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Projective manifold with the ample vector bundle $\overset{2}{\wedge} T_X$

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Our aim is to consider the following

Conjecture. Let X be a smooth projective variety over **C**. Assume that $\overset{2}{\wedge} T_X$ is ample. Then X is isomorphic to a projective space or a smooth quadric hypersurface.

Mori proved that smooth manifold with the ample tangent bundle is a projective space. Siu-Yau [S-Y] independently proved the Frankel conjecture that an n-dimensional compact Kaehler manifold of positive bisectional curvature is biholomorphic to the projective space. Here we must notice that the positivity of besectional curvature implies the ampleness of the tangect bundle. Next we have an interesting problem of determining the structure of variety with semi-ample tangent bundle. In differential geometry Mok [Mok] showed that if X is a compct manifold carrying a Kaehler metric with non-negative besectional curvature, then the universal covering is a product of C^k, projective space and Hermitian symmetric manifold of rank ≥ 2. Here we have to have in mind that the non-negative besectional curvature implies semi-ampleness of tangent bundle. In this meaning it seems to us that the above conjecture has significanse for the next step to study manifold with semi-ample tangent bundle.

In this paper noting that X in the above conjecture is a Fano variety, we investigate the structure of a Fano variety with following condition which appears as most important and complicated case in the conjecture. (see Proposition 1.6)

Main Theorem. Let X be an n-dimensional smooth projective Fano manifold over the complex number fields. Assume that for any extremal rational curve C on X, v^*T_X is isomorphic to $\mathcal{O}_{P^1}(2) \oplus \mathcal{O}_{P^1}(1)^{n-2} \oplus \mathcal{O}_{P^1}$ where $v: P^1 \longrightarrow C$ is the normalisation of C. Then if $n \geq 2$, X is a quadric hypersurface.

The outline of proof of Main theorem is as follows.

First we study the behavior of extremal rational curves in X and construct the parameter space Y of the extremal rational curves in X and its universal space Z which is P^1-bundle $Z \longrightarrow Y$. Next we see that Z is naturally contained in $P(\check{T}_X)$ by virtue of Theorem due to Fulton-Hansen. Moreover we find that the trace consisting of extremal rational curves in question through a point in X becomes a divisor and particularly it is a cone over an (n-2)-dimensional smooth quadric hypersurface. Therefore we get a conclusion.

We work over the algebraically closed field of zero characteristic. We use the customary terminology of algebraic geometry. $\mathcal{O}(a)$ denotes the line bundle $\mathcal{O}_{P^1}(1)^{\otimes a}$ on P^1. For a vector bundle E on a scheme S, $E^\vee$ denotes the dual vector bundle of E.

§.1 Preliminaries.

Throught this paper let X be an n-dimensional smooth Fano variety.

Take an extremal rational curve C_0 on X and the normalization $\varphi: P^1 \longrightarrow C_0$. Moreover we let H be an irreducible compoment of the Hilbert scheme $Hom(P^1, X)$ containing the morphism φ.

We assume

(1.1) for every v in H, v^*T_X is generated by global sections.

Thus noting $H^1(P^1, v^*T_X) = 0$,
we have following the property.

1) H is smooth.

2) X is swept by rational curves of H.

Let G be Aut P^1. Since the natural action of G on $Hom_k(P^1,X)$ induces the action σ of G on the connected component H,

$\sigma : G \times H \longrightarrow H$, $\sigma(g,v)x = v(g^{-1}x)$, $g \in G$, $v \in V$, $x \in P^1$,

G also acts on $H \times P^1$:

$$\tau: G \times H \times P^1 \longrightarrow H \times P^1, \quad \tau(g,v,x) = (\sigma(g,v),gx).$$

Let $Chow^dX$ be the Chow variety parameterising 1-dimensional effective cycles C of X with $C.\ K_X^{-1} = d$. Then we have a morphism $\alpha: H \longrightarrow Chow^mX$ with $v(P^1).\ K_X^{-1} = m$ $(v \in H)$.

Then in the same way as Lemma 9 in [Mo2], we can show

Proposition 1.2.

1) σ is a free action.

2) (Y,Γ) is the geometric quotient of H by in the sense of [Mu] where Y is the normalization of the closure of $\alpha(H)$ in $\mathrm{Chow}^m X$.

Thus H is a principal fiber bundle over Y with group G and Y is a non-singular projective variety.

Now the following is studied before the claim 8.2. in [Mo2].

Under the above notations, we have a G-invariant morphism:

$F: H \times P^1 \longrightarrow Y \times X$, $F(v,x) = (\Gamma(v), v(x))$, $v \in H$, $x \in P^1$.

Let $Z = \mathrm{Spec}_{Y \times X}[(F_* \mathcal{O}_H \times P^1)^G]$. Then Z is the geometric quotient $H \times P^1 / G$ and is a P^1-bundle $q : Z \longrightarrow Y$ in the etale topology. Moreover let $p: Z \longrightarrow X$ be a natural projection.

Hereafter we use the morphisms p, q very often.

In the above, we fixed an extremal rational curve C_0 on X and studied a family of rational curves on X which C_0 yields.

Next we fix a point P on which the curve C_0 is smooth and investigate a family of rational curves through the point P which C_0 induces. We let $i: o \longrightarrow P(\in X)$ be a map with a point o in P^1 and consider a Hilbert scheme $\mathrm{Hom}(P^1, X:i)$ which is reffered in [Mo]. Then $\mathrm{Hom}(P^1, X:i)$ is a closed subscheme in $\mathrm{Hom}(P^1,X)$. Now we take an irreducible component H_P of $\mathrm{Hom}(P^1,X:i)$ containing φ. Let $G_P = \{v \in \mathrm{Aut}\, P^1 | v(o) = o\}$. Then at the point P we consider the same situation as in 1.1

Since $H^1(P^1, v^*T_X \otimes \mathcal{O}(-1)) = 0$, we have

(1.1P) H_P is smooth.

Moreover we obtain

Proposition 1.2.P. Let σ_P: $G_P \times H_P \longrightarrow H_P$ be a canonical action induced by the action σ. Then we have

1) σ_P is a free action.

2) $(Y(P),\Gamma_P)$ is the geometric quotient of H_P by in the sense of [Mu] where Y(P) is the normalization of the closure of $\alpha(H_P)$ in $Chow^m X$. .

Thus H_P is a principal fiber bundle over Y(P) with group G_P and Y(P) is a non-singular projective variety.

