A SPECIAL FUCHSIAN SYSTEM CONNECTING SOME HILBERT PROBLEMS

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0.   **INTRODUCTION**

This survey draws an investigation line from a special Fuchsian system of differential equations to special diophantine equations. E. PICARD [26] began to study more intensively the following special system of differential equations:

(0.1) $$D_{ij}\, F(u,v) = 0 \text{ on } \mathbb{C}^2 \setminus \boxed{\triangle} = \mathbb{P}^2 - \boxtimes\,,$$

$$(i,j) = (1,1),(1,2),(2,2)\,,$$

$$D_{11} = \frac{\partial^2}{\partial u^2} + [9(u-1)u(v-u)]^{-1}\{3(-5u^2+4uv+3u-2v)\frac{\partial}{\partial u}+3(v-1)v\frac{\partial}{\partial v}+(u-v)\}\,,$$

$$D_{12} = \frac{\partial^2}{\partial u \partial v} + [3(u-v)]^{-1}\{\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\}\,,$$

$$D_{22} = \frac{\partial^2}{\partial v^2} + [9(v-1)v(u-v)]^{-1}\{3(u-1)u\frac{\partial}{\partial u}+3(-5v^2+4uv+3v-2u)\frac{\partial}{\partial v}-(u-v)\}\,.$$

$$\boxed{\triangle} := \{(u,v) \in \mathbb{C}^2;\ uv(u-1)(v-1)u-v) \neq 0\}\,,$$

$\boxtimes$:= six lines in $\mathbb{P}^2$ through pairs of four points in general

  position.

PICARD conjectured that the solutions of (1) should have an arithmetic meaning comparible with the role of elliptic integrals for plane cubic diophantine equations.

Fortunately D. HILBERT observed carefully the work of PICARD. We want to use some of the celebrated 23 problems of HILBERT [10] as an intuitive guide for a deeper arithmetic study of the solutions of (1). Via actual work of PARSHIN and VOJTA we discover at the end an interesting connection with FERMAT's equations.

Notations

$\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}$: complex, real, rational algebraic numbers, integers;

$\mathbb{P}^N$: (complex) projective space of dimension N;

$\mathbb{P}^N(L)$: Points of $\mathbb{P}^N$ with coordinates in the field $L \subseteq \mathbb{C}$;

$V(L) = V \cap \mathbb{P}^N(L)$, V a subvariety of $\mathbb{P}^N$;

$Gl_n^+(\mathbb{R})$: elements of $Gl_n(\mathbb{R})$ with positive determinant;

$\mathbb{P}Gl_n$: the projective group Gl_n / Gl_1;

$\mathbb{P}G$: the image of G in $\mathbb{P}Gl_n$, G a subgroup of Gl_n;

$U((2,1),A)$: the unitary group of a hermitean form of signature

(2,1) with coefficients in the ring $A \subseteq \mathbb{C}$ closed under complex

conjugation;

$\mathcal{O}_L$: ring of integers in the number field L;

$\Gamma_L = U((2,1),\mathcal{O}_L)$ the full PICARD modular group of the imaginary

quadratic number field L;

$K = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\rho)$: the field of EISENSTEIN numbers, ρ a primitive

third unit root;

$\mathbb{B}^2 = \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 < 1\}$ the standard complex 2-ball;

$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots , \quad q = \exp(2\pi i \tau)$

the elliptic modular function defined on

$\mathbb{H} = \{\tau \in \mathbb{C}; \text{Im}\,\tau > 0\}$ the POINCARE upper half plane;

$\mathbb{H}_g = \{Z \in Gl_g(\mathbb{C}); Z = {}^t Z, \text{Im}\, Z > 0\}$ the generalized SIEGEL upper

half plane;

$Sp(2g,A)$: symplectic group acting on $\mathbb{H}_g$, $A = \mathbb{R}$ or $\mathbb{Z}$;

$\mathcal{A}_g = Sp(2g,\mathbb{Z}) \backslash \mathbb{H}_g$: the (non-compact) moduli space of g-dimensional

(principally) polarized abelian varieties.

Hint: underlying on pages 5-9 should be : flat

1.   The Central Problem

A central role in HILBERT's program plays the following

12-th HILBERT PROBLEM (Extension of Kronecker's theorem to arbi-

trary algebraic domains of rationality): "... to find and discuss

those functions playing for arbitrary algebraic number fields the

same role as the exponential function for the field of rational

numbers and the elliptic modular function for imaginary quadradra-

tic fields".

We remember to the elliptic modular function $j: \mathbb{H} \longrightarrow \mathbb{P}^1$. On

$\mathbb{H}$ acts $\mathfrak{Sl}_2^+(\mathbb{R})$, $\mathbb{P}^1$ is the compactification of $\mathbb{H}/Sl_2(\mathbb{Z}) \cong \mathbb{C}$ and j

describes the quotient map. We call $\tau \in \mathbb{H}$ a   s i n g u l a r

m o d u l e , if it is an isolated fixed point of (a $g \in$) $\mathfrak{Sl}_2^+(\mathbb{Q})$.

1.1 Theorem. Let τ be a point of $\mathbb{H}(\overline{\mathbb{Q}})$. Then

(i) $j(\tau)$ is a transcendent number iff τ is not a singular module;

(ii) If τ is a singular module then $j(\tau)$ is a class field (abelian

extension, ray class field) of the imaginary quadratic field $\mathbb{Q}(\tau)$:

it is the HILBERT class field (maximal unramified abelian exten-

sion of $\mathbb{Q}(\tau)$, if $\mathbb{Z}1 + \mathbb{Z}\tau$ is a fractional ideal of $\mathbb{Q}(\tau)$.

