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# GKZ-decompositions for toric varieties

Hye Sook Park (朴 惠淑、東北大)

## 1 Introduction

We have defined the linear Gale transform in the context of $\mathbf{R}$-vector spaces and stated some properties of it in [12]. In this paper, we modify the definition in the context of $\mathbf{Q}$-vector spaces and apply it to compact toric varieties which have at most quotient singularities. For the definition of a toric variety, see [3], [9] and [10].

Let $N$ be a free $\mathbf{Z}$-module of rank $r$ and $\Xi$ a finite subset of primitive elements in $N$, such that $\Xi$ spans $N_{\mathbf{Q}} := N \otimes_{\mathbf{Z}} \mathbf{Q}$ over $\mathbf{Q}$. Then, as we show in Theorem 3.1, there exists a simplicial and admissible fan $\Delta_0$ in $N$, which is *full*, i.e., every $\xi \in \Xi$ gives rise to a one-dimensional cone in $\Delta_0$. Let $X_0 := T_N\mathrm{emb}(\Delta_0)$ be the corresponding toric variety. On the other hand, we can describe all GKZ-cones $\mathrm{cpl}(\Delta)$ in the GKZ-decomposition as in Theorem 3.4, where GKZ stands for the initials of Gelfand, Kapranov and Zelevinskij.

If $\Xi$ spans $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ positively over $\mathbf{R}$, then $X_0$ becomes a compact toric variety and the GKZ-cone $\mathrm{cpl}(\Delta_0)$ is equal to the cone spanned by the linear equivalence classes of numerically effective divisors on $X_0$. The support of the GKZ-decomposition is equal to the cone spanned by the linear equivalence classes of effective divisors on $X_0$.

Each fan $\Delta$ corresponding to a GKZ-cone $\mathrm{cpl}(\Delta)$ can be obtained from $\Delta_0$ by a finite succession of flops or star subdivisions as in [12, Theorem 3.12]. In this case, the corresponding toric variety has at most quotient singularities. Furthermore, as we show in Theorem 3.6, the union of $\mathrm{cpl}(\Delta)$'s with $\Delta$ obtained from $\Delta_0$ by finite successions of flops also is a convex polyhedral cone.

Now, let us state the outline of this paper.

In Section 1, we define the $\mathbf{Q}$-linear Gale transform, relate it to toric varieties and state some properties. This concept is very useful in dealing with toric varieties with small Picard numbers. For example, Kleinschmidt and Sturmfels [14] have proved that every $r$-dimensional compact toric variety $X$ with $\mathrm{Pic}(X) \leq 3$ must be projective. They also used Gale diagrams from a different point of view. We use the notion in a different way, that is, in connection with the Chow ring of a toric variety.

In Section 2, we introduce the GKZ-decomposition. [5] obtained some decompositions of $\mathbf{R}^N$ by using regular triangulations of integral polytopes corresponding to projective toric varieties. We have generalized and reformulated their results in [12]. We get some information on projective toric varieties when the corresponding fans are confined to have one-dimensional cones within some fixed set $\{\mathbf{R}_{\geq 0}\xi \mid \xi \in \Xi\}$.

In the last section, we first describe the dual cone of $\mathrm{cpl}(\Delta)$ when $\Delta$ is full, simplicial and admissible for a fixed $(N, \Xi)$. It is related to the Mori cone. Secondly, we apply the GKZ-decomposition to a fan which consists of all the faces of a strongly convex cone all of whose proper faces are simplicial.

## 2 Definitions

Throughout this paper, we fix a free $\mathbf{Z}$-module $N$ of rank $r$ over the ring $\mathbf{Z}$ of integers, and denote by $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ its dual $\mathbf{Z}$-module with a canonical bilinear pairing

$$\langle\ ,\ \rangle : M \times N \longrightarrow \mathbf{Z}.$$

We denote the scalar extensions of $N$ and $M$ to the field $\mathbf{R}$ of real numbers by $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ and $M_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$, respectively.

