数理解析研究所講究録
773巻 1991年 66-77

Conference on "Special Differential Equations"
RIMS, Kyoto Univ.
September 2-5, 1991

Appell Hypergeometric Function $F_2(a,b,b';c,c';x,y)$ and the Blowing Up Space of P^2

J. Sekiguchi
Department of Mathematics
University of Electro-Communications
Chofu, Tokyo 182, JAPAN

§1. Introduction. Appell's function $F_2(a,b,b';c,c';x,y)$ is a solution of the system of differential equations

$$[x(1-x)\partial_x^2-xy\partial_x\partial_y+\{c-(a+b+1)x\}\partial_x-by\partial_y-ab]u = 0$$
$$[y(1-y)\partial_y^2-xy\partial_x\partial_y+\{c'-(a+b'+1)y\}\partial_y-b'x\partial_x-ab']u = 0.$$

Let P^2 be the 2-dim. projective space and let $\xi=(\xi_1:\xi_2:\xi_3)$ be it homogeneous coordinate. Putting $(x,y)=(1-\xi_2/\xi_1,\xi_3/\xi_1)$, we may regard the system of differential equations above as that on P^2. Then it has singularities along the set

$$S =\{\xi\in P^2;\xi_1\xi_2\xi_3(\xi_2-\xi_3)(\xi_3-\xi_1)(\xi_1-\xi_2) = 0\}$$

which consists of six lines. Let Z be the blowing up of P^2 at four points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$, $(1:1:1)$ where three of lines of S intersect. Let π be the natural projection of Z to P^2 and put $\tilde{S} = \pi^{-1}S$. Then $\tilde{S}$ consists of ten lines and each intersecting point of some of ten lines is a normal crossing point. The purpose of this talk is to study the structure of the pull back of the system in question on the space Z.

§2. The blowing up space of P^2. We begin with constructing the space

Z concretely. (For the details, see [S].) A model of Z is defined by

$$\{(\xi,\eta,\zeta)\in P^2\times P^2\times P^2;\xi_1\eta_1=\xi_2\eta_2=\xi_3\eta_3,\ \xi_1\zeta_1+\xi_2\zeta_2+\xi_3\zeta_3=0,\zeta_1+\zeta_2+\zeta_3=0\}$$

and $\pi(\xi,\eta,\zeta)=\xi$ is the projection of Z to P^2. Moreover we define lines $L(ij)$ $(1\leq i<j\leq 5)$ of Z by

$L(12)$: $\xi_1=\xi_2=\xi_3,\ \eta_1=\eta_2=\eta_3$/ $L(13)$: $\xi_2=\xi_3=\eta_1=0$/
$L(14)$: $\xi_3=\xi_1=\eta_2=0$/ $L(15)$: $\xi_1=\xi_2=\eta_3=0$/ $L(23)$: $\xi_1=\eta_2=\eta_3=0$/
$L(24)$: $\xi_2=\eta_3=\eta_1=0$/ $L(25)$: $\xi_3=\eta_1=\eta_2=0$/ $L(34)$: $\xi_1=\xi_2,\ \eta_1=\eta_2$/
$L(35)$: $\xi_3=\xi_1,\ \eta_3=\eta_1$/ $L(45)$: $\xi_1=\xi_2,\ \eta_1=\eta_2$.

Then $\tilde{S}$ is the union of the lines defined above. It is clear from the definition that $L(ij)$ and $L(i'j')$ intersect if and only if i, j, i', j' are mutually different. In particular, if $\{i_1,i_2,i_3,i_4,i_5\}=\{1,2,3,4,5\}$, the lines intersecting with $L(i_1i_2)$ are $L(i_3i_4)$, $L(i_3i_5)$, $L(i_4i_5)$ and their intersection is normal crossing. In the sequel, we denote by $[ij][i'j']$ the intersecting point of two lines $L(ij)$ and $L(i'j')$ if they intersect. There are 15 normal crossing points of the set $\tilde{S}$. (See PICTURE I)

