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A duality for quasi-Gorenstein singularities

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Introduction

Let (X,x) be a germ of an n-dimenisoal normal isolated singularity, i.e., X is an n-dimensional normal Stein space and a point x is the only singularity of X. Let $\pi : (M,E) \longrightarrow (X,x)$ be a resolution of the singularity, where $E = \pi^{-1}(x)$. Then for $1 \le i \le n-1$, $\dim_{\mathbb{C}} (R^i\pi_*\mathcal{O}_M)_x$ is finite and is independent of the choice of the resolution (for example, see Yau [Y, Theorem 2.6, p.434]). We write $\dim_{\mathbb{C}} (R^i\pi_*\mathcal{O}_M)_x = h^i(X,x)$ for $1 \le i \le n-2$ and define the geometric genus of (X,x) to be $p_g(X,x) = \dim_{\mathbb{C}} (R^{n-1}\pi_*\mathcal{O}_M)_x$.

The analytic local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay if and only if $h^i(X,x) = 0$ for $1 \le i \le n-2$. The analytic local ring $\mathcal{O}_{X,x}$ is Gorenstein if and only if it is Cohen-Macaulay and quasi-Gorenstein, i.e., the canonical line bundle is trivial in a deleted neighborhood of x in $X - \{x\}$ (see [HO, Theorem 1.6, p.421]).

The purpose of this paper is to show the following theorems:

Theorem A. Suppose that (X,x) is a quasi-Gorenstein singularity. Then

(i) $h^i(X,x) = h^{n-(i+1)}(X,x)$ for $1 \le i \le n-2$,

(ii) If $n = 4m + 3$, then $h^{2m+1}(X,x)$ is even.

Theorem B. If (X,x) is a Gorenstein singularity of dimension $n = 2m + 1$, then

$$p_g(X,x) = T_n(c_1,c_2,\cdots,c_{n-1})[M] ,$$

where $T_n \in \mathbb{Q}[c_1,\ldots c_n]$ is the n-th Todd polynomial and $T_n[M]$ is the value of $T_n(c_1,\ldots,c_n)$ on the fundamental class $[M] \in H_{2n}(M,\partial M)$.

Theorem B is proved by Looijenga [Lo, 4.1g, p.299] under the conditon that the singularity (X,x) is smoothable.

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(1.1) Let (X,x) be a germ of an n-dimensional normal isolated singularity. By a theorem of Artin [A], (X, x) can be realized as a Zariski open subset of a projective variety Y with x as its only singularity. Let $\pi : M \longrightarrow X$ be a good resolution of the singular point. Then, in a natural manner, we get a desingularization $\rho : N \longrightarrow Y$ of Y by letting N to be $(Y - \{x\}) \cup M$. Let $E = \pi^{-1}(\{x\})$ and denote by D_i $(i=1,\ldots,r)$ the irreducible componets of E. These notations are used throghout the paper.

Note that M is a strongly pseudoconvex manifold and N is a non-singular compactification of M. We may also assume that $N - M$ consists of non-singular divisors in normal crossing.

(1.2) Let D be a non-singular divisor of $M \subset N$, and let $d \in H^2(N,\mathbb{Z})$ be the cohomology class represented by the oriented $(2n-2)$-cycle D. Denote by $[D]$ the line bundle defined by the integral divisor D. Then $c_1([D]) = d$.

The natural orientation of N defines an element of the 2n-dimensional integral homology group $H_{2n}(N,\mathbb{Z})$ called the fundamental cycle of N.

In general, following the notation in [Hi], for $a = \sum_{k=0}^{n} a_k \in H^*(N,\mathbb{C})$ with $a_k \in H^{2k}(N,\mathbb{C})$, we put

$$\kappa_n[a] = a_{2n},$$
$$\kappa_n(a) = a_{2n}[N] = \langle a_{2n},[N]\rangle,$$

$[N]$ denoting the fundamental 2n-cycle of N.