The transcendence proof is due to SIEGEL [34]. This is an ex-

tension of the 7-th HILBERT PROBLEM asking for the transcendence

of $e(\alpha) = \exp(\pi i \alpha)$ for $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, affirmatively solved by Gelfond

[8] and Schneider [28]. For a proof of (ii) we refer to SHIMURA

[33]. It is a part of KRONECKER's JUGENDTRAUM claiming the explicit construction of all abelian extensions of imaginary quadratic number fields by means of special values transcendendent functions. The JUGENDTRAUM appears as KRONECKER's problem in the first part of the 12-th HILBERT PROBLEM. It has been solved by TAKAGI [35] in 1921. Nowadays its solution is known as Main Theorem of Complex Multiplication, see [21].

2.   Ball Uniformization of Surfaces

Let X be an algebraic surface over $\mathbb{C}$, compact for a moment, with at most quotient or ball c u s p s i n g u l a r i t i e s . For a precise definition and classification of these singularities we refer to [16]. In this paper a special class of algebraic cycles is defined consisting of weigted irreducible curves and points on X. The weights are natural numbers or ∞. Finitely weighted points are q u o t i e n t p o i n t s. Infinitely weighted curves or points are c u s p c u r v e s or c u s p p o i n t s , respectively. For a precise definition/classification we refer to [16] again. We call these special cycles o r b i t a l c y c - l e s (on X). A pair $\underline{X} = (X,\underline{D})$, $\underline{D}$ an orbital cycle on X, is called an o r b i t a l s u r f a c e .

2.1 Example. Let $X = \mathbb{P}^2$, Q_1, Q_2, Q_3, Q_4 four points on $\mathbb{P}^2$ in general position, $L_{ij} = L_{ji}$ the line through Q_i, Q_j. By abuse of notations we define an orbital cycle by $\underline{D} = \sum_{1 \leq i < j \leq 4} 3L_{ij} + \sum_{k=1}^{4} \infty \cdot Q_k$, where the

weights appear as coefficients. The pair $(\mathbb{P}^2, \underline{D})$ is an orbital surface.

2.2 Remark. If X is a multi-blowing up of $\mathbb{P}^2$ and D is supported by lines, then HIRZEBRUCH called D a (linear) arrangement. see [3].

If we omit on X and D all infinitely weighted curves and points. then we get an o p e n o r b i t a l s u r f a c e $\underline{X}_f = (X_f, \underline{D}_f)$.

2.3 Main example. The group $\mathbb{U}((2,1,.\mathbb{C})$ acts an the 2-ball $\mathbb{B} \subset \mathbb{P}^2$ via projective transformations. Let Γ be an arithmetic subgroup of $\mathbb{U}((2,1),\mathbb{C})$. Then the quotient $X_f = \mathbb{B}/\Gamma$ is a (generally open) algebraic surface. The irreducible curves of the branch locus of the covering $\mathbb{B} \longrightarrow \mathbb{B}/\Gamma$ and the images of Γ-elliptic points on $\mathbb{B}$ are endowed with the corresponding ramification indices as weights (for a point after blowing up its preimages). Denote the corresponding (open) orbital cycle by $\underline{D}_f$. Then $\underline{X}_f = (X_f, \underline{D}_f)$ is an open orbital surface. We add the infinitely weighted compactification points of the Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ to the "closure" $\widehat{\underline{D}}_f$ of $\underline{D}_f$. The pairs $(X_f, \underline{D}_f)$ or $(\widehat{X}_f, \widehat{\underline{D}}_f)$ are called o p e n or c o m p a c t i f i e d o r b i t a l b a l l q u o t i e n t s u r f a c e s , respectively. sometimes they are denoted by $\underline{\mathbb{B}/\Gamma}$ or $\widehat{\mathbb{B}/\Gamma}$, respectively.

A fundamental question is: Which orbital surfaces are ball quotients ? A precise answer can be considered as a special but impor-

tant partial answer to the 22-nd HILBERT PROBLEM entitled "Uniformization of analytic relations by means of automorphic functions".

In order to find a constructive answer we introduced in [16] successively some invariants called (l o c a l) and g l o b a l) o r b i t a l h e i g h t s (respectively):

$$h_\varrho, h_\tau: \{\text{orbital curves}\} \longrightarrow \mathbb{Q} \ ,$$

$$H_\varrho, H_\tau: \{\text{orbital surfaces}\} \longrightarrow \mathbb{Q} \ .$$

O r b i t a l c u r v e s are defined to be irreducible 1-dimensional components $\underline{C}$ of (embedded !) orbital cycles $\underline{D}$ (with all weight informations of $\underline{D}$ along C). We give an implicit definition: If everything is smooth, compact and has trivial weight 1, then

$$h_\varrho(\underline{C}) = h_\varrho(C) = -(\text{Euler number of C}),$$

$$h_\tau(\underline{C}) = h_\tau(C) = -(\text{selfintersection index of } C \subset X),$$

$$H_\varrho(\underline{X}) = H_\varrho(X) = \text{Euler number of X},$$

$$H_\tau(\underline{X}) = H_\tau(X) = \text{signature of X}.$$

If $\underline{f}: \underline{X} \longrightarrow \underline{X}'$, $\underline{C} \longrightarrow \underline{C}'$ are finite coverings, then the following degree formulas hold:

$$H(\underline{X}) = \deg(f)H(\underline{X}') \ , \ h(\underline{C}) = \deg(\underline{f}|\underline{C})h(\underline{C}'), \ H = H_\varrho, H_\tau \ , \ h = h_\varrho, h_\tau.$$

Roughly spoken, a f i n i t e c o v e r i n g $\underline{f}$ is a usual finite covering f: X $\longrightarrow$ X' compatible with weights in the sense of Galois theory. The definition restricts to orbital curves. Proofs of existence via explicit definitions can be found in [13] and the related literature.