We now briefly review the action of $\mathfrak{S}_5$ on Z. The symmetric group $\mathfrak{S}_5$ on five letters is generated by permutations $s_j=(j,j+1)$ $(j=1,2,3,4)$. As is known, P^2 admits a birational action of $\mathfrak{S}_5$ in the following manner:

$$s_1\ :\ (\xi_1:\xi_2:\xi_3)\to(\xi_1^{-1}:\xi_2^{-1}:\xi_3^{-1}),$$
$$s_2\ :\ (\xi_1:\xi_2:\xi_3)\to(\xi_1:\xi_1-\xi_2:\xi_1-\xi_3),$$
$$s_3\ :\ (\xi_1:\xi_2:\xi_3)\to(\xi_2:\xi_1:\xi_3),$$
$$s_4\ :\ (\xi_1:\xi_2:\xi_3)\to(\xi_1:\xi_3:\xi_2).$$

Then, there is a holomorphic action $\tilde{g}_j$ $(j=1,2,3,4)$ on Z so that $\pi\cdot\tilde{g}_j = g_j\cdot\pi$ and this induces an $\mathfrak{S}_5$-action on Z. This action preserves the set $\tilde{S}$ invariant so that it induces the permutation of the ten lines. Because of the naming of the ten lines, we find that if $g\in\mathfrak{S}_5$ permute i, j $(1\leq i,j\leq 5)$ to i', j' respectively, then g maps $L(ij)$ to $L(i'j')$. In particular, the $\mathfrak{S}_5$-action on ten lines is transitive. Moreover, the $\mathfrak{S}_5$-action on the 15 normal crossing points are also transitive.

§3. **The idea of the study**. Under the identification $(x,y) = (1-\xi_2/\xi_1, \xi_3/\xi_1)$ given in §1, the system in question is defined in a neighbourhood of the point $P = [25][34]$ of the space Z. Modifying differential equations slightly, we introduce the system

$$[\mathfrak{d}_x(\mathfrak{d}_x+c-1)-x(\mathfrak{d}_x+\mathfrak{d}_y+a)(\mathfrak{d}_x+b)]u = 0,$$
$$[\mathfrak{d}_y(\mathfrak{d}_y+c'-1)-y(\mathfrak{d}_x+\mathfrak{d}_y+a)(\mathfrak{d}_y+b')]u = 0,$$

which we denote by $M(\Sigma)$ $(\Sigma= (a,b,b';c,c'))$ in the sequel. Here $\mathfrak{d}_x=x\partial_x$, $\mathfrak{d}_y=y\partial_y$. Needless to say, the system $M(\Sigma)$ is defined in a neighbourhood of the point P. Therefore, what we have to do first is to extend the system $M(\Sigma)$ on the whole space Z. To accomplish this program, it is sufficient to write down the system near each of 15 normal crossing points of $\tilde{S}$. To explain next purpose, we need some preparation. Let P_0 be one of 15 points in question and let (x_0,y_0) be a local coordinate at P_0 such that $(x_0,y_0) = (0,0)$ is P_0 and that $x_0 = 0$, $y_0 = 0$ are local defining equations of two lines of $\tilde{S}$. Moreover, let $R_ju = 0$ $(j = 1, 2, 3,\ldots)$ be the system of differential equations defined in a neighbourhood of P_0 which is the analytic continuation of $\mathcal{M}(\Sigma)$. Then it follows from the definition that there are holomorphic functions $f_j(x_0,y_0)$ near P_0 with the condition $f_j(0,0)=1$ and pairs of numbers (α_j,β_j) such that $\tilde{f}_j(x_0,y_0)=x_0^{\alpha_j}y_0^{\beta_j}f_j(x_0,y_0)$ $(j=1,2,3,4)$ form linearly independent solutions to $R_k u = 0$ $(k=1,2,3,\dots)$. It is important to determine the pairs (α_j,β_j) which are called *exponents* at P_0. Moreover, the restriction of the function $f_j(x_0,y_0)$ to each of the lines $x_0 = 0$, $y_0 = 0$ satisfies a certain ordinary differential equation which is called *the induced differential equation*. The next purpose is then to deterimine the exponents and the induced equations. The third purpose is to clarify the relationship among Appell's functions F_2, F_3 and Horn's function H_2.