Let $j : D \longrightarrow N$ be the embedding of D in N, and $c_i \in H^{2i}(N,\mathbb{Z})$ be the Chern classes of N. Every product $c_{j_1}c_{j_2}\cdots c_{j_r}$ of weight $n-s = j_1 + j_2 + \cdots + j_r$ defines an integer $c_{j_1}c_{j_2}\cdots c_{j_r}d^s[N]$, which is equal to $\langle j^*(c_{j_1}c_{j_2}\cdots c_{j_r}d^{s-1}),[D]\rangle$ if $s \geq 1$.

Denote the complex analytic tangent bundles of N, D by T_N, T_D. There is an exact sequence

$$0 \longrightarrow T_D \longrightarrow T_N|_D \longrightarrow [D]|_D \longrightarrow 0,$$

so we have

$$j^*c(T_N) = c(j^*T_N) = c(T_N|_D) = c(T_D)(1 + j^*d).$$

(multiplicity of the total Chern class)

Then any $j^*c_j(N)$ can be represented by the Chern classes of D

and j^*d. Thus $(c_{j_1}c_{j_2}\cdots c_{j_r}d^s)[N]$ is independent of the choice of the affine model of (X,x) and their non-singular compactifications if $s \geq 1$.

(1.3) Let f be a two dimensional cohomology class of $H^2(N,\mathbb{Z})$. Define $T(N,f)$ by

$$T(N,f) = < \kappa_n\left[e^f \prod_{i=1}^{n} \frac{\gamma_i}{1 - e^{-\gamma_i}}\right], [N] >.$$

This formula is to be understood as follows: There is a formal factorization

$$1+c_1x + \cdots + c_nx^n = (1+\gamma_1x)\cdots(1+\gamma_nx),$$

where $c_i \in H^{2i}(N,\mathbb{Z})$ are the Chern classes of N. Consider the term of degree n in f and the γ_i of the expression in square brackets. It is a symmetric function in the γ_i and is therefore a polynomial in f and the c_i with rational coefficients. If the multiplication is interpreted as the cup product in $H^*(N,\mathbb{Z})$, this polynomial defines as an element of $H^{2n}(N,\mathbb{Z})\otimes\mathbb{Q}$. The value of this element on the 2n-dimensional cycle of N determined by the natural orientation is denoted by $T(N,f)$.

(1.4) Following Laufer [L], we consider the sheaf cohomology with support at infinity. Let F be a line bundle on M. The sequence

$$0 \longrightarrow \Gamma(M,\mathcal{O}(F)) \longrightarrow \Gamma_\infty(M,\mathcal{O}(F)) \longrightarrow H^1_c(M,\mathcal{O}(F)) \longrightarrow \cdots$$

is exact. By Siu [Si], p. 374, any section of F defined near the boundary of M has an analytic continuation to $M - E$. Therefore there is a natural isomorphism $\Gamma_\infty(M,\mathcal{O}(F)) \simeq \Gamma(M-E,\mathcal{O}(F))$. By Hartshorne [H], p. 225, there exists an isomorphism:

$$H_c^1(M,\mathcal{O}(F)) \simeq H^{n-1}(M,\mathcal{O}(K-F))$$

where K denotes the line bundle determined by canonical divisors. Since M is strongly pseudoconvex, $H^1(M,\mathcal{O}(K-F))$ is a finite dimendional vector space. Hence by the inequality

$$\dim \Gamma(M-E,\mathcal{O}(F))/\Gamma(M,\mathcal{O}(F)) \le \dim H_c^1(M,\mathcal{O}(F)) = \dim H^{n-1}(M,\mathcal{O}(K-F)),$$

we have $\dim \Gamma(M-E,\mathcal{O}(F))/\Gamma(M,\mathcal{O}(F)) \le +\infty$. We define the Euler-Poincaré characteristic $\chi(M,\mathcal{O}(F))$ by

$$\chi(M,\mathcal{O}(F)) = \dim \Gamma(M-E,\mathcal{O}(F))/\Gamma(M,\mathcal{O}(F)) \quad \sum_{q=1}^{\infty} (-1)^q \dim H^q(M,\mathcal{O}(F)).$$

Now we have the following theorem of Riemann-Roch type.