2.4 Theorem ([16]). If $\underline{X}$ = (X,$\underline{D}$) = $\widehat{\mathbb{B}/\Gamma}$ is an orbital ball quotient, then it holds that

(2.5) $H_e(\underline{X}) = 3H_\tau(\underline{X}) > 0$, $h_e(\underline{C}) = 2h_\tau(\underline{C}) > 0$, $\underline{C} < \underline{D}$.

The discrete subgroup $\mathbb{P}\Gamma$ of $PU((2,1),\mathbb{C})$ is uniquely determined up to conjugation by the orbital surface $\underline{X}$.

So we found effective necessary conditions for an orbital surface to be a ball quotient. R.KOBAYASHI [20] gave sufficient conditions in another language. Until now there is no proof of the equivalence of Kobayashi's and our conditions. The following considerations indicate that they cannot be far away from each other.

The relations (2.5) can be understood as a system DIOPH(X,D) of diophantine equations and inequalities, if we write it down explicitly with all weights as variables. The coefficients depend only on geometric data of the supporting surface X and the supporting cycle D. It turns out that

2.6 Theorem ([16]). For any (admissible) pair (X,D) the system DIOPH(X,D) has at most finitely many solutions. These solutions can be calculated in an effective manner.

So a ball uniformization of a surface X with given branch locus D can only happen, if we find a solution of DIOPH(X,D) and there are, up to isomorphy, at most finitely many possibilities.

2.7 Corollary. If we fix infinite weights at the four triple points of $\triangle \subset \mathbb{P}^2$ defined in (0.1), then DIOPH($\mathbb{P}^2$, $\triangle$) has exactly one solution. The corresponding orbital surface ($\mathbb{P}^2$, $\underline{\triangle}$) coincides with that of Example 2.1.

one variable, "...to show that in any case there exists a linear differential equation of the FUCHSian class with given singular locus and prescribed monodromy group". The final solution of this HILBERT PROBLEM has been given by H. ROEHRL [27]. We change over and restrict ourselves to the second dimension.

3.1 Theorem (M.YOSHIDA [38]). Let $\underline{X} = (X,\underline{D})$ be an orbital surface uniformizable by the ball with quotient map $p: \mathbb{B} \longrightarrow X$. Then the inverse p^{-1} of p is a (multivalued) developing map of a FUCHSian system of linear partial differential equations.

It means that there is locally a fundamental system of solutions I_0, I_1, I_2 extending analytically to $X \smallsetminus D$, such that the multivalued map

$$(I_0 : I_1 : I_2): X \smallsetminus D \longrightarrow \mathbb{B} \subset \mathbb{P}^2,$$

$P \longmapsto (I_0(P):I_1(P):I_2(P))$, coincides with p^{-1} on $X \smallsetminus D$. The FUCHSian system is called the u n i f o r m i z i n g   e q u a - t i o n   of the orbital surface and the uniformizing group Γ is the m o n o d r o m y   g r o u p   of the system. Via solutions one gets a unitary representation of the fundamental group

$$\pi_1(X \smallsetminus D) \longrightarrow \mathbb{P}\Gamma \subset \mathbb{P}Gl_3(\mathbb{C}).$$

YOSHIDA [38] found an effective method in order to determine a corresponding Fuchsian system. Together with 2.8 one gets

3.2 Theorem. The system (0.1) is a uniformizing equation of the orbital surface $(\mathbb{P}^2, \underline{\mathbb{A}})$. Its monodromy group is the PICARD modular group $\Gamma(\sqrt{-3})$.

Now one can check KOBAYASHI's conditions to see that the ball uniformization of $(\mathbb{P}^2, \triangle)$ really exists. In [11] we proved more:

2.8 Proposition. Let $\Gamma(\sqrt{-3}) \subset U((2,1), \mathcal{O}_K)$, $K = \mathbb{Q}(\sqrt{-3})$, be the congruence subgroup corresponding to the ideal $(1-\varrho)$, $\varrho = e^{2\pi i/3}$, of $\mathcal{O}_K$. Then the orbital surfaces $\mathbb{B}/\Gamma(\sqrt{-3})$ and $(\mathbb{P}^2, \triangle)$ coincide.

For a proof it is convenient to use the following

2.9 Theorem ([16]). For the c_λ-volume $c_\lambda(\Gamma)$ of a fundamental domain of a ball lattice Γ it holds that

(2.10) $\qquad c_\lambda(\Gamma) = H_\varrho(\mathbb{B}/\Gamma)$, $c_\lambda(\Gamma)/3 = H_\tau(\mathbb{B}/\Gamma)$.

In [15] we presented an effective formula for $c_\lambda(\Gamma_M)$, M an imaginary quadratic number field, in terms of special value of L-series using arithmetic-geometric methods in the proof. Then one gets $H_\varrho(\mathbb{B}/\Gamma(\sqrt{-3}))$, $H_\tau(\mathbb{B}/\Gamma(\,-3))$, the Chern numbers $c_\lambda(\widehat{\mathbb{B}/\Gamma(\sqrt{-3})}) = 3$, $c_\Lambda^2(\widehat{\mathbb{B}/\Gamma(\sqrt{-3})}) = 9$ after classification of elliptic points and cusps, and finally $\mathbb{B}/\Gamma(\sqrt{-3}) = \mathbb{P}^2$ by the theory of surface classification.