Remark 1.3. By the above argument, we get a canonical morphism $\theta_P: Y(P) \longrightarrow Y$ whose degree is one.

To show that every Fano variety is algebraically simply connected we have

Proposition 1.4. Let Z and U be smooth projective varieties and f: U ---→ Z an ètale finite morphism. Assume that $\chi(U,\mathcal{O}_U) = 1$. Then, f is an isomorphism.

Proof. The assumption says that $f^*T_Z = T_U$.
Thus, Hirzebruch Atiyah-Singer Riemann-Roch theorem implies that $\deg f \times \chi(Z, \mathcal{O}_Z) = \chi(U,\mathcal{O}_U) = 1$. Hence f is a isomorphism.

q.e.d.

Corollary 1.4.1. Any smooth projective Fano variety Z defined over the complex number field is algebraically simply connected.

Proof. Let f: U $\longrightarrow$ Z be a finite etale morphism from a algebraic scheme. Then we see that U is a smooth projective variety.

Since $f^*K_Z = K_U$, U is a Fano variety. Hence by virtue of Kodaira's vanishing Theorem, we get $H^i(Z,\mathcal{O}_Z) = 0$ for $1 \le i \le \dim Z - 1$. Thus, Proposition 1.4 asserts that f is an isomorphism.

q.e.d.

Finally in this section we study the type v^*T_X of the ample vector bundle $\overset{2}{\Lambda} T_X$ for each point v in H.

Proposition 1.5. Let X be a smooth projective variety . Assume that $\overset{2}{\Lambda} T_X$ is ample. Then we have

1) X is Fano variety.

2) Let C be an extremal rational curve on X with $v: P^1 \longrightarrow C$ the normalisation of C. Assume additionally that $n \ge 5$. Then v^*T_X is one of the following:

α) $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-1}$

β) $\mathcal{O}(3) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$.

γ) $\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus n-3} \oplus \mathcal{O}$.

δ) $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$.

Proof. Since $\overset{2}{\Lambda} T_X$ is ample, so is $\deg \overset{2}{\Lambda} T_X$ which is $-(n-1)K_X$. Thus we get the first part. Next letting $v^*T_X = \oplus \mathcal{O}(a_i)$ with $a_1 \ge a_2 \ge \dots \ge a_n$, we have $a_1 \ge 2$.

Noting $v^*\overset{2}{\Lambda} T_X = \sum_{i<j} (a_i \oplus a_j)$ and it is ample, we see $a_i + a_j$ is positive. By virtue of Theorem 4 in [Mo2], $\deg v^*T_X \le n + 1$. Thus we get the remainder.

q.e.d.

§. 2 The property of singular curves ℓ_y.

Let us maintain notations C_0, X, Z, Y ,p and q stated in §.1 and set $pq^{-1}(y)$ as ℓ_y.

In this section we study how many curves in the set $\{\ell_y \mid y \in Y\}$ of extremal rational curves on X are singular and what the type of the singularity is.

First let us begin with the definition with respect to singular curves.

A nodal (or, cuspidal) curve means the one imaged by a rational singular curve C with only one node (or cusp) point P via a birational morphism v. Moreover the point v(P) of the curve v(C) is said to be nodal (or, cuspidal) point respectively.

Let $\mathcal{N}$ be a set $\{y \in Y \mid \ell_y$ is a nodal curve$\}$ and $\mathcal{C}$ a set $\{y \in Y \mid \ell_y$ is a cuspidal curve$\}$. Moreover let $\mathcal{NC}$ be $\mathcal{N} \cap \mathcal{C}$.

Remark. 2.1. The set $\mathcal{N} \cup \mathcal{C}$ is the closed subset in Y and there $\mathcal{C}$ is closed.

In order to investigate how many nodal curves exist in X, we need several results. First let us state a condition.

(2.2) Let E be a direct sum of line bundles $L_1 \oplus L_2$ on an irreducible reduced curve C. Set P(E) as S and the section $P(L_i)$ as C_i. Now let φ be a morphism from S to a variety so that a fiber of a canonical projection $\pi : S \longrightarrow C$ goes to a curve via φ. Then we have

Lemma 2.3. Under the above condition 2.2, let M be a quotient line bundle of E which yields a section C_3 of π. Assume that $\varphi(C_3)$ is a point and $\dim \varphi(S) = 2$. Then the morphism φ is obtained by a linear system of the line bundle $(\mathcal{O}_{P(E)}(1) \otimes \pi^*(-M))^{\otimes a}$ with some positive integer a. Moreover $L_1 \otimes M^{-1}$ and $L_2 \otimes M^{-1}$ is semi-ample and either of them is ample.

Proof. Let $\mathcal{O}_{P(E)}(a) \otimes \pi^*N$ $(= W)$ be a line bundle which gives the morphism φ where N is a line bundle on C. First since φ(a fiber of π) is a curve, a is positive. Moreover since $W_{|C_3} = \mathcal{O}_C$, we have $N = -aM$. Hence we infer that $W = (\mathcal{O}_{P(E)}(1) \otimes \pi^*(-M))^{\otimes a} = \mathcal{O}_{P(E(-M))}(1)^{\otimes a}$. On the other hand $W_{|C_i}$ is semi-ample by the assumption. As $W_{|C_i}$ is $(L_i \otimes M^{-1})^{\otimes a}$, $L_i \otimes M^{-1}$ is semi-ample. Moreover $\dim \varphi(S) = 2$ implies that the self-intersection of $(\mathcal{O}_{P(E)}(1) \otimes \pi^*(-M))^{\times a}$ $(= a^2 \sum_i \deg(L_i \otimes M^{-1}))$ is positive. Thus we get the last part.

q.e.d.

Lemma. 2.4. Let the condition be as in 2.2. Assume that $\varphi(C_i)$ is a curve for $i = 1, 2$. Then φ is a finite morphism.

Proof. Assume φ is not finite, in a word, there is a point t in $\varphi(S)$ such that $\varphi^{-1}(t)$ contains a curve D projected to the curve C by π.

First we consider the case that D is a section of π. Then E has a quotient line bundle M on C corresponding to the section D. Let $E' = E(-M)$ and $L_i' = L_i \otimes M^{-1}$. Since $\varphi(C_i)$ is a curve, $W_{|C_i}$

in the previous lemma is ample, which implies that L_1' is ample. Thus we get a contradiction that E' has a quotient trivial line bundle which induces D.

