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<tr>
<td>Author(s)</td>
<td>SATO, Takeshi</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1991), 773: 59-65</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1991-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/82398">http://hdl.handle.net/2433/82398</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
A differential equation associated with the Horrocks-Mumford bundle

佐藤 猛 (Takeshi SATO)
(東大 理)

0. Introduction

Let $X$ be a bounded symmetric domain and let $\Gamma$ be a group which acts on $X$ discontinuously. $M$ denotes the quotient space $X/\Gamma$. $\pi$ is the projection from $X$ to $M$. We consider the inverse map $\pi^{-1}$ of the projection $\pi$. We call it the developing map.

\[ X : \text{a bounded symmetric domain} \]
\[ \downarrow \pi \]
\[ M = X/\Gamma \]

Let me give a problem.

**Problem.** Describe the developing map $\pi$ in terms of differential equation.

Let me give a classical example. Let $X$ be the upper half plane $H$ and let $\Gamma$ be Schwarz's triangle group i.e. its fundamental region is the sum of two hyperbolic triangles. We name its angles $\pi/n_1$, $\pi/n_2$ and $\pi/n_3$. And we assume that $n_1$, $n_2$ and $n_3$ are integers greater than 1. Then the quotient space $M$ is isomorphic to one-dimensional complex projective space $P_1(\mathbb{C})$.

\[ X = H \]
\[ \Gamma : \text{a Schwarz's triangle group} \]
In this case we have an answer to the problem. We consider a hypergeometric differential equation on $P_1(\mathbb{C})$.

\[ x(x-1)\frac{d^2 z}{dx^2} + \{\gamma + (\alpha + \beta - 1)x\} \frac{dz}{dx} - \alpha \beta z = 0 \]

And we assume that the parameters $\alpha$, $\beta$ and $\gamma$ satisfy the following conditions.

\[ |1 - \gamma| = \frac{1}{n_1}, \quad |\gamma - \alpha - \beta| = \frac{1}{n_2}, \quad |\alpha - \beta| = \frac{1}{n_3}. \]

Let $w_1$ and $w_2$ be the linearly independent solutions of the hypergeometric equation. Let $p$ be the multivalued map from $P_1(\mathbb{C})$ to $H$ that corresponds $w_1(z)/w_2(z)$ to $z$.

\[ p : P_1(\mathbb{C}) \rightarrow H \]
\[ z \quad \mapsto \quad \frac{w_1(z)}{w_2(z)} \]

**Theorem.** (Gauss, Schwarz) The map $p$ gives the developing map $\pi^{-1}$.

We shall consider the case that $X$ is Siegel upper half space $\mathcal{H}_2$ of genus two and $M$ is the three-dimensional complex projective space $P_3(\mathbb{C})$. 
1. Horrocks-Mumford bundle

We give a survey on the geometry of Horrocks-Mumford bundle. Sometimes we abbreviate Horrocks-Mumford to HM. The HM-bundle $\mathcal{F}$ is a holomorphic vector bundle of rank two on the four-dimensional projective space $P_4(\mathbb{C})$.

\[
\begin{array}{c}
\mathcal{F} \\
\downarrow \\
P_4(\mathbb{C})
\end{array}
\]

We don’t explain how to construct HM-bundle, because we do not need it for the following argument. (See [HoMu].) So we only give some properties of the HM-bundle without proof.

The space $S$ of its holomorphic sections is four-dimensional. For generic section $s$ in $S$, the zero set $X_s$ of $s$ is an abelian surface with $(1,5)$-polarization and level-5-structure. So we have a map $p$ from $S$ to the moduli space of such a abelian surfaces. It maps $s$ to $X_s$. Horrocks and Mumford proved that this map is birational.