3.   Ball Uniformization and Differential Equations

In [38] M.YOSHIDA succeeded to solve a higher-dimensional version of the RIEMANN-HILBERT problem. The background is HILBERT's 21-st PROBLEM "Proof of the existence of linear differential equations with prescribed monodromy groups" set up for functions of

4.　　　　　　　GAUSS-MANIN Connection

YOSHIDA's general approach lifting the Gauss-Schwarz of

FUCHSian equations to higher dimensions has a classical origin

in the work of PICARD and APPELL. Especially for linear arrange-

ments of low-dimensional projective spaces a more immediate result

known as PTDM-Theorem (due to PICARD, TERADA, MOSTOW, DELIGNE)

would be sufficient for our purposes. We refer to [38], [3] and

further literature given there. But we prefer to change over from

the analytic viewpoint to an algebraic-geometric approach in order

to find "algebraic solutions" of special FUCHSian equations repre-

sented by integrals on algebraic curves along cycles depending on

on parameters u,v. The general framework of the corresponding al-

gebraic theory is known as G a u s s - M a n i n c o n n e c -

t i o n of algebraic families of algebraic manifolds. For more

details we refer to [11] in order to understand the rather expli-

cit theory of algebraic families of curves involving differential

equations.

Let $\mathscr{C}$/T be a smooth algebraic family of smooth algebraic va-

rieties all defined over the complex numbers, say. The relative

DE RHAM complex is a sequence

$$\tilde{\Omega}_{\mathscr{C}/T}: \quad \mathscr{O}_T \xrightarrow{d} \Omega^1_{\mathscr{C}/T} \xrightarrow{d} \Omega^2_{\mathscr{C}/T} \xrightarrow{d} \ \dots$$

Using open (affine, say) coverings one defines the ČECH complexes

$$C^{\cdot}(\Omega^q_{\mathscr{C}/T}): \quad C^0(\Omega^q_{\mathscr{C}/T}) \xrightarrow{\delta} C^1(\Omega^q_{\mathscr{C}/T}) \xrightarrow{\delta} C^2(\Omega^q_{\mathscr{C}/T}) \ \dots$$

in the usual manner. Taking the limit along refinements of open

coverings one gets the ČECH - DE RHAM bicomplex $C^{\cdot\cdot}(\Omega_{\mathscr{C}/T})$. The

DE RHAM cohomology groups $\mathbb{H}^i_{DR}(\mathcal{C}/T)$ of the family $\mathcal{C}/T$ are the hypercohomology groups of $C^{\cdot}$ ($\underline{\Omega}_{\mathcal{C}/T}$) defined as cohomology groups of the corresponding total Čech - de Rham complex C^{tot} ($\underline{\Omega}_{\mathcal{C}/T}$). The construction applies to all restricted families $\mathcal{C}_u/U$, U an open part of T. On this way one gets the D E R H A M c o h o - h o m o l o g y s h e a v e s $\mathcal{H}^i_{DR}(\mathcal{C}/T)$ on T.

We restrict ourselves now to curve families $\mathcal{C}/T$. For our pur-poses it suffices to assume that T is an affine part of a projec-tive space $\mathbb{P}^N$. Let $\mathcal{D}_T$ be the sheaf of differential operators on T. Then the de Rham cohomology sheaf $\mathcal{H}^1_{DR}(\mathcal{C}/T)$ is not only an $\mathcal{O}_T$-module but also a $\mathcal{D}_T$-module sheaf. Looking for a family with a section $\bar{\omega}$ in $\mathcal{H}^1_{DR}(\mathcal{C}/T)$ satisfying the differential equations (0.1) with $\bar{\omega}$ instead of F one can take the P I C A R D c u r - v e f a m i l y

$$\mathcal{C}/\mathbb{P}^2 \smallsetminus \mathbb{A} : Y^3 = X(X-1)(X-u)(X-v)$$

and $\bar{\omega}$ represented by the differential form $\omega = dx/y$ depending on u,v. Taking integrals along cycles one gets an "algebraic" fun-damental system of solutions

(4.1)

$$I_k(t) = \int_{\alpha_k(t)} \omega(t), \quad k = 1,2,3, \quad t = (u,v) \in \mathbb{P}^2 \smallsetminus \mathbb{A}, \quad \omega = dx/y$$

Altogether we found the developing map of the FUCHSian system (0.1) in an explicit and algebraic manner. Looking back to the geometric

starting point and

to the results of the earlier sec-tions we receive

4.2 Theorem. The quotient map $p: \mathbb{B} \longrightarrow \mathbb{P}^2$ with covering group $\Gamma(\sqrt{-3})$ is inverted by $(I_0 : I_1 : I_2): \mathbb{P}^2 \setminus \triangle \longrightarrow \mathbb{B}$ on $\mathbb{P}^2 \setminus \triangle$ with cycloelliptic integrals $I_k(t)$ described in (4.1) along linearly independent cycle families $\alpha_0(t)$, $\alpha_1(t)$, $\alpha_2(t)$.

4.3 Remark. Historically p^{-1} was first known [26]. The monodromy group was known to be a sublattice of $\Gamma(\sqrt{-3})$ generated by five elements, see [1].

5.   Moduli Space of PICARD Curves.

5.1 Definition. A P I C A R D c u r v e is an algebraic (complex compact) curve isomorphic to one of the following plane curves of equation type

(5.2)

$$Y^3 = \prod_{i=1}^{4} (X - e_i) = X^4 + G_2 X^2 + G_3 X + G_4 \qquad \text{(affine)},$$

$$WY^3 = \prod_{i=1}^{4} (X - e_i W) = X^4 + G_2 W^2 X^2 + G_3 W^3 X + G_4 W^4 \quad \text{(projective)},$$

Notice that $\sum_{i=1}^{4} e_i = 0$.

One proves that the n o r m a l f o r m (5.2) of a PICARD curve C is well-defined up to a common factor of the e_i's, if C is smooth, that means $e_i \neq e_j$ for $i \neq j$, see [14]. In other words:

5.3 Proposition-Definition. The m o d u l i s p a c e o f s m o o t h PICARD c u r v e s is $(\mathbb{P}^2 \setminus \triangle)/S_4$. The m o d u l i s p a c e o f PICARD c u r v e s is $\mathbb{P}^2/S_4$.