Let $\Xi$ be a finite subset of *primitive* elements in $N$, such that $\Xi$ spans $N_{\mathbf{Q}} := N \otimes_{\mathbf{Z}} \mathbf{Q}$ over the field $\mathbf{Q}$ of rational numbers. Let $Z$ be the $\mathbf{Q}$-vector space with a basis $\{e_\xi \mid \xi \in \Xi\}$, which is in one-to-one correspondence with $\Xi$. By sending $e_\xi$ to $\xi \in \Xi$, we get a surjective linear map $Z \to N_{\mathbf{Q}}$. Let $Z^* := \mathrm{Hom}_{\mathbf{Q}}(Z, \mathbf{Q})$ be the dual space with the dual basis $\{e_\xi^* \mid \xi \in \Xi\}$. Then we have the dual injective linear map $M_{\mathbf{Q}} := M \otimes_{\mathbf{Z}} \mathbf{Q} \to Z^*$ which sends $m \in M_{\mathbf{Q}}$ to $\sum_{\xi \in \Xi} \langle m, \xi \rangle e_\xi^*$. The cokernel $G^{\mathbf{Q}} := Z^*/M_{\mathbf{Q}}$ of the injective map is a $\mathbf{Q}$-vector space of dimension $\#\Xi - r$. For each $\xi \in \Xi$, we denote by $g(\xi) \in G^{\mathbf{Q}}$ the image of $e_\xi^* \in Z^*$. Then by definition, the defining relations among the elements in $g(\Xi) := \{g(\xi) \mid \xi \in \Xi\}$ are

$$\sum_{\xi \in \Xi} \langle m, \xi \rangle g(\xi) = 0 \qquad \text{for all} \qquad m \in M_{\mathbf{Q}}.$$

More symmetrically, they can be written as

$$\sum_{\xi \in \Xi} \xi \otimes g(\xi) = 0 \qquad \text{in} \qquad N_{\mathbf{Q}} \otimes_{\mathbf{Q}} G^{\mathbf{Q}},$$

which we call the *defining relation.* We call the pair $(G^{\mathbf{Q}}, g(\Xi))$ the $\mathbf{Q}$-*linear Gale transform* of $(N_{\mathbf{Q}}, \Xi)$.

We regard $G^{\mathbf{Q}}$ as a subset of its scalar extension $G := G^{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$. Hence $(G, g(\Xi))$ is the linear Gale transform of $(N_{\mathbf{R}}, g(\Xi))$ in the sense of [12]. We define a cone $G_{\geq 0}$ in $G$ by

$$\mathrm{G}_{\geq 0} := \sum_{\xi \in \Xi} \mathbf{R}_{\geq 0} g(\xi).$$

If $\Xi$ positively spans $N_{\mathbf{R}}$ over $\mathbf{R}$, that is, $N_{\mathbf{R}} = \sum_{\xi \in \Xi} \mathbf{R}_{\geq 0}\xi$, then we easily see that $G_{\geq 0}$ becomes a strongly convex cone.

**Example.** Let $\Delta$ be a complete and simplicial fan with $\{n(\rho) \mid \rho \in \Delta(1)\} = \Xi$, where $n(\rho)$ is the unique primitive element in $N$ contained in each one-dimensional cone $\rho$. Let $X := T_N\mathrm{emb}(\Delta)$ be the corresponding compact toric variety. Since $\Delta$ is assumed to be complete and simplicial, we have a perfect pairing in the Chow ring for $\Delta$ (cf. [11])

$$A^{r-1}(X)_{\mathbf{Q}} \times A^1(X)_{\mathbf{Q}} \longrightarrow A^r(X)_{\mathbf{Q}} \cong \bigwedge^r M_{\mathbf{Q}} \xrightarrow{[\,]} \mathbf{Q},$$