Next let us consider the general case.
Let D be a curve in $\varphi^{-1}(t)$ and let us consider the fiber product $D \times_C P(E) = X'$. Then D yields a section of the canonical projection $X' \longrightarrow D$. Since $X' \simeq P(f^*E)$ with $f = \pi_{|D}$, we get the same setting as the first case. Thus we are done.

q.e.d.

Corollary.2.5. Let the condition and assumption be as in Lemma 2.3. Assume $C_1 \cap C_3 = \phi$. Then $\varphi(C_1)$ is a curve and $C_2 = C_3$.

Proof. Assume that $\varphi(C_1)$ is a point. Then by Lemma 2.3, we see that $M = L_1$ and $L_2 \otimes L_1^{-1}$ is ample. Moreover C_1 and C_3 belong to the member of the complete linear system of the line bundle $\mathcal{O}_{P(E)}(1) \otimes \pi^*(-L_2)$. Thus $C_1.C_3$ is $\deg(L_1 - L_2)$. On the other hand the intersection C_1 and C_2 is zero. Therefore we have a contradiction. Next assume $C_2 \neq C_3$. By the same argument as the above we infer that $\varphi(C_2)$ is a curve. Thus Corollary 2.4 implies that φ is finite, which contradicts to the assumption.

q.e.d.

Proposition. 2.6. Let M be a smooth projective variety, $\pi: S \longrightarrow C$ a P^1-bundle over an irreducible projective curve C and $f: S \longrightarrow M$ a morphism with $\dim f(S) = 2$. We assume that

1). For each point c in C, $\pi^{-1}(c)$ is transformed to a curve.

2). f is not finite.

Then we have

1) The set $\{s \in f(S) | \dim f^{-1}(s) \geq 1\}$ consists of only the point A.

2) one dimensional part of $f^{-1}(A)$ consists of only one rational section of π.(Here a rational section D of π means that $\pi_{|D} : D \longrightarrow C$ is of degree one.)

Proof. By taking the normalisation $\bar{C}$ of the curve C and by the base change of $\bar{C} \times_C S$, we can suppose that C is smooth. There is a point A in f(S) so that $f^{-1}(A)$ contains an irreducible component D which is of one-dimension. Now assume that D is not a section of π. Let $\bar{D}$ be the normalisation of D . Then a canonical morphism $j: \bar{D} \longrightarrow C$ induces a P^1-bundle $\bar{\pi} : \bar{D} \times_C S (=\bar{S}) \longrightarrow \bar{D}$ and a section D_1 of $\bar{\pi}$. Letting $h: \bar{S} \longrightarrow S$ the morphism induced by the morphism j, $h^{-1}(D_1)$ has another irreducible curve D_2 ($\neq D_1$) and the image of D_1 and D_2 by $hf: \bar{S} \longrightarrow M$ is the same point A. Now note that $f^{-1}(A)$ yields another curve D_3 in $\bar{S}$ which intersects with neither D_1 nor D_2. Therefore by Corollary 2.5. we have a contradiction.

Next we assume that the set $\{s \in f(S) | \dim f^{-1}(s) \geq 1\}$ consists of more than one elements. Then we can find three sections which are disjoint with each other (by suitable base change). Thus we have a contradiction by 2.5. Thus we complete the proof of 1).

q.e.d.

The above results provide us with the following

Proposition 2.7. (chark = 0) Let $\pi: T \longrightarrow V$ be a P^1-bundle over a smooth projective variety V and $\varphi: T \longrightarrow U$ a morphism. Assume that

1) every fiber of π goes to a curve via φ

2) there is an irreducible divisor D of T which collapses to a point A in U via φ.

3) the restriction of the morphism φ to T - D is quasi-finite.

Then D is a section of π. Moreover if the characteristic of the ground field of zero, there is an ample line bundle M on V so that $T \simeq P(\mathcal{O} \oplus M)$ and $P(\mathcal{O})$ corresponds to a section D.

Proof. The assumption 1) implies that the morphism $\pi_{|D}$ is finite. By 2) in Proposition 2.6 and Zariski Main Theorem we infer that D is a section of π.
Thus the section D gives a rank-2 vector bundle E on V and the quotient line bundle M with an exact sequence on V:

$$0 \longrightarrow M \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0$$

where $P(\mathcal{O})$ determines a section D canonically.

By the proof in Lemma 2.3, we infer φ is obtained by high power of $\mathcal{O}_{P(E)}(1)$. Thus E corresponds to an element σ in $H^1(V,M)$. Now take an irreducible divisor G of T which does not intersect with D and if G is singular, make the desingularization $f: \bar{G} \longrightarrow G$ of G. Then the fiber product $P(E) \times_V \bar{G}$ has another section $\bar{\bar{G}}$ which does not intersect with the section induces by G. Thus f^*E splits to $\mathcal{O} \oplus f^*M$. This says that there is a canonical homomorphoism $f^*: H^1(V,M) \longrightarrow H^1(\bar{G},f^*M)$ with $f^*\sigma = 0$. By Proposition 4.17[F], we have $\sigma = 0$ (in characteristic zero). Since $\mathcal{O}_{P(E)}(1)_{|P(M)} \simeq M$, the remainder is trivial. Thus we get the proof.

q.e.d.

Therefore we have an important

Sublemma. Let $\mathscr{C}$, $\mathscr{N}$ be as above. Assume that there is a point p in X and a curve T in $\mathscr{N}$ so that for each y in T, ℓ_y passes through the point P. Then $\bar{T} \cap \mathscr{C}$ is not empty.

Proof. We suppose the contrary. Then for every point y in $\bar{T}$ every ℓ_y has no cuspidal point. Letting $g: \bar{C} \longrightarrow C$ the normalisation of C, we see that the smooth ruled surface $\bar{C} \times_C q^{-1}(C)$ has two sections C_1, C_2 which do not intersect and moreover it has another section C_3 which goes to the point P. Thus the ruled surface is isomorphic to $P(L_1 \oplus L_2)$ with two line bundles L_1 L_2 on $\bar{C}$ so that each line bundle L_i corresponds to the section C_i. Since φ is finite on each C_i, φ is finite by Lemma 2.5. Thus we get a contradiction.

q.e.d.

Corollary. 2.12. Let X be a Fano variety enjoying the assumption 1.1. Suppose that $\mathscr{C}$ is empty. Then dim $\mathscr{N} \leq n - 1$. Namely, there is an open subset U in X such that for every y in Y, ℓ_y is smooth in U.