6.   The Relative SCHOTTKY Problem

Smooth PICARD curves C have genus 3. The Jacobian (variety) of

C is denoted by J(C) and Jac(C) denotes the canonically polarized

Jacobian of C. The correspondence C $\longmapsto$ Jac(C) induces a biratio-

nal map jac: $\mathbb{P}^2/S_4 \cdot \rightarrow \overline{\mathcal{A}_3}$, where $\overline{\mathcal{A}_3}$ is the moduli space of prin-

cipally polarized abelian threefolds. Its restriction to $\mathbb{R}^2 \setminus \triangle$

is an open embedding by TORELLI's Theorem and 5.3. We want to uni-

formize the map jac in an effective manner, that means we look for

a commutative diagram

(6.1)

The vertical arrows denote quotient maps by the arithmetic groups

$\Gamma = \mathbb{U}((2,1),\mathcal{O}_K)$ or Sp(6,$\mathbb{Z}$), respectively. The r e l a t i v e

SCHOTTKY p.r o b l e m f o r PICARD c u r v e s asks for the

explicit knowledge of $*$ in terms of period matrices.

The new ball $\mathbb{B} \subset \mathbb{P}^2$ corresponds to the hermitean (2,1)-metric

on $\mathbb{C}^3$ defined by $\begin{pmatrix} 0 & 0 & \overline{\gamma} \\ 0 & 1 & 0 \\ \gamma & 0 & 0 \end{pmatrix}$; $\langle \ , \ \rangle$ denotes the hermitean product.

Now we define successively the maps

(6.2)

$$*: \; \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \; , \; (A,B,C) \longmapsto (A,B,-\tfrac{?}{3}A,C,\tfrac{?}{3}B,\tfrac{?}{3}C) \; ,$$

$$P: \; \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathrm{Mat}_{3\times 6}(\mathbb{C}) \; , \; (\alpha,\ell,\nu) \longmapsto \begin{pmatrix} \overline{*\alpha} \\ \overline{*\ell} \\ \overline{*\nu} \end{pmatrix},$$

π : the restriction of P to triples with the conditions:

$$\langle \alpha, \alpha \rangle < 0, \quad \ell \perp \alpha, \quad \kappa \perp \alpha, \quad \ell, \nu \text{ linearly independent};$$

$$*: \; \mathbb{B} \longrightarrow \mathbb{H}_3 \; , \; \mathbb{P}\alpha \longmapsto Gl_3(\mathbb{C}) \backslash \pi(\alpha, \ell, \nu)$$

The images of π (or $*$) are called t y p i c a l p e r i o d.

m a t r i c e s (p o i n t s). The orthogonality conditions _

come from the RIEMANN period relations.

6.3 Theorem ([14]). A principally polarized abelian threefold is

the Jacobian of a PICARD curve if and only if it corresponds to a

typical period point in $\mathbb{H}_3$.

The proof needs the construction of t y p i c a l symplectic

bases of $H_1(C,\mathbb{Z})$ and the use of GALOIS-invariant bases of $H^0(C,\Omega_C^1)$

on smooth PICARD curves C due to PICARD [26].

7.   Effective TORELLI Theorem

We would like to make TORELLI's Theorem for PICARD curves ef-

fective by means of transcendent functions in analogy to the el-

liptic modular function j. Our problem is to find for given $\tau \in \mathbb{B}$

(or $*\tau \in *\mathbb{B} \subset \mathbb{H}_3$) the normal equation (see (5.2)) of a PICARD cur-

ve C_τ corresponding to the moduli point $p(\tau) \in \mathbb{P}^L$. In other words

we look for an explicit analytic description of the quotient maps

in diagram (6.1).

7.1 Definition. a holomorphic function $f: \mathbb{B} \longrightarrow \mathbb{C}$ is called a

PICARD m o d u l a r f o r m $(\Gamma' - f o r m)$ o f w e i g h t

m, if there is a sublattice Γ' of a PICARD modular group such that

(7.2) $\gamma^*(f) = j_\gamma^m \cdot f$ for all $\gamma \in \Gamma'$,

where $j_\gamma(\tau)$ is the JACOBI determinant of γ at $\tau \in \mathbb{B}$.

7.3 Theorem ([11]). There is a basis t_1, t_2, t_3 of the space of

$S\Gamma(\overline{\sqrt{-3}})$-forms of weight 1 such that

$$\tau \longmapsto (t_1(\tau) : t_2(\tau) : t_3(\tau) : t_4(\tau))$$

with $t_1 + t_2 + t_3 + t_4 = 0$ is the quotient map $p: \mathbb{B} \longrightarrow \mathbb{P}^L$.

The proof goes through the fine surface classification of

$X = \mathbb{B}/S\Gamma(\sqrt{-3})$ and identification of the modular forms with ~~see~~ *ele-*

~~tions~~ *ments* of $H^0(\widetilde{X}, \mathcal{O}(mK_{\widetilde{X}} + mT))$, $\widetilde{X}$ the minimal smooth compactification

and T the compactifying curve. The surface $\widetilde{X}$ is described by the

weighted homogeneous equation

(7.4) $s^3 = (s_2^L - s_1^L)(s_2^L - s_0^L)(s_1^L - s_0^L)$, $wt(S_\iota) = 1$, $wt(S) = 2$.

This is a nice example fitting in the 22-nd HILBERT PROBLEM again,

see section 2. Namely, the ring of $S\Gamma(\sqrt{-3})$-forms is $\mathbb{C}[s_0, s_1, s_2, s]$

with generators satisfying the same relations as described in

(7.4) The compactification theory of BAILY-BOREL [2] is the ge-

neral framework of solution of HILBERT's 22-nd PROBLEM, by means

of automorphic forms, but the explicit solution remains to be a case-by-case problem.