§4. The isotropy group of the point P. In the sequel, we assume that *the parameters* a,b,b',c,c' *are "generic" so that the arguments below go well.*

Let H be the isotropy subgroup of $\mathfrak{S}_5$ at the point P. Then H is generated by g_3 and g_2g_4. In particular, $H \simeq \mathbb{Z}^2\times\mathbb{Z}^4$. Writing down the actions of g_3, g_2g_4 with respect to the local coordinate (x,y), we find that $g_3:(x,y) \to (x/(x-1),y/(1-x))$, $g_2g_4:(x,y) \to (y,x)$. Corresponding to the H-action, solutions to $\mathcal{M}(\Sigma)$ are transformed to other solutions. Then we obtain well-known Kummer type formulas for $F_2(a,b,b';c,c';x,y)$ (cf. [AK]):

$$\begin{aligned}F_2(a,b,b';c,c';x,y) &=(1-x)^{-a}F_2(a,c-b,b';c,c';x/(x-1),y/(1-x))\\ &=(1-x)^{-a}F_2(a,b,c'-b';c,c';x/(1-y),y/(y-1))\\ &=(1-x-y)^{-a}F_2(a,c-b,c'-b';c,c';x/(x+y-1),y/(x+y-1))\end{aligned}$$

Also H acts on the space of parameters as follows: $g_3: b \leftrightarrow c-b$, $g_2 g_4$: $b \leftrightarrow b'$, $c \leftrightarrow c'$. This means that *the system* $M(\Sigma)$ *admits an* H-*action.*

§5. **Analytic continuation of** $M(\Sigma)$ **near the points** A, B. We concentrate our attention to the two points A = [13][25], B = [13][24] on $L(13)$. It follows from the definition that among 15 points, 12 points except P, Q = [45][23], R = [24][35] are transformed to A or B by the H-action. We are going to write down analytic continuations of the system $M(\Sigma)$ near points A, B.

(I) *The system near the point* A.

We take $(x_A, y_A) = (\xi_2/\xi_1, \xi_3/\xi_2)$ as a local coordinate at A. From the definition, $x = 1-x_A$, $y = x_A y_A$ and $x_A = 0$, $y_A = 0$ are local defining equations of lines $L(13)$, $L(25)$, respectively. We introduce a system of differential equations $M_A(\Sigma)$ on the (x_A, y_A)-space by

$$[(\partial_{x_A}-\partial_{y_A})(\partial_{x_A}+a+b-c)-x_A(\partial_{x_A}+a)(\partial_{x_A}-\partial_{y_A}+b)]u = 0,$$
$$[\partial_{y_A}(\partial_{y_A}+c'-1)-y_A(\partial_{x_A}-\partial_{y_A})(\partial_{y_A}+b')-x_A y_A(\partial_{x_A}+a)(\partial_{y_A}+b')]u = 0.$$

which is same as $M(\Sigma)$ by the coordinate transformation $(x,y) \to (x_A, y_A)$ on an open dense subset the (x_A, y_A)-space where the Jacobian is non-singular. There are fundamental solutions of $M_A(\Sigma)$ of the forms: $f_{A,1}(x_A, y_A)$, $x_A^{c-a-b} f_{A,2}(x_A, y_A)$, $x_A^{1-c'} y_A^{1-c'} f_{A,3}(x_A, y_A)$, $x_A^{c-a-b} y_A^{1-c'} f_{A,4}(x_A, y_A)$, such that each $f_{A,j}(x_A, y_A)$ is holomorphic near the point A and that $f_{A,j}(0,0) = 1$. By computing the induced equations, or by direct computation, we obtain the concrete forms of restrictions of $f_{A,j}(x_A,y_A)$ to the lines $L(13)$ $(x_A = 0)$, $L(25)$ $(y_A = 0)$:

$$f_{A,1}(x,0) = {}_2F_1(a,b;a+b-c+1;x),$$
$$f_{A,2}(x,0) = {}_2F_1(c-a,c-b;c-a-b+1;x),$$
$$f_{A,3}(x,0) = {}_2F_1(a-c'+1,b;a+b-c-c'+2;x),$$
$$f_{A,4}(x,0) = {}_2F_1(c+c'-a+1,c-b;c+c'-a-b;x),$$

$$f_{A,1}(0,y) = 1,\ f_{A,2}(0,y) = {}_2F_1(a+b-c,b';c';y),$$
$$f_{A,3}(0,y) = 1,\ f_{A,4}(0,y) = {}_2F_1(b'-c'+1,a+b-c-c'+1;2-c';y).$$