Proof. If dim $\mathscr{N} \leq n-1$, there is nothing to prove. Assume that dim $\mathscr{N} \geq n$. Then by Corollary 2.11, we have a contradiction.

Proposition 2.8. Let X, Y, p and q be as in § 1 and let $\ell_y = pq^{-1}(y)$. Then for every point P in X, the set $\{y \in Y \mid P$ is a nodal point of $\ell_y\}$ is a finite set.

Proof. We have this proposition by 2) of Proposition 2.6.

(2.9) We study the dimension of $\mathcal{N}$ and $\mathcal{C}$ in terms of the length $i(\leq n+1)$ of the extremal rational curve C_0 in §.1.

First we have

Proposition.2.10. Assume that $\mathcal{N}$ (or, $\mathcal{C}$) is not empty. Then if the length of C_0 is i, $\mathcal{N}$ is of at least i - 1 dimension.

Morever assume that $\mathcal{N}\mathcal{C}$ is not empty. Then if the length of C_0 is i, $\mathcal{N}\mathcal{C}$ is of at least max $\{i - n , 0\}$ dimension.

Proof. Let C be a curve with only one node. Then $\dim \mathrm{Hom}(C, X) \geq n\,\chi(C,\mathcal{O}_C) + (-K_X . C) = i$. Moreover since Hom(C,X) has a $\mathbf{C}^x$ -action, this is done. Since $\mathrm{Aut}(P^1, 2(o),\infty)$ is a finite group and the Euler-Poincare characteristic of a rational curve with only two double points is -1, we get the latter part.

q.e.d.

Corollary.2.11. Assume that dim $\mathcal{N} \geq n$. Then $\mathcal{C}$ is not empty and $\mathcal{C} \cap \bar{\mathcal{N}}$ is not empty. Here for a subset A in Y, $\bar{A}$ denotes the closure of A.

Proof. We have only to show the following:

§. 3 Fano variety X where v^*T_X is $\mathcal{O}(2)^{\oplus a} \oplus \mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c}$ and the morphism $g: Z \longrightarrow P(\Omega_X)$

We maintain notations H, Y, Z, H_P, $Y(P)$ defined in §.1.

(3.1) Assume that for every element v in H, v^*T_X is $\mathcal{O}(2)^{\oplus a} \oplus \mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c}$,namely the set $\mathcal{E}$ in §.2 is empty.

Under the assumption we show in this section that the induced morphism $g: Z \longrightarrow P(\Omega_X)$ is a closed embedding and ,moreover, we study the basic property of X obtained in case of $a = c = 1$ and $b = n - 2$.

First let us begin with the observation of the morphism $p:Z \longrightarrow X$ below Proposition 1.2.

Proposition 3.2. Let us maintain the assumption 3.1. Then the morphism $p: Z \longrightarrow X$ is smooth. Moreover every fiber of p is irreducible.

Proof. By Proposition 1.2, it suffices to show that the morphism $s: P^1 \times H \longrightarrow X$ is smooth, namely the induced homomorphism $s_* \; T_{P^1 \times H} \longrightarrow s^* T_X$ is surjective. Since v^*T_X is generated by global sections for every point v in H , the canonical isomorphism between $H^0(P^1, v^*T_X)$ and the Zariski tangent space $T_{H,v}$ provides us with the surjectivity s_* on $P^1 \times \{v\}$, which yields the desired fact. Thus since $p:Z \longrightarrow X$ is smooth, take the Stein factorisation p': $\mathrm{Spec}_X \, p_*\mathcal{O}_Z \longrightarrow X$ of p. Then it is a finite etale morphism. Since X is a Fano variety, the morphism p' is an isomorphism one by Corollary 1.4.1. (char k = 0)

q.e.d.

The P^1-bundle $Z \longrightarrow Y$ yields an exact sequence

$0 \longrightarrow T_* \xrightarrow{i} T_Z \longrightarrow q^*T_Y \longrightarrow 0.$

On the other hand the morphism $p: Z \longrightarrow X$ gives a homomorphism $p_*: T_Z \longrightarrow p^*T_X$. Thus we consider the composite homomorphism ip_*

(3.3) $T_* \longrightarrow p^*T_X$

Thus since the above situation 3.1 means that for any point v in H

(3.4) the morphism $v: P^1 \longrightarrow X$ is unramified, the homomorphism f in 3.3 is injective as a vector bundle on Z. Hence we get a morphism $g: Z \longrightarrow P(\Omega_X)$ satisfying the following diagram:

(3.4.1)
$$\begin{array}{cccc} Z & \xrightarrow{g} & P(\Omega_X) & \\ {\scriptstyle q}\searrow & & \swarrow{\scriptstyle p} & \\ & Y & & X \end{array}$$

where η is a tautological line bundle of T_X and $g^*\eta \simeq T_*^\vee$.

Now we show that the morphism g is a closed embedding.

First we prepare notations.

(3.5) For a point x in X, let $Y_x = qp^{-1}(x)$ and $Z_x = q^{-1}qp^{-1}(x)$. Moreover let $L_y = q^{-1}(y)$ and $\ell_y = p(L_y)$.

Hereafter till the end of this paper, these are used very often.

Now if we recall Proposition 2.8 and 3.4, we have a

Remark 3.5.1. For every point x in X there is a point y in Y so that ℓ_y is smooth at the point x.

(3.6) Now let us study the property of the morphism g on $p^{-1}(x)$, written by g_x. Proposition 3.2 says that $p^{-1}(x)$ is smooth and irreducible. On the other hand $\theta_x(Y(x))$ (see Remark 1.3) contains $q(p^{-1}(x))$ by the construction of Y and $Y(x)$ and the dimmension of these two subvariety coincide with each other. Thus since $q\colon p^{-1}(x) \longrightarrow qp^{-1}(x)$ is finite and birational, ther is a natural isomorphism: $p^{-1}(x) \longrightarrow Y(x)$.

Thus let us study the morphism g_x.

Let H_x be as in §.1 and let us define a morphism

$\Phi : H_x \longrightarrow V(\Omega_{X,x})$

induced by the canonical morphism : $H_x \times P^1 \longrightarrow X$

$$H_x \quad v \longrightarrow dv_{*,o}\left(\frac{d}{dt}\right) \in V(\Omega_{X,x})$$

where t is a local parameter of P^1 at the fixed point o.

Now hereafter we assume that

(3.7) $n \geq 4$, $b > c$ and for each point v in H_x, v^*T_X is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c}$.

Now let us show that the morphism g_x is unramified.