We change over from the left to the right quotient map in diagram (6.1) and guess that the theory of theta functions is helpful to describe the modular forms t_i in 7.3 more explicitly in terms of Fourier series as it is known for the elliptic modular function $j(\tau)$, see e.g. [21].

Theta functions $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}$ with characteristics $a, b \in \mathbb{Q}^g$ are holomorphic functions on $\mathbb{C}^g \times \mathbb{H}_g$. Explicitly the theta functions

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}: \mathbb{C}^g \times \mathbb{H}_g \longrightarrow \mathbb{C}$$

are defined by

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp\{\pi i\,{}^t(n+a)\Omega(n+a) + 2\pi i\,{}^t(n+a)(z+b)\}$$

The restrictions $\vartheta \mid 0 \times \mathbb{H}_g$

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(\Omega) = \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(0, \Omega)$$

are called t h e t a   c o n s t a n t s (with characteristics).

7.5 Theorem (Feustel [6] , Shiga [39]).

Let $\theta_i(\Omega) = \vartheta_i(0, \Omega)$, $i = 0,1,2$, be the theta constants on $\mathbb{H}_3$ restricting the theta functions

$$\vartheta_k = \vartheta\begin{bmatrix} 0 & 1/6 & 0 \\ k/3 & 1/6 & k/3 \end{bmatrix}(z, \Omega) \, , \quad k = 0,1,2, \quad z \in \mathbb{C}^3.$$

Set

(7.6) $\mathrm{Th}_1 = \theta_0^3 + \theta_1^3 + \theta_2^3 \, , \quad \mathrm{Th}_2 = -3\theta_0^3 + \theta_1^3 + \theta_2^3 \, ,$

   $\mathrm{Th}_3 = \theta_0^3 - 3\theta_1^3 + \theta_2^3 \, , \quad \mathrm{Th}_4 = \theta_0^3 + \theta_1^3 - 3\theta_2^3 \, .$

~~and~~ Then

(7.7) $\mathrm{th}_i(\tau) = \mathrm{Th}_i(*\tau), \quad i = 1,2,3,4, \quad \tau \in \mathbb{B},$

is aspecial choice of $\mathsf{S}\Gamma(\sqrt{-3})$-forms inverting PICARD's integral

map I: $\mathbb{P}^1 \setminus \triangle \longrightarrow \mathbb{B}$ as ~~described~~ *explained* in 7.3.

7.8 Remark. The $\mathsf{S}\Gamma(\sqrt{-3})$-forms th_i are n o r m a l i z e d to be

compatible with the action of $\Gamma/\mathsf{S}\Gamma(\sqrt{-3}) \cong S_4 \times (\mathbb{Z}/3\mathbb{Z})$, S_4 symmetric

group, in the following sense (see [12] or [6]):

    Up to a character the group $S_4 \times (\mathbb{Z}/3\mathbb{Z})$ acts on $\{th_1, \ldots, th_4\}$

    via permutations of indices.

7.9 Corollary. Each PICARD curve C has a normal form

(7.10) $\quad C_\tau: Y^3 = \prod_{i=1}^{4} (X - th_i(\tau)) = X^4 + G_2(\tau)X^2 + G_3(\tau)X + G_4(\tau)$

for a suitable $\tau \in \mathbb{B}$.

    FEUSTEL's proof of Theorem 7.7 needs preperatory work of RIE-

MANN (RIEMANN constants), PICARD [26], ALEZAIS [1], MUMFORD [24],

SHIGA's preperatory work for [29] and HOLZAPFEL [11]. The idea is

to show that some theta candidates coming out directly from PICARD

curves satisfy all the functional equatiens defining Γ-forms of

N e b e n t y p u s described in (7.2) and 7.8. Since generators

of Γ are explicitly known, this is an effective finite problem.

8.   Special Values of the PICARD Modular Theta Function

8.1 Definition. A s i n g u l a r m o d u l e on $\mathbb{B}$ is an isola-

ted fixed point of (an element) of $U((2,1),K)$. If the Jacobian

$J_\tau = J(C_\tau)$, $\tau \in \mathbb{B}$ can be decomposed up to isogeny into simple abe-

lian varieties with complex multiplication, then τ is called a $\mathfrak{D}$CM --module $(\text{Decomposed Complex Multiplication})$.

8.2 Proposition (FEUSTEL, unpublished; maybe to read off from [9] with some effort). The point $\tau \in \mathbb{B}$ is singular module iff it is a $\mathfrak{D}$CM-module.

We denote the preimage of $\triangle$ along p: $\mathbb{B} \longrightarrow \mathbb{P}^2$ by $\diamondsuit$. It consists of infinitely many discs in $\mathbb{B}$ (see [11]).

8.3 Theorem

(A) ([14]). If $\sigma \in \mathbb{B}$ is a singular module, then

(8.4) $\qquad th(\sigma) = (th_1(\sigma):th_2(\sigma):th_3(\sigma):th_4(\sigma))$

is an algebraic point of $\mathbb{P}^2$.

(T) ([30],[34]). If $\tau \in \mathbb{B}(\overline{\mathbb{Q}}) \setminus \diamondsuit$, then $th(\tau)$ is a transcendent point of $\mathbb{P}^2$, this means that $th(\tau) \notin \mathbb{P}^2(\overline{\mathbb{Q}})$.

The first part is an application of the Theorem of SHIMURA-TANIYAMA stating the algebraicity of moduli points of (polarized) abelian CM-varieties. The second part comes out from WUESTHOLZ' transcendence theory, see [36], as announced in [14]. Conjecturally the exclusion of $\diamondsuit$ in (T) can be omitted by the method of jumping to elliptic curves used in [14] for the complete proof of (A).