The system $\mathcal{M}_A(\Sigma)$ seems unfamiliar but by changing coordinate systems, we find that $\mathcal{M}(\Sigma)$ is reduced to a system contained in Horn's list. In fact, the following hold:

$$f_{A,2}(x_A,y_A) = (1-x_A)^{b-c}H_2(a+b-c,b',c-b,1-b,c';y_A,x_A/(1-x_A)),$$
$$f_{A,4}(x_A,y_A)$$
$$= (1-x_A)^{b-c}H_2(a+b-c-c'+1,b'-c'+1,c-b,1-b,2-c';y_A,x_A/(1-x_A)),$$

where $H_2(a,b,c,d,e;x,y)$ is one of Horn's functions (cf. [EMOT, p.224]). Since the element $g_4g_3g_2g_3g_4 = (2\ 5)$ (pemutation of 2, 5) is contained in H and fixes the point $A = [13][25]$, we obtain a Kummer type formula for the function H_2:

$$H_2(a+b-c,c'-b',c-b,1-b,c';y_A/(y_A-1),x_A(1-y_A)/(1-x_A))$$
$$= (1-y_A)^{a+b-c}H_2(a+b-c,b',c-b,1-b,c';y_A,x_A/(1-x_A)).$$

(II) *The system near the point* B.

We take $(x_B,y_B) = (\xi_3/\xi_1,\xi_2/\xi_3)$ as a local coordinate at B. From the definition, $x_B = x_Ay_A$, $y_B = 1/y_A$ and $x_B = 0$, $y_B = 0$ are local defining equations of lines $L(13)$, $L(24)$, respectively. We introduce a system of differential equations $M_B(\Sigma)$ on the (x_B,y_B)-space by

$$[\vartheta_{y_B}(\vartheta_{x_B}+a+b-c)-x_By_B(\vartheta_{x_B}+a)(\vartheta_{y_B}+b)]u = 0,$$

$$[\vartheta_{y_B}(\vartheta_{x_B}-\vartheta_{y_B}+b')+y_B(\vartheta_{x_B}-\vartheta_{y_B})(\vartheta_{x_B}-\vartheta_{y_B}+c'-1)$$
$$-x_By_B(\vartheta_{x_B}+a)(\vartheta_{x_B}-\vartheta_{y_B}+b')]u = 0,$$

$$[(\vartheta_{x_B}+a+b-c)(\vartheta_{x_B}-\vartheta_{y_B})(\vartheta_{x_B}-\vartheta_{y_B}+c'-1)$$
$$-x_B(\vartheta_{x_B}+a)(\vartheta_{x_B}-\vartheta_{y_B}+b')(\vartheta_{x_B}-\vartheta_{y_B}+a-c+1)]u = 0$$

which is same as $M_A(\Sigma)$ by the coordinate transformation $(x_A,y_A) \to (x_B,y_B)$ on an open dense subset of the (x_B,y_B)-space where the Jacobian is non-singular. We note that the third differential equation above follows from the former two when $y_B \neq 0$. As fundamental solutions of $M_B(\Sigma)$, we take the following ones: $f_{B,1}(x_B,y_B)$, $x_B^{c-a-b}f_{B,2}(x_B,y_B)$, $x_B^{1-c'}f_{B,3}(x_B,y_B)$, $x_B^{c-a-b}y_B^{b'+c-a-b}f_{B,4}(x_B,y_B)$, where each $f_{B,j}(x_B,y_B)$ is holomorphic near B and $f_{B,j}(0,0) = 1$. By computing the induced equations, we find that

$$f_{B,1}(x,0) = {}_3F_2(a,b',a-c+1;a+b-c+1,c';x),$$
$$f_{B,2}(x,0) = {}_3F_2(c-b,1-b,b'+c-a-b;c-a-b+1,c+c'-a-b;x),$$
$$f_{B,3}(x,0) = {}_3F_2(a-c'+1,b'-c'+1,a-c-c'+2;a+b-c-c'+2,2-c';x),$$
$$f_{B,4}(x,0) = 1,$$

$$f_{B,1}(0,y) = 1,\ f_{B,2}(0,y) = {}_2F_1(a-b,a+b-c-c'+1;a+b-b'-c+1:y),$$
$$f_{B,3}(0,y) = 1,\ f_{B,4}(0,y) = {}_2F_1(b',b'-c'+1;b'+c-a-b+1;y).$$