Theorem 1.5 ([W]). For any integral divisor D with the first Chern class d on M, the equality

$$\chi(M,\mathcal{O}([D])) - \chi(M,\mathcal{O}) = T(N) - T(N,d)$$

holds.

(2.1) Let (X,x) be a normal n-dimensional isolated singularity. The geometric genus $p_g(X,x)$ is defined to be the dimension of $\dim_{\mathbb{C}}(R^{n-1}\pi_*\mathcal{O}_M)_x$ where $\pi : M \longrightarrow X$ is a resolution of the singularity.

Theorem 2.2 (Laufer-Yau[Y]). Let (X,x) be a normal n-dimensional isolated singularity. Suppose that x is the only singularity of X and X is a Stein space. Let $\pi : M \longrightarrow X$ be a resolution of the singularity. Then

$$\dim H^{n-1}(M,\mathcal{O}) = \dim \Gamma(M - E,\mathcal{O}(K))/\Gamma(M,\mathcal{O}(K))$$

where $E = \pi^{-1}(\{x\})$.

Definition 2.3. Let (X,x) be a normal isolated singularity. We say (X,x) is quasi-Gorenstein if there exists a holomorphic n-form ω defined on a deleted neighborhood of x, which is nowhere vanishing on this neighborhood.

(2.4) Assume that (X,x) is a quasi-Gorenstein singularity. Then there exists a nowhere vanishing holomorphic n-form ω defined on $X - \{x\}$. Let K_∞ be the part of the divisor of $\pi^*\omega$ on N which is supported on $N - M$. Then $(\omega) \sim K + K_\infty$. Let k, $k_\infty \in H^2(N,\mathbb{Z})$ be the cohomology class represented by the cycle K, K_∞ respectively.

(3.1) Let $\{ T_k(c_1,\dots,c_k) \}$ be the multiplicative sequence with characteristic power series

$$Q(x) = \frac{x}{1 - e^{-x}} .$$

The polynomial T_k are called Todd polynomial. For small n,

$$T_1 = \frac{1}{2}c_1,$$
$$T_2 = \frac{1}{12}(c_2 + c_1^2),$$
$$T_3 = \frac{1}{24}c_1c_2.$$

Lemma 3.2. Let n be a positive integer, then

$$\sum_{k=0}^{n-1} \frac{(-c_1)^{n-k}}{(n-k)!} T_k(c_1,\dots,c_k) = \{(-1)^n-1\}T_n(c_1,\dots,c_n).$$

Proof.

$$\sum_{k=0}^{n} \frac{(-c_1)^{n-k}}{(n-k)!} T_k(c_1,\dots,c_k) = \kappa_n\left[e^{-c_1}\prod_{i=1}^{n}\frac{\gamma_i}{1-e^{-\gamma_i}}\right]$$
$$= \kappa_n\left[e^{-(\gamma_1+\cdots+\gamma_n)}\prod_{i=1}^{n}\frac{\gamma_i}{1-e^{-\gamma_i}}\right] = \kappa_n\left[\prod_{i=1}^{n}\frac{\gamma_i}{e^{\gamma_i}-1}\right]$$
$$= \kappa_n\left[\prod_{i=1}^{n}\frac{-\gamma_i}{1-e^{-(-\gamma_i)}}\right] = T_n(-c_1,c_2,\dots,(-1)^i c_i,\dots,(-1)^n c_n)$$
$$= (-1)^n T_n(c_1,\dots,c_n)$$

Corollary 3.3 ([Hi]). $T_k(c_1,\dots,c_k)$ is divisible by c_1 for k odd.

Lemma 3.4. $T(N) - T(N,k) = \{1-(-1)^n\}T_n(-k,c_2,\dots,c_n)[N]$.

