, B. van der Marel, On $p$-adic monodromy. It turns out to be conjugate to:

**case $p \equiv \pm 1 \text{mod} 5$**

\[
\left\{ \begin{pmatrix} \eta a & 0 \\ \eta^2 b & \eta^{-2} b \\ 0 & \eta^{-1} a \end{pmatrix} \right\} \quad a, b \in \mathbb{Z}_p^*, \quad \eta \in \mu_5
\]

**case $p \equiv \pm 2 \text{mod} 5$**

\[
\left\{ \begin{pmatrix} \eta a & 0 \\ \eta^2 a^\sigma & \eta^{-2} a^\sigma \\ 0 & \eta^{-1} a \end{pmatrix} \right\} \quad a \in \mathcal{W}(IF_{p^2})^*, \quad \eta \in \mu_5
\]