On the other hand there is another way to construct such a moduli space. The quotient space of Siegel upper half space $\mathcal{H}_2$ by certain discontinuous group $\Gamma_{1,5}$ gives this moduli space. We omit the description of the group $\Gamma_{1,5}$. (See [HL].)

\[
P_3(\mathbb{C}) \cong P(S) \\
[s] \leftrightarrow X_s \in \left\{ \begin{array}{l}
 \text{abelian surface} \\
 (1,5)\text{-polarization} \\
 \text{level-}5\text{-structure}
\end{array} \right\} \cong \mathcal{H}_2/\Gamma_{1,5}
\]

Then we obtain the following diagram.

\[
\begin{array}{c}
\mathcal{H}_2 \\
\downarrow \pi \\
P_3(\mathbb{C})
\end{array} \leftarrow \Gamma_{1,5}
\]

This projection $\pi$ has branch locus.
Proposition. [BHM] The projection $\pi$ branches along the surface $D$ with the branch index two, where $D$ is given by

\[
\begin{align*}
&x_1^{10} - 5x_1^8x_2 + 20x_1^7x_2^2x_3 - 15x_1^7x_3^2 \\
&- 10x_1^6x_2^2 - 45x_1^6x_2x_3^3 + 5x_1^6x_3 + 16x_1^5x_2^5 \\
&- 140x_1^5x_2^3x_3 + 155x_1^5x_2x_3^2 + 27x_1^5x_3^5 - 2x_1^5 \\
&- 40x_1^4x_2^4x_3^2 + 50x_1^4x_2^3 + 295x_1^4x_2^2x_3 - 75x_1^4x_2x_3 \\
&- 15x_1^4x_3^4 - 80x_1^3x_2^6 + 220x_1^3x_2^4x_3 + 25x_1^3x_2^3x_3^4 \\
&- 515x_1^3x_2^2x_3^2 - 180x_1^3x_2x_3^5 + 5x_1^3x_2 + 50x_1^3x_3^3 \\
+ 200x_1^2x_2^5x_3^2 - 15x_1^2x_2^4 - 315x_1^2x_2^3x_3^3 + 155x_1^2x_2^2x_3 \\
+ 220x_1^2x_2x_3^4 - 10x_1^2x_3^2 - 180x_1^2x_2^5x_3 - 125x_1^2x_2^4x_3^4 \\
+ 295x_1^2x_2^3x_3^2 + 200x_1^2x_2^2x_3 - 15x_1^2x_3^5 - 140x_1^2x_2x_3^3 \\
- 80x_1^3x^6 - 5x_1^3x_3 + 27x_2^5 + 25x_2^4x_3^3 \\
- 45x_2^3x_3 - 40x_2^2x_3^4 + 20x_2x_3^2 + 16x_3^5 + 1.
\end{align*}
\]

They find this by studying the degeneration of abelian surfaces. We will answer the problem for this diagram.

2. Uniformizing differential equation

The Siegel upper half space $\mathcal{H}_2$ of genus two is isomorphic to the non-compact dual of the three-dimensional hyperquadrics $Q^3$ in four-dimensional projective space $P_4(\mathbb{C})$. Therefore $\mathcal{H}_2$ is naturally embedded in hyperquadrics.

We consider a system of differential equations (EQ) on $P_3(\mathbb{C})$ of rank five i.e. it has exactly five linearly independent solutions. Let $s_0, \ldots, s_4$ be the five linearly independent solutions. Then we obtain a multi-valued map $\Phi$ from $P_3(\mathbb{C})$ to $P_4(\mathbb{C})$. It maps $x \in P_3(\mathbb{C})$ to the ratio $[s_0(x) : \cdots : s_4(x)]$ of the solutions.

\[
\begin{array}{ccc}
\mathcal{H}_2 & \hookrightarrow & Q^3 \\
\downarrow & \pi & \nearrow \Phi \\
P_3(\mathbb{C}) & \\
\end{array}
\]
**Definition.** When the above diagram is commutative, we call this equation the uniformizing differential equation.