The system $\mathcal{M}_B(\Sigma)$ seems unfamiliar but by taking another coordinate system, we find that $\mathcal{M}_B(\Sigma)$ is reduced to a system for Appell's function F_3, that is, the following hold:

$$\begin{aligned} &f_{B,4}(x_B,y_B) \\ &= (1-x_By_B)^{b-c}F_3(b,c-b;b'-c'+1,1-b;b'+c-a-b+1;y_B,x_By_B/(x_By_B-1)). \end{aligned}$$

It is clear from the expression above that $f_{B,4}(x_B,y_B)$ is constant on the line $L(24):y_B = 0$.

§6. **The structure of the system on Z.** In the previous section, we defined systems near points A, B. Moreover, under the H-action, any of 12 normal crossing points of $\tilde{S}$ except P, Q, R is transformed to A or B. Therefore, we can define a system near such a point. In this way, we can construct a system $\tilde{\mathcal{M}}(\Sigma)$ on the space Z which is an analytic continuation of the system $\mathcal{M}(\Sigma)$.

It is possible to compute the exponents at each of the 15 points. The result is summerized in TABLE I.

Note on TABLE I. Let P_0 be the intersecting point of two lines $L(ij)$, $L(i'j')$. We take a local coorninate (x_0,y_0) near P_0 with the conditions (1) $P_0 = (0,0)$, (2) $L(ij) = \{x_0 = 0\}$, $L(i'j') = \{y_0 = 0\}$. Then there are four linearly independent solutions to $\tilde{\mathcal{M}}(\Sigma)$ of the form $x_0^{\alpha_j}y_0^{\beta_j}h_j(x_0,y_0)$ (j = 1, 2, 3, 4) such that each $h_j(x_0,y_0)$ is holomorphic in a neighbourhood of $(x_0,y_0) = (0,0)$ and that $h_j(0,0) = 1$. In TABLE I, the point in question is written by [ij][i'j'] and the pairs (α_j,β_j) (j = 1, 2, 3, 4) are written in the right hand side of the same line.

TABLE I

Intersecting point	Exponents
[34][25]	(0,0),(1-c,0),(0,1-c'),(1-c,1-c')
[13][25]	(0,0),(c-a-b,0),(1-c',1-c'),(c-a-b,1-c')
[13][24]	(0,0),(c-a-b,0),(1-c',0),(c-a-b,b'+c-a-b)
[13][45]	(0,0),(c-a-b,0),(1-c',0),(c-a-b,c+c'-a-b-b')
[14][25]	(a,0),(b,0),(a-c'+1,1-c'),(b,1-c')
[14][23]	(a,a),(b,a),(a-c'+1,a),(b,b+b')
[14][35]	(a,0),(b,0),(a-c'+1,0),(b,b+c'-a-b')
[12][34]	(0,0),(c'-a-b',0),(1-c,1-c),(c'-a-b',1-c)
[12][35]	(0,0),(c'-a-b',0),(1-c,0),(c'-a-b',b+c'-a-b')
[12][45]	(0,0),(c'-a-b',0),(1-c,0),(c'-a-b',c+c'-a-b-b')
[15][34]	(a,0),(b',0),(a-c+1,1-c),(b',1-c)
[15][23]	(a,a),(b',a),(a-c+1,a),(b',b+b')
[15][24]	(a,0),(b',0),(a-c'+1,0),(b',b'+c-a-b)
[23][45]	(a,0),(a,0),(b+b',0),(a,c+c'-a-b-b')
[35][24]	(0,0),(0,0),(0,b'+c-a-b),(b+c'-a-b',0)

The following results are consequences of the arguments in §§4,5.