where $A^k(X)_{\mathbf{Q}}$ is the scalar extension to $\mathbf{Q}$ of the homogeneous part $A^k(X)$ of degree $k$ in the Chow ring $A(X)$ . Furthermore, if we denote by $T_N\mathrm{Div}(X)_{\mathbf{Q}}$ the scalar extension to $\mathbf{Q}$ of the group of $T_N$-invariant Weil divisors and by $V(\rho)$ the closure of the $T_N$-orbit $\mathrm{orb}(\rho)$ corresponding to each cone $\rho \in \Delta(1)$, then by [10, Proposition 2.1 and Corollary 2.5] we have

$$T_N\mathrm{Div}(X)_{\mathbf{Q}} = \bigoplus_{\rho\in\Delta(1)} \mathbf{Q}V(\rho) \qquad \text{and} \qquad \mathrm{Pic}(X)_{\mathbf{Q}} = A^1(X)_{\mathbf{Q}}.$$

So we have mutually dual short exact sequences of $\mathbf{Q}$-vector spaces:

$$\begin{array}{ccccccccc} 0 & \leftarrow & N_{\mathbf{Q}} & \leftarrow & (T_N\mathrm{Div}(X))^*_{\mathbf{Q}} & \leftarrow & A^{r-1}(X)_{\mathbf{Q}} \otimes_{\mathbf{Q}} \left(A^r(X)_{\mathbf{Q}}\right)^* & \leftarrow & 0 \\ 0 & \rightarrow & M_{\mathbf{Q}} & \rightarrow & T_N\mathrm{Div}(X)_{\mathbf{Q}} & \rightarrow & A^1(X)_{\mathbf{Q}} & \rightarrow & 0, \end{array}$$

where $(T_N\mathrm{Div}(X)_{\mathbf{Q}})^*$ (resp. $(A^r(X)_{\mathbf{Q}})^*$) denotes the dual space of $T_N\mathrm{Div}(X)_{\mathbf{Q}}$(resp. $A^r(X)_{\mathbf{Q}}$). Let us denote by $v(\rho)$ the rational equivalence class of the $T_N$-invariant Weil divisor $V(\rho)$. Then $A^1(X)_{\mathbf{Q}}$ is generated over $\mathbf{Q}$ by the set $\{v(\rho) \mid \rho \in \Delta(1)\}$. Thus the pair

$$\left( A^1(X)_{\mathbf{Q}},\ \{v(\rho) \mid \rho \in \Delta(1)\} \right)$$

is the $\mathbf{Q}$-linear Gale transform of $(N_{\mathbf{Q}}, \{n(\rho) \mid \rho \in \Delta(1)\})$. The defining relation becomes

$$\sum_{\rho\in\Delta(1)} n(\rho) \otimes v(\rho) = 0 \qquad \text{in} \qquad N_{\mathbf{Q}} \otimes_{\mathbf{Q}} A^1(X)_{\mathbf{Q}}.$$

By the properties of the $\mathbf{Q}$-linear Gale transform, some of the properties of $N_{\mathbf{Q}}$ can be translated as those of $A^1(X)_{\mathbf{Q}}$. Namely, in the same notation as above, we have the following:

**Proposition 2.1** (cf. [12]) *Let $\Delta$ be a complete and simplical fan.*

(1) *Let $\rho_1, ..., \rho_r \in \Delta(1)$. Then $\{n(\rho_1), ..., n(\rho_r)\}$ is a $\mathbf{Q}$-basis for $N_{\mathbf{Q}}$ if and only if $\{v(\rho) \mid \rho \in \Delta(1), \rho \neq \rho_1, ..., \rho_r\}$ is a $\mathbf{Q}$-basis for $A^1(X)_{\mathbf{Q}}$.*

(2) *The cone $(A^1(X)_{\mathbf{R}})_{\geq 0} := \sum_{\rho\in\Delta(1)} \mathbf{R}_{\geq 0} v(\rho)$ is strongly convex.*

(3) $\sum_{\rho\in\Delta(1)}\alpha_\rho n(\rho) = 0$ *holds for some* $\alpha_\rho \in \mathbf{Q}$ *if and only if there exists a* $\gamma \in A^{r-1}(X)_{\mathbf{Q}}$ *such that* $\alpha_\rho = [\gamma \cdot v(\rho)]$.