Our problem is to find the uniformizing differential equation. Let $x_1, x_2$ and $x_3$ be inhomogeneous coordinates of $P_3(\mathbb{C})$ and let $z$ be a solution of UDE. Since the rank of UDE is five, every derivative of $z$ can be expressed by linear combination of five basis. So we fix the basis $\{z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \frac{\partial z}{\partial x_3}, \frac{\partial^2 z}{\partial x_1 \partial x_3}\}$. There are no essential reason why we choose the base $\frac{\partial^2 z}{\partial x_1 \partial x_3}$. Then the uniformizing differential equation can be written in the following form.

\begin{equation}
\frac{\partial^2 z}{\partial x_i \partial x_j} = g_{ij} \frac{\partial^2 z}{\partial x_1 \partial x_3} + \sum A_{ij}^k \frac{\partial z}{\partial x_k} + A_{ij}^0 z
\end{equation}

**PROPOSITION.** The conformal class of the tensor $\varphi = \sum g_{ij} dx_i dx_j$ does not depend on the choice of local chart. And the pull-back of the tensor field by the projection $\pi$ gives the canonical conformal structure on $\mathcal{H}_2$ which is given by $^t(dz)A(dz)$, where $A$ is the matrix which defines the hyperquadrics i.e. $Q^3 = \{z \in P_4(\mathbb{C}); ^t z A z = 0\}$.

\[ \pi^*(\phi) \cong ^t(dz)A(dz) \]

So in order to obtain the coefficients $g_{ij}$, we have to express $^t(dz)A(dz)$ in terms of the inhomogeneous coordinates $x_1, x_2$ and $x_3$.

Let $\theta$ be a function s.t. $\det(e^\theta g_{ij}) = 0$ and let $\Gamma_{ij}^k$, $R_{ij}$ and $R$ be Christoffel symbol, Ricci tensor and Scalar curvature with respect to $e^\theta g_{ij}$ respectively. $S_{ij}$ is the Schouten tensor defined by

\[ S_{ij} = R_{ij} - \frac{R}{4} e^\theta g_{ij}. \]

Now we introduce a theorem due to Sasaki and Yoshida.

**THEOREM.** Let $\varphi$ be conformally flat. When we put

\[ A_{ij}^k = \Gamma_{ij}^k - g_{ij} \Gamma_{13}^k \]
$A_{ij}^{0} = S_{ij}^{k} - g_{ij}S_{13}$

Then (♠) is integrable and of rank five. And the image of $\Phi$ is in a hyperquadrics.

$\text{Im}(\Phi) \subset Q^3$.

So if we have the coefficients $g_{ij}$, we can calculate other coefficients $A_{ij}^{k}$ according to the theorem. In order to calculate $g_{ij}$, the following properties of $\varphi$ are effective.

1. The tensor $\varphi$ is conformally flat.
2. Each $g_{ij}$ is a polynomial of degree 4
3. $\sum_{i=1}^{3} \frac{\partial D}{\partial x_i} \cdot \Delta_{ij} \equiv 0 \text{ (mod } D)$

where $\Delta_{kl}$ is the $(k, l)$-cofactor of the matrix $g_{ij}$.
4. $\det\{g_{ij}\} = D$.
5. The tensor field $\varphi$ is invariant under the action of the alternating group $\mathfrak{A}_5$ of degree five.

So these conditions enable us to obtain the coefficients $g_{ij}$.

**Main Theorem.** The coefficients $g_{ij}$ of UDE are given by

\[
\begin{align*}
g_{11} &= -2(x_1^2 x_2^2 + x_1^2 x_3 - 2x_1 x_2 x_3^2 - x_1 + 3x_2^3 - 2x_2 x_3) \\
g_{12} &= 2x_1^3 x_2 - 3x_1^2 x_3^2 + 2x_1 x_2^2 + 4x_1 x_3 - 1 \\
g_{13} &= x_1^3 - x_1^2 x_2 x_3 - x_1 x_2 + 5x_2^2 x_3 - 4x_3^2 \\
g_{22} &= -2(x_1^4 - x_1^2 x_2 - 5x_1 x_3^2 + x_3) \\
g_{23} &= 3(x_1^3 x_3 - x_1^2 - 5x_1 x_2 x_3 + x_2) \\
g_{33} &= -2(x_1^3 x_2 - 5x_1 x_2^2 - x_1 x_3 + 1) \\
g_{21} &= g_{12}, \ g_{31} = g_{13}, \ g_{32} = g_{23}. 
\end{align*}
\]

Of course it is not so difficult to calculate $A_{ij}^{k}$ if we use computer. However we omit them because they are very complicated.
REFERENCES