(6.1) Four linearly independent solutions at P = [25][34] are expressed by Appell's function F_2.

(6.2) The four points A = [13][25], [14][25], [12][34], [15][34] form an H-orbit. Let A_0 be one of the four points. Then two of the fundamental solutions at A_0 are expressed by Horn's function H_2.

(6.3) We consider the four lines $L(24)$, $L(35)$, $L(23)$, $L(45)$ which form an H-orbit. From the definition, Q is the intersection of $L(23)$ and $L(45)$ and R is the intersection of $L(24)$ and $L(35)$.

Let L_0 be one of the four lines above. Then there is a unique solution f to the system $\widetilde{M}(\Sigma)$ (up to a constant factor) defined in a neighbourhood of the line L_0 such that "the restriction" of f to the line L_0 is non-zero constant. In other word, we consider the space Z_{L_0} which is obtained from Z by blowing down along the line L_0. Then there is a solution f' to the system on Z_{L_0} (obtained from $\widetilde{M}(\Sigma)$) such that f is the pull back of f'. The solution f is expressed by Appell's function F_3. In this sense, *Appell's function* F_3 *is attached to a line on* Z.

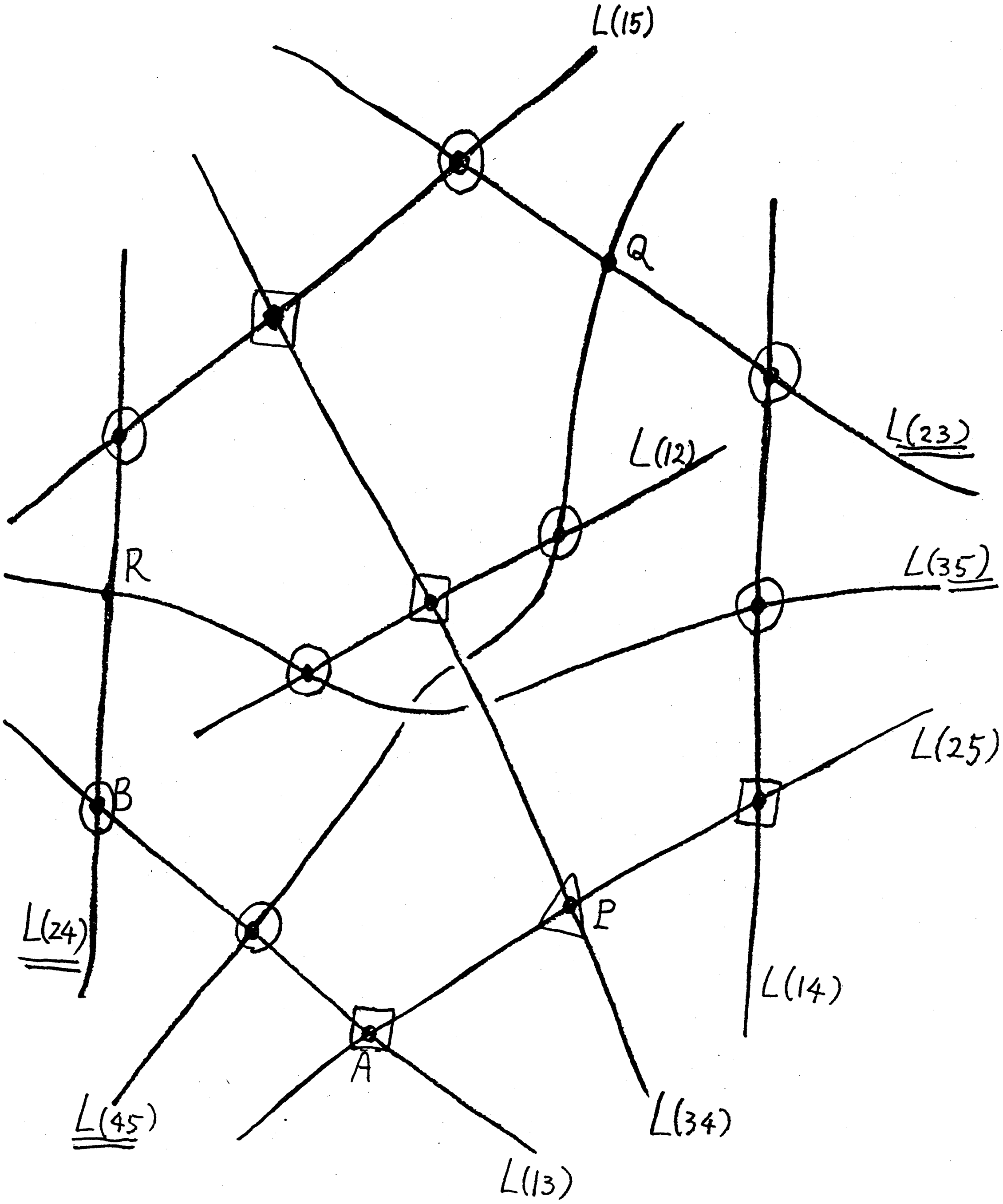
(6.4) The exponents of each of 15 normal crossing points except Q, R are multiplicity free. Now we concentrate our attention to the point R. Then there are two linearly independent holomorphic solutions at R (cf. TABLE I). It is hard to separate these two solutions. This is an obstruction to the determination of connection relations among fundamental set of solutions at 15 normal crossing points.