Proof. By definition $T(N) = T_n(c_1,\dots,c_n)[N]$ and $T(N,k) = \langle\kappa_n\left[e^{k}\prod_{i=1}^{n}\frac{\gamma_i}{1-e^{-\gamma_i}}\right], [N]\rangle$, and hence it suffices to show

$$T_n(c_1,\dots,c_n) - \kappa_n\left[e^{k}\prod_{i=1}^{n}\frac{\gamma_i}{1-e^{-\gamma_i}}\right]$$
$$= T_n(c_1,\dots,c_n) - \sum_{j=0}^{n}\frac{k^{n-j}}{(n-j)!} T_j(c_1,\dots,c_j)$$
$$= -\sum_{j=0}^{n-1}\frac{k^{n-j}}{(n-j)!} T_j(c_1,\dots,c_j)$$
$$= -\sum_{j=0}^{n-1}\frac{k^{n-j}}{(n-j)!} T_j(-k-k_\infty,\dots,c_j)$$
$$= -\sum_{j=0}^{n-1}\frac{k^{n-j}}{(n-j)!} T_j(-k,\dots,c_j) \qquad \left[\ k\cdot k_\infty = 0\ \right]$$
$$= \{1-(-1)^n\}T_n(-k,\dots,c_n).$$

(4.1) From Lemma 3.4, applying Theorem 1.5 to the case [D] = K, we have the following:

Corollary 4.2. Let (X,x) be a normal isolated singularty of dimension n. If (X,x) is quasi-Gorenstein, then

$$\{1-(-1)^n\}\{p_g(X,x) - T_n(-k,c_2,\ldots,c_{n-1})[N]\}$$
$$= h^1(X,x) - h^2(X,x) + \cdots + (-1)^{n-1}h^{n-2}(X,x).$$

Proof. $\chi(M,K) - \chi(M) =$
$= p_g(X,x) - \{ h^1(X,x) - h^2(X,x) + \cdots + (-1)^n h^{n-1}(X,x) \}$
$= \{1-(-1)^n\}p_g(X,x) - \{ h^1(X,x) - h^2(X,x) + \cdots +$
$(-1)^{n-1}h^{n-2}(X,x) \}$

On the other hand, from Lemma 3.4

$$T(N) - T(N,k) = \{1-(-1)^n\}T_n(-k,\ldots,c_n)[N].$$

Hence we obtain the corollary by Theorem 1.5.

Corollary 4.3. Let (X,x) be a normal isolated singularty of dimension $2m + 1$. If (X,x) is quasi-Gorenstein, then

$$2\{p_g(X,x) - T_{2m+1}(-k,c_2,\ldots,c_{2m},c_{2m+1})[N]\}$$
$$= h^1(X,x) - h^2(X,x) + \cdots + h^{2m-1}(X,x).$$

Corollary 4.4. Let (X,x) be a normal isolated singularty of odd dimension. If (X,x) is Gorenstein, then

$$p_g(X,x) = T_n(-k,c_2,\ldots,c_{n-1})[N].$$

Proof. As is well known, $h^i(X,x) = 0$ for $1 \leq i \leq n-1$; [Ya, Theorem 2.6, p.434].

Corollary 4.5. Let (X,x) be a normal isolated singularity of even dimension. If (X,x) is quasi-Gorenstein, then

$$h^1(X,x) - h^2(X,x) + \cdots - h^{n-2}(X,x) = 0,$$

i.e., $\chi(M,\mathcal{O}) = h^{n-1}(X,x) = p_g(X,x)$.

Corollary 4.6. Let (X,x) be a normal isolated singularity of dimesnion 3. If (X,x) is quasi-Gorenstein, then

$$2\{ p_g(X,x) - \frac{-k\cdot c_2}{24}[N] \} = h^1(X,x),$$

i.e., the dimension of the second local cohomology group of $\mathcal{O}_{X,x}$ is even.

Corollary 4.7. Let (X,x) be a normal isolated singularity of dimesnion 4. If (X,x) is quasi-Gorenstein, then

$$h^1(x,x) = h^2(X,x).$$

Remark 4.8. A quasi-homogeneous cone over a three dimensional abelian variety satisfies the comdition of this Corollary.

Theorem 5.1. If (X,x) is a quasi-Gorenstein normal isolated singularity of dimension n, then $h^i(X,x) = h^{n-(i+1)}(X,x)$.