First $v\colon P^1 \longrightarrow v(P^1)$ is umramified. Thus we see that the image $\Phi(H_x)$ is contained in $V(\Omega_{X,x}) - \{0\}$, which induces morphism $H_x \longrightarrow P(\Omega_{X,x}) \simeq P^{n-1}$. . Since this morphism is G_x-invariant, we have the induced morphism $Y(x) \longrightarrow P^{n-1}$, which is just the morphism g_x itself by the construction of the morphism.

Now by the assumption 3.7, $v^*T_X \otimes \mathcal{O}(-2)$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus b} \oplus \mathcal{O}(-2)^{\oplus c}$ and therefore, dim $H^0(P^1, v^*T_X \otimes \mathcal{O}(-2))$ = 1. Thus $\dim_v \Phi^{-1}\Phi(v) \leq 1$. On the other hand the algebraic group G_x acts on H_x and dim $H^0(P^1, T_{P^1} \otimes \mathcal{O}(-2)) = 1$, and therefore $\dim_v \Phi^{-1}\Phi(v) = 1$. Noting that $H^0(P^1, v^*T_X \otimes \mathcal{O}(-2))$ is the Zariski tangent space of $\Phi^{-1}\Phi(v)$ at v we infer that $\Phi^{-1}\Phi(v)$ is smooth and therefore every fiber of Φ is smooth. Thus we see that g_x is umramified.

Thus we have

Proposition 3.7.1. Under the notation 3.4, assume 3.7. Then g is of maximal rank on every point v in Z. Moreover, for each point x in X, g_x is a closed embedding.

Proof. The former is shown.
The latter is due to the following Theorem by W.Fulton anf J.Hansen.

Theorem (Proposition 2 [F-H]) Let V be an irreducible variety of dimension n, h: V $\longrightarrow P^m$ an unramified morphism with $m < 2n$. Then f is a closed embedding.

q.e.d.

Moreover we get

Corollary 3.8. Let the notation and condition be as in 3.4. Assume 3.7. Then g is a closed embedding.

Proof. For each point x in X, the restricted morphism of g

$g_x: p^{-1}(x) \longrightarrow P(\Omega_{X,x})$ is also of maximal rank at every point z in $p^{-1}(x)$. Thus g_x is a closed embedding by virtue of Hulton-Hansen's Theorem in Proposition 3.7.1, which implies that g is injective. Since g is a finite morphism, it is a closed embedding.

q.e.d.

Now to study the structure of Z_x, we prepare a notation.

(3.9) Let $\sigma_x: B(X,x) \longrightarrow X$ be the blow up of X with the point x as the center. For a subvariety W in X, $\sigma_x{}^{-1}[W]$ denotes the proper transform of W by σ_x.

Now by Corollary 2.12 and 3.1, we take

(3.9.1) a point A in X so that ℓ_y is smooth at the point A for any y in $qp^{-1}(A)$.

Let us consider the morphism $p:Z \longrightarrow X$ restricted to Z_A, which is written as p_A. Noting that $p_A{}^{-1}(A)$ is a Cartier divisor in Z_A, by the universality of blowing-up we get

(3.10) a morphism $m: Z_A \longrightarrow B(X,A)$ with $m\,\sigma_A = p_A$ and $m(p_A{}^{-1}(A)) = \sigma_A^{-1}(A) \cap \sigma_A^{-1}[X_A]$.

Set $\bar{X}_A = \sigma_A^{-1}[X_A]$, $F = \sigma_A^{-1}(A)$.

Now let us study the behavior of the morphism m on $p_A^{-1}(A)$.

Take a point y in Y_A. Let $\bar{\ell}_y$ be the proper transform of ℓ_y

by σ_A and $h: P^1 \longrightarrow \bar{\ell}_y$ the normalisation of ℓ_y.

(3.11) First we remark that for each point y in Y_A

1) $m^{-1}[\ell_y]$ intersects with F transversally,

2) Since $p^*T_{X|L_y}$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus b} \oplus \mathcal{O}^{\oplus c}$,
$m^*T_{B(X,A)|L_y}$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}^{\oplus b} \oplus \mathcal{O}(-1)^{\oplus c}$.

To show this, it is sufficient to use the following result in Appendix B.6.10. in [H]

(#) Let $X \subset Y$ and $Y \subset Z$ be regular imbeddings. Let $\bar{Z}$ be blowing-up of Z at X, $\bar{Y}$ the blowing-up of Y at X and E the exceptional divisor of X via the morphism $f: \bar{Z} \longrightarrow Z$. Then
$N_{\bar{Y}/\bar{Z}} = f^* N_{Y/Z} \otimes \mathcal{O}_{\bar{Z}}(-E)$.

3) $\sigma_A^{-1}(A) \cap \sigma_A^{-1}[X_A]$ is a smooth hypersurface in $\sigma_A^{-1}(A)$ ($\simeq P^{n-1}$) and it is canonically isomorphic to $p^{-1}(A)$.

We study the morphism $Z_A \longrightarrow m(Z_A)$. By (2) of 3.11, m is of maximal rank at each point z in Z_A. Precisely speaking, the homomorphism $m_*: T_{Z_A} \longrightarrow m^*T_B$ is injective as a vector bundle. Moreover we see that $m^{-1}(\sigma_A^{-1}(A) \cap \sigma_A^{-1}[X_A])$ is $p^{-1}(A)$. Thus the morphism m is isomorphism around $p^{-1}(A)$.

As a result summarizing the above, we get

Proposition 3.12. Let the notation as in 3.9.1. Then $Z_A \longrightarrow m(Z_A)$ and $Z_A \longrightarrow p_A(Z_A)$ are birational morphisms. More precisely, there is an open neighborhood U ($\supset p^{-1}(A)$) in Z_A so that $m: U \longrightarrow m(U)$ is an isomorphism and p_A is an immersion on

$U - p^{-1}(A)$. Moreover $Z_A - p^{-1}(A) \longrightarrow p(Z_A) - \{A\}$ is finite.

Proof. We have only to show the last part. If there exists a point $x(\neq A)$ such that $\dim p_A^{-1}(x) \geq 1$, take a curve C in Z_A with $p_A(C) = x$. Setting $C' = q(C)$, we see that for any y in C' ℓ_y passes through A and x, which yields a contradiction by the argument in Theorem 4 [Mo] because such a ℓ_y is extremal rational curve.

q.e.d.