The proofs of (1) and (2) are the same as those of [12, Propositions 1.1 and 1.3]. (3) is clear, because the defining relation gives rise to all the $\mathbf{Q}$-linear relations among the elements in $g(\Xi)$.

We refer the reader to [8] and [12] for more properties.

# 3 GKZ-decomposition

**Definition.** Suppose that $\Delta$ is a simplicial fan in $N$ such that the support $|\Delta|$ is convex and spans $N_{\mathbf{R}}$ over $\mathbf{R}$.(Note that $\Delta$ may not be complete.)

An $\mathbf{R}$-valued function $h$ on $|\Delta|$ is called a $\Delta$-*linear support function* if $h$ is $\mathbf{Z}$-valued on $N \cap |\Delta|$ and if $h$ is linear on each cone $\sigma \in \Delta$. We denote by $\mathrm{SF}(N, \Delta)$ the additive group consisting of all $\Delta$-linear support functions.

If $\Delta$ is simplicial, then $\mathrm{SF}(N,\Delta) \otimes_{\mathbf{Z}} \mathbf{Q}$ is isomorphic to $T_N\mathrm{Div}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ via the homomorphism sending $h \otimes q$ to $\left(\sum_{\rho\in\Delta(1)}(-h(n(\rho)))V(\rho)\right) \otimes q$ for $h \in \mathrm{SF}(N,\Delta)$ and $q \in \mathbf{Q}$. Let us denote $\mathrm{PL}(\Delta) := \mathrm{SF}(N,\Delta) \otimes_{\mathbf{Z}} \mathbf{R}$.

A function $\eta$ in $\mathrm{PL}(\Delta)$ is said to be *convex* if

$$\eta(w + w') \leq \eta(w) + \eta(w') \qquad \text{for all} \qquad w, w' \in |\Delta|.$$

A function $\eta \in \mathrm{PL}(\Delta)$ is said to be *strictly convex with respect to* $\Delta$ if there exists an $m_\sigma \in M_{\mathbf{R}}$ for each $\sigma \in \Delta$ such that

$$\begin{array}{rcll} \eta(w) & = & \langle m_\sigma, w\rangle & \text{if } w \in \sigma \\ \eta(w) & > & \langle m_\sigma, w\rangle & \text{otherwise.} \end{array}$$

A fan $\Delta$ is said to be *quasi-projective* if there exists an $\eta \in \mathrm{PL}(\Delta)$ which is strictly convex with respect to $\Delta$. If a fan $\Delta$ is complete and quasi-projective, then $\Delta$ is said to be *projective*.

We denote by $\mathrm{CPL}(\Delta)$ the cone consisting of all convex functions in $\mathrm{PL}(\Delta)$.

Since we assume that the support $|\Delta|$ spans $N_{\mathbf{R}}$ over $\mathbf{R}$, we can embed $M_{\mathbf{R}}$ into $\mathrm{PL}(\Delta)$. In fact, it can be embedded into the subset $\mathrm{CPL}(\Delta) \subset \mathrm{PL}(\Delta)$. If we regard $M_{\mathbf{R}}$ as a subset of $\mathrm{CPL}(\Delta)$ in this way, then we have $\mathrm{CPL}(\Delta) \cap (-\mathrm{CPL}(\Delta)) = M_{\mathbf{R}}$. Also by using the toric Kleiman-Nakai criterion (cf. [12, Theorem 2.3]), we see that a fan $\Delta$ is quasi-projective if and only if $\mathrm{CPL}(\Delta)$ spans $\mathrm{PL}(\Delta)$ over $\mathbf{R}$.

Let us now fix a finite subset $\Xi$ of primitive elements in $N$ such that $\Xi$ spans $N_{\mathbf{R}}$ over $\mathbf{R}$. We consider all possible fans in the following sense and compare them.