Using more carefully the SHIMURA-TANIYAMA theory of complex multiplication of abelian varieties one discovers a strong quality of special values $th(\tau)$ in the case (A) fitting in HILBERT's 12-th

PROBLEM.

8.5 Theorem (Explicit construction of SHIMURA class fields of cubic

extensions of EISENSTEIN numbers by special values of Theta con-

stants (see [18]). Let σ be a singular

module ~~with simple~~ J_σ. *such that* $\nearrow$ *has complex multiplication* Then we have, with the notations below, a

tower of algebraic number fields

$$F'_\sigma(th(\sigma)) \; / \; F'(th(\sigma))^{S_4(\sigma)} \; / \; F'_\sigma \; / \; K;$$

where the middle extension is abelian (SHIMURA class field), which

is unramified, if the additional ideal condition (I) is satisfed.

8.6 Notations (see [32] or [22]). The endomorphism algebra $F_\sigma \cong$

$End(J_\sigma) \otimes \mathbb{Q}$ is the cubic extension $K(\sigma)$ of K. Its reflex field is

is denoted by F'_σ. We set

$$F'_\sigma(th(\sigma)) \; = \; F'_\sigma(\ldots, th_i(\sigma)/th_j(\sigma), \ldots) \; , \; 1 \leqslant i,j \leqslant 4$$

in the parantheses. The symmetric group acts on the generators

$th_i(\sigma)/th_j(\sigma)$ via permutation of indices. All such permutation,

which are extendable to an automorphism of $F'_\sigma(th(\sigma))/F'$ form the

the group $S_4(\sigma)$. Let (F_σ, Φ_σ), $\Phi_\sigma = \sum_{i=1}^{3} \wp_i$ with field embeddings

$\wp_i : F \longrightarrow \mathbb{C}$, *be* the type of $Jac(C_\sigma)$. Then *one has* $J_\sigma \cong \mathbb{C}^3 / \Phi_\sigma(\mathcal{M}_\sigma)$ for

a suitable $\mathbb{Z}$-lattice $\mathcal{M}_\sigma$ of F_σ. The ideal condition in 8.5 is:

(I) $\mathcal{M}_\sigma$ is a (fractional) ideal of F_σ.

9.   Connection With FERMAT Equations

FREY [7] discovered a deep connection between FERMAT's Last

Theorem and the arithmetic of elliptic curves. PARSHIN [25] joint

it with the BOGOMOLOV—MIYAOKA—YAU inequality transfered to arith-
metic surfaces. The corresponding inequality for invariants of
arithmetic surfaces is not proved until now. We will call it the
PARSHIN problem (PAR), see below. PARSHIN proved the following re-
markable

9.1 Theorem ([9]). If the PARSHIN problem has an affirmative
answer, then the statement of FERMAT's Last Theorem is true for
almost all FERMAT equations $x^p + y^p = z^p$.

"Almost all" means: all up to a finite number. The conclusing
statement of 9.1 is also known as "Asymptotic FERMAT Theorem".
Since the proof goes through elliptic curves, we asked for a simi-
lar connection with Picard curves. Surprisingly we found a rigorous
reduction of PARSHIN's Theorem 9.1, which could be useful for a
more effective approach. We announce

9.2 Theorem ([19]). If the PARSHIN problem has an affirmative
answer only for arithmetic surfaces of KODAIRA—PICARD type, then the
asymptotic FERMAT Theorem holds.

We finish with the presentation of some necessary definitions
and hints for the proof. The best reference is LANG's book [23].
Let X/S be an arithmetic surface. B = spec $\mathcal{O}$, $\mathcal{O} = \mathcal{O}_L$. La

number field. We say that X/B is o f KODAIRA – PICARD t y p e ,

if the general fibre X_L is a PICARD curve or a smooth biquadratic

covering of a smooth PICARD curve branched over exactly one point.

PARSHIN's problem is to prove

(PAR) There are universal constants a_0, a_1, a_2 (a_0 depending on the

genus g of general fibre) such that for all semistable arithmetic

surfaces X/B it holds that

(Ar.BMY) $(\omega_{X/B}^L) \leq a_2 \sum_{v \in M_L} \delta_v + a_1 (2g-2)\log|D_{L/Q}| + a_0.$

$D_{L/Q}$ the discriminant

$(\omega_{X/B}^L)$ is the selfintersection of the relative canonical sheaf,

M_L is the set of all (finite and infinite) places of L. For δ_v we

refer to [15]. The most complicated contributions δ_v, $v \in M_\infty$, are

described in terms of special values of Theta constants in the

framework of FALTINGS' basic theory of arithmetic surfaces [5].

For the proof of 9.2 we followed the line of VOJTA's reduction

in [39] of PARSHIN's Theorem 9.1 to KODAIRA-PARSHIN covers of the

FERMAT curve of genus 3.

Until now an explicit calculation of (global and local) inva-

riants of arithmetic surfaces needed in (Ar.BMY) seems to be only

possible for the elliptic case ([5]) and genus 2 ([4]) because

of a good knowledge of the connections with modular forms. A rush

to the KODAIRA-PICARD types would accomplish one side of the deep

mathematical thinking and feeling of the old masters PICARD and

HILBERT.