§7. **Connection formulas**. Let P_1, P_2 be normal crossing points of $\widetilde{S}$ and let $f_{P_i,j}$ (j = 1,2,3,4) be fundamental solutions to $\widetilde{M}(\Sigma)$ at P_i (i = 1, 2). We put $F_{P_i} = (f_{P_i,1}, f_{P_i,2}, f_{P_i,3}, f_{P_i,4})$ (i = 1, 2). Then there is a matrix $C_{P_1,P_2}(\Sigma)$ of order 4 depending only on Σ such that $F_{P_2} = F_{P_1} C_{P_1,P_2}(\Sigma)$. It is an interesting problem to determine these matrices $C_{P_1,P_2}(\Sigma)$ for various P_1, P_2. This is reduced to that for the case where both P_1, P_2 lie on a same line of $\tilde{S}$.

Now we assume that P_1, P_2 lie on a line L_0 of $\tilde{S}$ but $P_i \neq Q$, R (i = 1, 2). Then in virtue of the results of §4, we find that "the restriction" of $f_{P_i,j}$ to the line L_0 is expressed by $_2F_1(a,b;c;x)$ or $_3F_2(a_1,a_2,a_3;b_1,b_2;x)$, where P_1, P_2 correspond to the points $x = 0, \infty$ on the line L_0, respectively. Noting this, with the help of connection formulas among fundamental solutions at $x = 0$ and those at $x = \infty$ for hypergeometric functions $_2F_1(a,b;c;x)$ and $_3F_2(a_1,a_2,a_3;b_1,b_2;x)$, we find that all the matrix coefficients of $C_{P_1,P_2}(\Sigma)$ are expressed in terms of products of Gamma functions of variable Σ.

Next we assume that P_1, P_2 lie on a line L_0 of $\tilde{S}$ and one of P_1, P_2 equal Q or R, say $P_1 = R$. In this case, we can compute $C_{P_1,P_2}(\Sigma)$ if we can detrmine connection formulas among fundamental solutions of $_3F_2(a_1,a_2,a_3;b_1,b_2;x)$ at $x = 0$ and those at $x = 1$. But as is known, to compute the last connection formulas, we need the special value $_3F_2(a_1,a_2,a_3;b_1,b_2;1)$ at $x = 1$.

REFERENCES [AK] P. Appell and J. Kampe de Feriet : Fonctions Hypergeometriques et Hyperspheriques, Gauthier-Villars, 1926/ [EMOT] A. Erdelyi et al.: Higher Transcendental Functions, Vol. I, Robert E. Krieger Publishing Company, 1981/ [S] J. Sekiguchi : The birational action of $\mathfrak{S}_5$ on P^2 and the icosahedron, preprint.



PICTURE I

$\left\{ \begin{array}{l} \triangle \cdots F_2 \\ \square \cdots H_2 \\ \bigcirc \cdots F_3 \end{array} \right.$ (cf. § 6)

F_3 : attached to $\underline{\underline{L}}(ij)$