**Definition.** A fan $\Delta$ in $N$ is said to be *admissible* for $(N, \Xi)$ if

(i) $\Delta$ is quasi-projective,

(ii) $|\Delta| = \sum_{\xi\in\Xi} \mathbf{R}_{\geq 0}\xi$ and

(iii) $\Delta(1) \subset \{\mathbf{R}_{\geq 0}\xi \mid \xi \in \Xi\}$.

We denote by $\Xi(\Delta)$ the subset consisting of those elements in $\Xi$ which are of the form $n(\rho)$ for some $\rho \in \Delta(1)$. Note that $\Xi(\Delta) \neq \Xi$ may happen. For any given $\Xi$, however, there always exists a simplicial fan $\Delta$ such that $\Delta$ is admissible for $(N, \Xi)$ with $\Xi(\Delta) = \Xi$, as we now prove by using the concept *pulling*(cf. [6]).

**Definition.** Let $P$ be a convex polytope in $\mathbf{R}^r$ with the vertex set $\mathrm{ver}(P) = \Xi$.
For $\xi \in \Xi$ and $c > 1$, the convex hull $P_* := \mathrm{conv}\,((\mathrm{ver}(P) \setminus \{\xi\}) \cup \{c\xi\})$ is said to be *obtained from $P$ by pulling $\xi$ to $c\xi$* if $(\xi, c\xi] \cap H = \phi$ for the hyperplane $H$ determined by any facet of $P$, where $(\xi, c\xi] := \{a\xi \mid 1 < a \leq c\}$.

Eggleston, Grübaum and Klee [4] described all the faces of $P_*$ explicitly. Using a similar concept of *pushing* instead of *pulling* of vertices, Klee [7] constructed a simplicial convex polytope $P_*$ from a given convex polytope $P$.

**Theorem 3.1** *Let $\Xi$ be a finite subset of primitive elements in $N$ such that $\Xi$ spans $N_{\mathbf{R}}$ over $\mathbf{R}$. Then there exists a simplicial and admissible fan $\Delta$ in $N$ which is* full, *that is, $\Xi(\Delta) = \Xi$. In the two-dimensional case, such a fan $\Delta$ is unique.*

In order to prove this theorem, we use the following lemma:

**Lemma 3.2** *Suppose that $\Delta$ is an $r$-dimensional simplicial fan with convex support. Then $\Delta$ is quasi-projective if and only if there exists $c_\xi > 0$ for each $\xi \in \Xi(\Delta)$ such that the convex hull* $\mathrm{conv}(\{c_\xi \cdot \xi \mid \xi \in \Xi(\Delta)\} \cup \{0\})$ *gives rise to the same fan as $\Delta$ by projection from* 0.

Proof of Theorem 3.1. Let us denote $P_0 := \mathrm{conv}(\Xi \cup \{0\})$. If $\mathrm{ver}(P_0) \neq \Xi$ (or $\Xi \cup \{0\}$, if $\Delta$ is not complete), then we can find $x_\xi > 0$ for each $\xi \in (\Xi \setminus \mathrm{ver}(P_0))$ such that

$$P := \mathrm{conv}\,(\mathrm{ver}(P_0) \cup \{x_\xi\xi \mid \xi \in \Xi \setminus \mathrm{ver}(P_0)\} \cup \{0\})$$

becomes a convex polytope with $\mathrm{ver}(P) = \Xi$ (or $\Xi \cup \{0\}$, if $\Delta$ is not complete).