References

[1] Alezais,M.R., Sur une classe de fonctions hyperfuchsiennes, Ann. Ec. Norm. 19, Ser.3 (1902), 261-323

[2] Baily,W.L., Borel,A., Compactifcation of arithmetic quotiends of bounded symmetric domains, Ann. of Math. 84, 442-528 (1966)

[3] Barthel,G., Hirzebruch,F., Hoefer,Th., Geradenkonfigurationen und algebraische Flaechen, Asp. Math. D4, Vieweg, Braunschweig, 1987

[4] Bost,J.-B., Mestre,J.-F., Moret-Bailly,L., Sur le calcul explicite des "classes de Chern" des surfaces arithmetiques de genre 2, Rapp- de Rech. LMENS 84-5, 1989

[5] Faltings,G., Calculus on arithmetic surfaces, Ann. Math. 119 (1984), 387 - 424

[6] Feustel, J.-M., Ringe automorpher Formen auf der komplexen Einheitskugel und ihre Erzeugung durch Thetakonstanten, Prepr. Ser. P-Math-13, AdW DDR, IMath (1986)

[7] Frey,D., Links between stable elliptic curves and certain diophantine equations, Ann. Univ. Saraviensis, 1986, vol.1 No. 1

[8] Gelfond,A.O., Sur le septieme probleme de Hilbert, Izv. Akad. Nauk SSSR, ser. Phys.-Math. 4(1934), 623-630

[9] Hashimoto,K., Elliptic conjugacy classes of the Siegel modular group and unimodular hermitian forms over the ring of cyclotomic integers, J. Fac. Sci. Univ. Tokyo 33 (1986), 57-82

[10] Hilbert,D., Mathematische Probleme, Nachr. Kgl. Ges. d. Wiss. Goettingen, Math.-Phys. Kl., 3 (1900), 253-297

[11] Holzapfel,R.-P., Geometry and arithmetic around Euler partial differential equations, Dt. Verl.d. Wiss., Berlin / Reidel Publ. Comp., Dordrecht, 1986

[12] Holzapfel,R.-P., On the Nebentypus of Picard Modular Forms, Math. Nachr. 139 (1988), 115-137

[13] Holzapfel,R.-P., Discrete analysis of surface coverings I,II, Rev. Roum. Math. 33 (3) (1988), 197-232, 33 (4) (1988),305-348

[14] Holzapfel,R.-P., An arithmetic uniformization for arithmetic points of the plane by singular moduli, Journ. Ramanujan Math. Soc. 3, No. 1 (1988), 35-62

[15] Holzapfel,R.-P., Volumes of fundamental domains of Picard modular groups, Proc. Workshop Arithm. of Complex manifolds, SLN on Math. 1399, Springer, 1988

[16] Holzapfel,R.-P., An effective finiteness theorem for ball latti-ces, SLN 1447 (1990)., 203-236

[17] Holzapfel,R.-P., Ball model for Hilbert's twelvth problem, Pre-print-Reihe MPI. 90-95. Bonn,1990

[18] Holzapfel,R.-P., On algebraic values of the Picard modular functions
Proc. Symposium on Special Differ. Equ., Arata, Aug.1991

[19] Holzapfel,R.-P., The 2-ball and some diophantine equations, to appear (in Math. Nachr.)

[20] Kobayashi,R., Uniformization of complex surfaces, Preprint, Univ. Tokyo, 1990

[21] Lang,S., Elliptic functions, Addison-Wesley Publ. Comp., London. Amsterdam,...,Tokyo, 1973

[22] Lang,S., Complex multiplication, Grundl. Math. Wiss. 255, Sprin-ger, 1983

[23] Lang,S., Introduction to Arakelov theory. Springer, New Yorck, 1988

[24] Mumford,D., Tata Lectures on Theta, Birkhaeuser. Boston,Basel, Stuttgart, 1983

[25] Parshin;A.N., On the use of branched coverings in the theory of diophantine equations, Mat. Sbornik 180, no.2 (1989), 244-259,

[26] Picard,E., Sur des fonctions de deux variables independentes analogues aux fonctions modulaires, Acta math. 2 (1883), 114-135

[27] Roehrl,H., Das Riemann-Hilbertsche Problem der Theorie der line-aren Differentialgleichungen, Math. Ann. 133 (1957), 1-25

[28] Schneider,Th., Transzendenzuntersuchungen periodischer Funktio-nen I. Transzendenz von Potenzen, Journ. f. reine u. angew. Math. 172 (1934), 65-69

[29] Schneider,Th., Transzendenzuntersuchungen periodischer Funktio-nen I. Transzendenz von Potenzen, Journ. f. reine u. angew. Math.

[30] Shiga,H., On the representation of the Picard modular function by θ-constants I-II, Publ. RIMS, Kyoto Univ. 24 (1988), 311-360

[30] Shiga,H., Transcendency for the values of certain modular func-tions of several variables at algebraic points, Preprint. China Univ., 1990

[31] Shiga H., On the transcendency of the values of the modular function at algebraic points, Preprint. China Univ., 1990

[32] Shimura,G., Taniyama,Y., Complex multiplication of abelian va-
rieties and its applications to number theory, Publ. Math. Soc.
Japan, No. 6, 1961

[33] Shimura,G., Introduction to the arithmetic theory of automorphic
functions, Iwanami Shoten Publishers and Princeton University
Press, 1971

[34] Siegel,C.-L., Transcendental numbers, Ann. of Math. Studies 16,
Princeton Univ. Press, 1949

[35] Takagi,T., Ueber eine Theorie des relativ-Abelschen Zahlkoer-
pers, Journ. Coll. Science, Tokyo, 41 (1920), 1-132.

[36] Wuestholz,G., Algebraische Punkte auf analytischen Untergruppen
algebraischer Gruppen, Ann. of Math. 129 (1989), 501-517

[37] Vojta,P., Diophantine inequalities and Arakelov theory, Appendix
in Lang's book [23]

[38] Yoshida,M., Fuchsian Differential Equations, Aspects of Math.,
Vieweg, Braunschweig, 1987

Karl-Weienstrass-Inst. Berlin