Now recalling the set $\mathscr{E}$ of our Fano varity X in question is empty and combining Corollary 2.12 and Proposition 3.2, we get

Corollary 3.13. Let A be a point in 3.9.1. Then Y_A is a smooth subvariety in Y, Z_A a P^1-bundle over Y_A and $p^{-1}(A)$ is a section in Z_A. More precisely there is an ample line bundle M on Y_A so that $Z_A \simeq P(\mathcal{O} \oplus M)$, the restricted morphism of p to Z_A is given by the tautological line bundle of $\mathcal{O}_A \oplus M$ and $P(\mathcal{O}_{Y_A})$ is $p^{-1}(A)$.

Finally we assume that $b = n-2$ and $c = 1$.

Then we show that

(3.14) There is a point $\bar{A}$ in U (3.9.1) so that $p(Z_{\bar{A}})$ is a normal Cartier divisor with only one isolated singularity $\bar{A}$. Then a natural map $\bar{p}_{\bar{A}}$: $Z_{\bar{A}} - p^{-1}(\bar{A}) \longrightarrow p(Z_{\bar{A}}) - \bar{A}$ is an isomorphism.

For a variety T, Sing T denotes the singular part of T.

Assume that (#) for every point A in U, $p(Z_A)$ is non-normal equivalently, $\operatorname{codim}_{p(Z_A)} \operatorname{Sing} p(Z_A) = 1$ because $p(Z_A)$ is a Cartier divisor in X.

Thus for every point x in X, $\operatorname{codim}_{p(Z_x)} \operatorname{Sing} p(Z_x) = 1$.

First the induced morphism $Z_x - p^{-1}(x) \longrightarrow p(Z_x)$ by p is quasi-finite and birational.

Let $\bar{S}(Z_x)$ be a set $\{z \in Z_x |$ there is a point $z'(\neq z)$ such that $p(z) = p(z')\}$ and $S(Z_x)$ the closure of $\bar{S}(Z_x) - p^{-1}(x)$ (note if $q(z) = q(z')$, $\ell_{q(z)}$ is a nodal curve at $p(z)$).

Then we have the property:

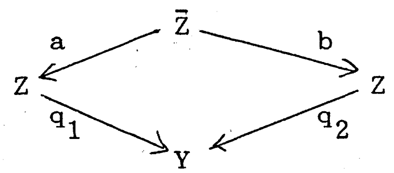
1) For a point x in X, $\operatorname{Sing} p^{-1}(x)$ is a finite or empty set and therefore so is $S(Z_x) \cap p^{-1}(x)$.

2) If A is a point in U, $S(Z_A) \cap p^{-1}(A)$ is empty. Therefore $(\operatorname{Sing} p(Z_A)) - \{x\} = p(S(Z_A))$ (note that $p^{-1}(A)$ is smooth and $p^{-1}(A) X_Y Z \simeq Z_A$).

Now take an irreducible divisor J ($\subset \bar{Z}$) in the closure of $h^{-1}(\operatorname{Sing} h(\bar{Z})) - \Delta$.

Let $\bar{p}_x : p^{-1}(x) X_Y Z \longrightarrow Z_x$ be a canonical morphism (note that $\bar{p}_A$ is a isomorphism for A in U). . Then $\bar{p}_x(J \cap p^{-1}(x) X_Y Z)$ is contained in $\bar{S}(Z_x)$

Now let $\bar{Z}$ be the fiber product $Z X_Y Z$ of Z and Z over Y and Δ the diagonal of $\bar{Z}$. Then there is a canonical morphism $h: \bar{Z} \longrightarrow X \times X$ by $(z, z') \longrightarrow (p(x), p(x'))$.



Then we can easily show that

1) $\bar{Z}$ is $\bigcup_{x \in X} p^{-1}(x) \times_Y Z$ as a set,

2) $\Delta \cap (Z_x \times \{x\}) \underset{\text{canonically}}{\simeq} p^{-1}(x) \times \{x\}$

3) $\dim (J \cap \Delta \cap (Z_x \times \{x\}) = 0$ or -1.

Now we get

Proposition $\Delta \cap J$ is empty.

Proof. $\Delta \cap J \cap (Z_A \times \{A\})$ is empty by the assumption of A. Note that $\operatorname{codim}_{\bar{Z}} J = 1$ and moreover that $\Delta \cap J$ is empty or its codimension in $\bar{Z}$ is of ≤ 2.
Thus we are done.

q.e.d.

Now since the diagonal Δ is a section of b, we have an exact sequence on Z:

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow L \longrightarrow 0$$

with a rank-2 vector bundle E on Z and a line bundle L on Z where $\bar{Z}$ and Δ are canonically isomorphic to P(E) and P(L) respectively.

Now take the desingularisation $\bar{J}$ of J and consider the fiber product $\bar{J} \times_Z \bar{Z}$ ($\simeq \bar{J} \times_Z P(E)$). Then Δ and J yield disjoint two sections with respect to the P^1-bundle b in the above fiber product. In the same way as in

Corollary 2.7, we infer that E splits to $\mathcal{O}$ + L by virtue of Proposition 4.17 [F]. Now restricting the Z-isomorphism $P(\mathcal{O} \oplus L) \simeq \bar{Z}$ (with respect to the morphism b) to $q_2^{-1}(y)$, we get an isomorphism $P(\mathcal{O} \oplus L_{|P^1}) \simeq P^1 \times P^1$ and therefore $L_{|P^1} \simeq \mathcal{O}$. Thus taking the direct image $R^0 q_{2*}$ of $\mathcal{O} \oplus L$, we get a rank-2 vector bundle $\mathcal{O}_Y \oplus N$ where N is a line bundle $q_{2*}L$ on Y. Thus we infer that $P(\mathcal{O}_Y \oplus N) \simeq Z$. Hence since $Z_A \simeq P((\mathcal{O} \oplus N)_{Y_A})$, $P(\mathcal{O}_{Y_A})$ is a unique section in Z_A and $M \simeq N_{|Y_A}$ by Corollary 3.13. which contradicts to the fact that $Z = \bigcup_{A \in X} p^{-1}(A)$.

Hence we proved 3.14.

Finally in this section we state

Proposition 3.15. For x and x' in X, $p(Z_x)$ is linearly equivalent to $p(Z_{x'})$ and the line bundle is base point free.