Note that this convex polytope $P$ may have a facet which is not an $(r-1)$-simplex. But if we use a method similar to that in [4, Theorem 2.1] and [7, collary 2.5], we can find a $c_\xi > 0$ for each $\xi \in \Xi$ such that every facet of the new convex polytope $P_*$, which is obtained from $P$ by pulling $\xi$ to $c_\xi \xi$ for any $\xi \in \Xi$, is an $(r-1)$-simplex. Let us define

$$\sigma_F := \bigcup_{x \in F} \mathbf{R}_{\geq 0} x$$

for any facet $F$ of $P_*$ with $0 \notin F$. Then it is clear that $\sigma_F$ is an $r$-dimensional cone. Now we define

$$\Delta := \{\text{the faces of } \sigma_F \mid F : \text{a facet of } P_* \text{ with } 0 \notin F\}.$$

Then $\Delta$ becomes a simplicial fan with $\Xi(\Delta) = \Xi$. It is clear that $\Delta$ is quasi-projective, by Lemma 3.2.

The second statement is clear. q.e.d.

Recall the exact sequence of $\mathbf{Q}$-vector spaces

$$0 \longrightarrow M_{\mathbf{Q}} \longrightarrow Z^* = \bigoplus_{\xi \in \Xi} \mathbf{Q} e_\xi^* \longrightarrow G^{\mathbf{Q}} \longrightarrow 0.$$

For any simplicial and admissible fan $\Delta$, we define the cone $\widetilde{\mathrm{CPL}}(\Delta)$ in $Z_{\mathbf{R}}^* := Z^* \otimes_{\mathbf{Q}} \mathbf{R}$ to be the set of all elements $x = \sum_{\xi \in \Xi} x_\xi e_\xi^* \in Z_{\mathbf{R}}^*$ satisfying the following: There exists an $\eta \in \mathrm{CPL}(\Delta)$ such that

$$x_\xi \geq \eta(\xi) \quad \text{for all} \quad \xi \in \Xi \qquad \text{and that} \qquad x_\xi = \eta(\xi) \quad \text{for all} \quad \xi \in \Xi(\Delta).$$

$\widetilde{\mathrm{CPL}}(\Delta)$ contains the nontrivial vector subspace $M_{\mathbf{R}}$. We denote by $\mathrm{cpl}(\Delta)$ the image of $\widetilde{\mathrm{CPL}}(\Delta)$ in $G$. Then $\mathrm{cpl}(\Delta)$ is a maximal-dimensional strongly convex cone, that is,

$$\mathrm{cpl}(\Delta) \cap (-\mathrm{cpl}(\Delta)) = \{0\}$$

and

$$\dim \mathrm{cpl}(\Delta) = \dim G = \#\Xi - r,$$

since $\Delta$ is assumed to be simplicial and quasi-projective.

**Theorem 3.3** (cf. [12, Proposition 3.3 and Theorem 3.5]) *Let $\Xi$ be a finite subset of primitive elements in $N$. Assume that $\Xi$ spans $N_{\mathbf{R}}$ over $\mathbf{R}$. Then we get:*

$$\bigcup_{\Delta} \widetilde{\mathrm{CPL}}(\Delta) = M_{\mathbf{R}} + \sum_{\xi \in \Xi} \mathbf{R}_{\geq 0} e_\xi^*$$

*and*

$$\bigcup_{\Delta} \mathrm{cpl}(\Delta) = \sum_{\xi \in \Xi} \mathbf{R}_{\geq 0} g(\xi) = G_{\geq 0},$$

*where $\Delta$ runs through all the simplicial fans admissible for $(N, \Xi)$.*

**Remark.** V. Batyrev pointed out that this theorem can be regared as one on the existence and uniqueness of the Zariski decomposition of effective divisors, and suggests a possible nice formulation of the problem for general higher-dimensional algebraic varieties and arithematic varieties.

In view of the above remark, we now reproduce our earlier proof in [12] in algebro-geometric language.

Proof. It is enough to prove only the first statement.