Proof. Let $\pi : M \longrightarrow X$ be a resolution of the singularity. By (2,2) of [La], we have the following exact sequence

$$H^1(M,\mathcal{O}(K)) \longrightarrow H^1_\infty(M,\mathcal{O}(K)) \longrightarrow H^2_c(M,\mathcal{O}(K)) \longrightarrow H^2(M,\mathcal{O}(K)) \longrightarrow \cdots$$
$$\cdots \longrightarrow H^{n-2}_\infty(M,\mathcal{O}(K)) \longrightarrow H^{n-1}_c(M,\mathcal{O}(K)) \longrightarrow H^{n-1}(M,\mathcal{O}(K)) .$$

By the vanishing theorem of Grauert-Riemenschneider, $H^i(M,\mathcal{O}(K))$

= 0 for $i \geq 1$. Therefore

$$H^i_\infty(M,\mathcal{O}(K)) \simeq H^{i+1}_c(M,\mathcal{O}(K)) \quad \text{for} \quad 1 \leq i \leq n-2.$$

Since $H^i(M,\mathcal{O})$ is Serre dual to $H^{n-i}_c(M,\mathcal{O}(K))$,

$$H^i_\infty(M,\mathcal{O}(K)) \simeq \left(H^{n-(i+1)}(M,\mathcal{O})\right)^* \quad \text{for} \quad 1 \leq i \leq n-1.$$

Consider the following exact sequence

$$H^1_c(M,\mathcal{O}) \longrightarrow H^1(M,\mathcal{O}) \longrightarrow H^1_\infty(M,\mathcal{O}) \longrightarrow$$

$$\cdots \quad \cdots \quad \cdots$$

$$\cdots \longrightarrow H^{n-2}(M,\mathcal{O}) \longrightarrow H^{n-2}_\infty(M,\mathcal{O}) \longrightarrow H^{n-1}_c(M,\mathcal{O}) .$$

By Serre duality, we know that $H^i_c(M,\mathcal{O})$ is the strong dual of $H^{n-i}(M,\mathcal{O}(K))$ which is zero by the vanishing theorem of Grauert-Riemenschneider for $i \neq n$. So $H^i_c(M,\mathcal{O}) = o$ for $i \neq n$. It follows that

$$H^i(M,\mathcal{O}) \simeq H^i_\infty(M,\mathcal{O}) \quad \text{for} \quad 1 \leq i \leq n-1 .$$

Since the singularity is quasi-Gorenstein, there exists a holomorphic n-form ω defined on a deleted neighborhood of $x \in X$, which is nowhere vanishing on this neighborhood. Cupping, or wedging, with $\tilde{\omega} = \pi^*\omega$, we have a morphism

$$H^i_\infty(M,\mathcal{O}) \longrightarrow H^i_\infty(M,\mathcal{O}(K)).$$

The morphism is an isomorphism, since "at ∞" $\tilde{\omega} = \omega$ doesn't vanish. Therefore $h^i(X,x) = h^{n-(i+1)}(X,x)$.

Proposition 5.2. If (X,x) is a quasi-Gorenstein singularity of dimension $n = 4m + 3$, then $T_n(-k,c_2,\ldots,c_n)[N]$ is an integer.

Proof. By Cororllary 4.3

$$2\{p_g(X,x) - T_n(-k,c_2,\ldots,c_n)[N]\} = h^1 - h^2 + \cdots + h^{4m+1}.$$

Then, since $h^i = h^{4m+i}$,

$$2\{p_g(X,x) - T_n(-k,c_2,\ldots,c_n)[N]\} = 2\{h^1 - h^2 + \cdots - h^{2m}\} + h^{2m+1}$$

On the other hand, $H^{2m+1}(M,\mathcal{O})$ has a non-degenerate skew-symmetric bilinear form, then the dimension of $H^{2m+1}(M,\mathcal{O})$ is even. Thus the number $T_n(-k,c_2,\ldots,c_n)[N]$ is an integer.

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