Proof. Let D_x be $p(Z_x)$. Then we see easily that $\{D_x | x \in X\}$ is an algebraic family, and therefore these element are linearly equivalent to each other because X is a Fano variety.
Assume there is a base point P of the line bundle. By the definition of D_x, we immediately infer that ℓ_y passes through the point P for every element y in Y, in a word X is swept out by these curves ℓ_y. This implies that for an element v in the variety H $v^*T_X \otimes \mathcal{O}(-1)$ is generated by global sections, which yields a contradiction to the definition v in H.

q.e.d.

§.4 The proof of Main theorem ($n \neq 4$)

First in case of $n = 2$, X is a Del Pezzo surface. Moreover the assumption implies that the surface has no exceptional rational curve of first kind. Thus we infer X is a smooth quadric surface.

Next assume $n = 3$. Then we need

Theorem [W2] Let X be a Fano manifold and $\ell(X)$ $\min\{-K_X . C \mid C$ is a rational curve on $X\}$. If $\ell(X) > \dim X / 2 + 1$, then $\mathrm{Pic}\, X \simeq \mathbf{Z}$. Thus we see that X is a Fano manifold of first kind. Hence X is the desired thing thanks to Corollary 2.6 in [W1].

First we state a well-known result

Lemma 4.1. Let V be an effective ample divisor on an irreducible projective local complete intersection W. Assume that W has at most rational singularities. The canonical map
$\mathrm{Pic}\, W \longrightarrow \mathrm{Pic}\, V$ is isomorphism if $\dim W \geq 4$ and
$\mathrm{Pic}\, W \longrightarrow \mathrm{Pic}\, V$ has torsion free cokernel if $\dim W = 3$.

For the proof see Lefschetz theorem of Hamm [H] and Lemma 0.3.2. in [F-So].

(4.2) Now we study the structure of $p(Z_{\bar{A}})$ in 3.14 ,written by D. We see easily that D has at most rational singularities.

For the purpose we state the property of D. Since $Z_{\bar{A}} \simeq P(\mathcal{O}_{Y_{\bar{A}}} \oplus M)$ by 3.13, we have

1) $\mathrm{Pic}\, D \simeq \mathbf{Z}\, \mathcal{O}_D(S)$ where S is the image of the section P(M) via p

2) The canonical homomorphism $\mathrm{Pic}\, X \simeq \mathrm{Pic}\, D$ by Lefschetz's

Theorem.

3) The canonical homomorphism Pic D $\simeq$ Pic S by Lefschetz'of Hamm [H].

Note that $p^{-1}(\bar{A})$ ($\simeq$ S) is a smooth hypersurface of degree d in $P(\Omega_{X,A})$.

Thus let f be a homogenuous polynomial in $k[x_0,\ldots,x_{n-1}]$ where $S \simeq \mathrm{Proj}\ k[x_0,\ldots,x_{n-1}]/(f)$ in P^{n-1} and the weight of x_i = 1 for every i.

Hereafter before having an argument in case of n = 4, we treat the case $n \geq 5$.

Remarking Pic S $\simeq \mathbf{Z}$ (dim S $\geq$ 3), we have

Proposition. 4.3. Let $\mathcal{O}_D(S)_{|S}$ be $\mathcal{O}_S(c)$. Then c = 1. Namely, D is a hypersurface in P^n which is isomorphic to $\mathrm{Proj}\ k[x_0,\ldots,x_n]/(\bar{f})$ in P^n where the weight of x_n = 1, $\bar{f}$ is a homogeneous polynomial (= $x_n^d + a_{n-1}x_n^{d-1} + \ldots + a_1x_n + f$) of degree d , a_i a homogeneous polynomial of degree d - i in $k[x_0,\ldots,x_n]$ and $\bar{f}(x_0,\ldots,x_{n-1},0) = f$.

Proof. The former is trivial by 1) and 3) of 4.2. The latter is obtained by virutue of Theorem 3.6 in [Mo1].

q.e.d.

Therefore we see that the above S is an intersection of D and a hyperplane in P^n.

Next the intersection number of the fiber of q and P(M)in $Z_{\bar{A}}$ is one. Since $p_{\bar{A}}$ is birational,

(4.4) the image ℓ_y of any fiber of $q: Z_{\bar{A}} \longrightarrow Y_{\bar{A}}$ via p is a line ($\subset D$) in P^{n+1}.

Now using Theorem 3.6 in [Mo1] again, we see that X is isomorphic to Proj $k[x_0,\dots,x_{n+1}]/(F)$ in the weighted projective space $\mathbf{Q}(1,\dots,1,c)$ where F is a weighted homogeneous polynomial $(= x_{n+1}^e + b_{e-1}x_{n+1}^{e-1} + \dots + b_1x_{n+1} + \bar{f})$ of degree d in $k[x_0,\dots,x_{n+1}]$ (= ce), b_i a homogeneous polynomial of degree d-ic in $k[x_0,\dots,x_n]$ and $F(x_0,\dots,x_n,0] = \bar{f}$. Moreover by virtue of Theorem 3.7, Pix X $\simeq \mathbf{Z}L$ with the ample generater L.

On the other hand we know

(4.5) $K_X = (d - (n + 1 + c))L$ by virtue of Proposition 3.3 in [Mo1].

Now we show

(4.6) $-K_X = nL$.

In fact let $-K_X = aL$. Thus

$n = (\ell_y . -K_X) = (\ell_y . aL) = a(\ell_y . L_D)_D = a$ by 4.4.

Hence combining 4.5 and 4.6, we have c = 1 and e = d = 2. Thus we are done.

§.5 The case $n = 4$.

First let us consider the case that S is a surface of degree d in P^3.

Since $\mathrm{Pic}\ X \simeq \mathbf{Z}\, L$ by Theorem due to [W2] stated in § 4, $K_X \simeq -aL$ and $\mathcal{O}_X(D) = cL$ with some positive integers a, c. Then we infer that

(5.1) the canonical sheaf ω_D of D is $(c - a)L_D$ and $K_S = (c - a + 1)L_{D|S}$.

Thus, letting $\mathcal{O}_{P^3}(u)_{|S} = \mathcal{O}_S(u)$, we can show

Proposition 5.1.1. Assume $d \neq 4$. Then $L_{D|S} = \mathcal{O}_S(1)$.

Proof. Note that $K_S = \mathcal{O}(d-4)$. Since the cokernel C of

$$0 \longrightarrow \mathbf{Z}\mathcal{O}_S(1) \longrightarrow \mathrm{Pic}\ S$$

is torsion free by Lefschtz's Theorem, K_S yields a zero element in C and therefore so does $L_{D|S}$ by 5.1. Thus we infer that $L_{D|S} = \mathcal{O}_S(h)$ with some integer h. Moreover by Lemma 4.1 we get $h = 1$.

q.e.d.

Hence, we see that D is a hypersurface in P^4 and X a weighted hepersurface in $P(1,1,1,1,1,c)$ in the same way as in § 4. Thus we can get the desired result in case of $n = 4$ and $d \neq 4$.

Finally we consider the case of $d = 4$. Consequently we show that this case does not occur.

Now take a point $\bar{A}$ in U in 3.9.1. Then letting D be $p(Z_x)$,

D is a cone with at most one isolated singular points at $\bar{A}$.

Let N be the normal bundle of S in D.

Now recall that a smooth variety which is a cone over a smooth is a projective space. Thus if D were P^3, K_D is $-4L_{|D}$ and therefore K_X is $-bL$ with $b \geq 5$ by adjunction formula, which implies that the intersection of $-K_X$ with extremal rational curve is not less that 5. This is a contradiction to the assumption of Main Theorem.

Hence we study the cone singularity (D,x). For the purpose let (R,m) be the local ring $\mathcal{O}_{D,x}$. Letting $T = \bigoplus_{t\geq 0} H^0(S, tN)$, R is the localisation of T at T_0 where $T_0 = \bigoplus_{t\geq 1} H^0(S, tN)$.

Here D is a Cartier divisor with isolated singularity in 4-dimensional smooth variety X. Hence we have a property

(5.2) $\dim H^0(S,N) \leq \dim m / m^2 = 4$.

From now on let us determine the value c with $\mathcal{O}_X(D) = c\,L$.

Since N ($= N_{S/D}$) is ample on K3 surface S, $h^0(S,N) = N^2/2 + 2$ by Riemann-Roch Theorem. Thus by proposition 5.2, we get $N^2 = 2$ or 4. Remarking that $N^2 = (S^3)_D = L^3.D = cL^4$ and $a = c + 1$ by 5.1, we have a table

(5.3)

N^2	2		4		
c	1	2	1	2	4
L^4	2	1	4	2	1
a	2	3	2	3	5

Since $a(L.\ell_y) = -K_X.\ell_y = 4$ by our assumption, the case of $c = 2, 4$ is ruled out.

Thus we have one possibility.

(5.4) $c = 1$, in a word $\mathcal{O}_X(D) \simeq L$. Thus $\mathcal{O}_X(D)$ has base point free by Proposition 3.15.

Now we have two exact sequences on D and X:

(5.4.1) $0 \longrightarrow \mathcal{O}_D \longrightarrow L_D \longrightarrow N \longrightarrow 0$

(5.4.2) $0 \longrightarrow \mathcal{O}_X \longrightarrow L \longrightarrow L_D \longrightarrow 0$

Now we show

Proposition 5.5. $H^1(D,\mathcal{O}_D) = 0$ and $H^1(X,\mathcal{O}_X) = 0$.

Proof. $H^1(X,\mathcal{O}_X) = H^{n-1}(X,K_X) = 0$ by Serr's duality and Kodaira's vanishing. We have another exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

Again Kodaira's Vanishing Theorem yields $H^i(X,\mathcal{O}_X(-D)) = 0$ for $i = 1,2$. Thus we get the remainder.

q.e.d.

Thus two exact sequences yields

(5.6.1) $0 \longrightarrow H^0(D,\mathcal{O}_D) \longrightarrow H^0(D,L_D) \longrightarrow H^0(S,N) \longrightarrow 0$

(5.6.2) $0 \longrightarrow H^0(X,\mathcal{O}_X) \longrightarrow H^0(X,L) \longrightarrow H^0(D,L_D) \longrightarrow 0$

Since $\mathcal{O}_X(D)$ has base point free by Proposition 3.15. the ample line bundle N on S is base point free. Hence we study a morphism $h: S \longrightarrow P(H^0(S,N))$.

Now we recall

Proposition 5.7.(Proposition 2 in [Ma])

Let L be an invertible sheaf on a K3 surface F such that L has non-zero section with $L^2 = d > 0$ and such that L has no fixed components. Letting g the rational map induced by $|L|$, then $|L|$ has no base point and either

1) $g:F \longrightarrow P(H^0(F,L))$ is a birational map onto a projectively normal surface of degree d or

2) $g:F \longrightarrow P(H^0(F,L))$ is a rational map of degree 2 onto a projectively rational surface of degree $d / 2$.

We devide into two cases: α) $N^2 = 4$ β) $N^2 = 2$.

Let us consider the case α).

Thus applying the quartic surface S to 1) in 5.7, we see that the morphism $h: S \longrightarrow P(H^0(S,L))$ ($\subset P^3$) is a finite birational morphism onto a projectively normal surface and therefore h is an isomorphism and $h(S)$ is a quartic surface. Consequently X is an quartic smooth 4 fold. Noting $L = \mathcal{O}_{P^5}(1)_{|X}$, $K_X = -2L$ and that a line ℓ on X is a extremal rational curve we have $-K_X . \ell = 2$, which yields a contradiction to our assumption.

Next by 5.7, we infer h is a double covering from S to normal quadric surface. Consequently we have a double covering $f: X \longrightarrow Q$ with a normal quadric hypersurface Q in P^5. Now we have $K_X = f^*\omega_Q + R$ with the branched locus R. Then we see that $R = 2L$ and therefore $f(R)$ is a complete intersection of Q and another quadric hypersurface Q' in P^5. Thus we can take a line ℓ in $f(R)$. Letting C the inverse image of ℓ via f, we infer that C is an extremal rational curve. On the other hand we see that $K_X.C = f^*\mathcal{O}_Q(-4).C + R.C = -2$. This induces a contradiction.

Consider the case β).

In case 1) in 5.7, $h: S \longrightarrow P^2$ is finite birational morphism and therefore an isomorphism which contradicts $N^2 = 2$.

Next $h:S \longrightarrow P^2$ is a double covering and consequently X is a double covering $f: X \longrightarrow P^4$. Hence we have an equality: $K_X = f^*K_{P^4} + R$ with the branched locus R. Thus we see that f(R) is a cubic 3-fold. Now take a line ℓ in f(R). In the same way as in case α), for the inverse image C of ℓ, $K_X.C = f^*\mathcal{O}_{P^4}(-5).C + R.C = -2$. This yields a contradiction.

Thus we could show that the case that S is a quartic surface does not occur.

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