As we have seen in Theorem 3.1, for a given set $\Xi$ there exists a simplicial and admissible fan $\Delta_0$ in $N$ such that $\Xi = \{n(\rho) \mid \rho \in \Delta_0(1)\}$. Now we fix $\Delta_0$ and denote by $X_0 = T_N\text{emb}(\Delta_0)$ the corresponding toric variety. Then we have a short exact sequence of $\mathbf{Q}$-vector spaces.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{\mathbf{Q}} & \longrightarrow & Z^* = T_N\text{Div}(X_0)_{\mathbf{Q}} & \longrightarrow & G^{\mathbf{Q}} = A^1(X_0)_{\mathbf{Q}} & \longrightarrow & 0. \\ & & & & \| & & \| & & \\ & & & & \bigoplus_{\rho\in\Delta_0(1)} \mathbf{Q}V(\rho) & & \sum_{\rho\in\Delta_0(1)} \mathbf{Q}v(\rho) & & \end{array}$$

Hence, for any simplicial and admissible fan $\Delta$, $\widetilde{\text{CPL}}(\Delta)$ can be regarded as a subcone of $T_N$-invariant $\mathbf{R}$-divisors on $X_0$. What we have to do is to show

$$\bigcup\{\widetilde{\text{CPL}}(\Delta) \mid \Delta : \text{ simplicial and admissible }\} = M_{\mathbf{R}} + \sum_{\rho\in\Delta_0(1)} \mathbf{R}_{\geq 0}V(\rho).$$

Let $\Delta$ be a simplicial fan admissible for $(N, \{n(\rho) \mid \rho \in \Delta_0(1)\})$. Then by the definition of $\widetilde{\text{CPL}}(\Delta)$, there exists a $T_N$-invariant principal divisor $P$ on $X_0$ for any divisor $D \in \widetilde{\text{CPL}}(\Delta)$ such that $D - P$ is a $T_N$-invariant effective divisor on $X_0$. So the left hand side is contained in the right hand side.

To prove the opposite inclusion, let us denote

$$D := \sum_{\rho\in\Delta_0(1)} x_\rho V(\rho) \quad \in \quad M_{\mathbf{R}} + \sum_{\rho\in\Delta_0(1)} \mathbf{R}_{\geq 0}V(\rho) \quad \subset \quad \bigoplus_{\rho\in\Delta_0(1)} \mathbf{R}V(\rho).$$

Consider the convex polyhedral cone

$$E(D) := \mathbf{R}_{\geq 0}(0, 1) + \sum_{\rho\in\Delta_0(1)} \mathbf{R}_{\geq 0}(n(\rho), x_\rho) \quad \subset \quad |\Delta_0| \times \mathbf{R}.$$

Since $D$ is an element in the set $M_{\mathbf{R}} + \sum_{\rho\in\Delta_0(1)} \mathbf{R}_{\geq 0}V(\rho)$, there exists a unique function $\eta_D : |\Delta_0| \to \mathbf{R}$ such that the epigraph

$$\text{epi}(\eta_D) := \{\ (w, c) \in |\Delta_0| \times \mathbf{R} \ \mid \ c \geq \eta_D(w)\ \}$$

is equal to the cone $E(D)$. Namely, there exists a $T_N$-invariant divisor

$$P' := \sum_{\rho\in\Delta_0(1)} \eta_D(n(\rho))V(\rho)$$

on $X_0$ such that $D-P'$ is a $T_N$-invariant effective divisor on $X_0$ with the smallest number of positive coefficients.

Construct a fan $\Delta_D$ by projecting the faces of $\mathrm{epi}(\eta_D)$ using the first projection $\mathrm{pr}_1 : |\Delta_0| \times \mathbf{R} \to |\Delta_0|$. By construction, $\eta_D$ is strictly convex with respect to this $\Delta_D$. Hence $\Delta_D$ is admissible for $(N, \{n(\rho) \mid \rho \in \Delta_0(1)\})$.

If $\Delta_D$ itself is simplicial, then clearly we have $D \in \widetilde{\mathrm{CPL}}(\Delta_D)$. Such a fan $\Delta_D$, however, is not simplicial in general, but we can obtain a simplicial and admissible fan $\Delta'_D$, which is a subdivision of $\Delta_D$, by the same method as that used in the proof of Theorem 3.1. Since $\Delta_D(1) \subset \Delta'_D(1)$ and $\eta_D$ is strictly convex with respect to $\Delta_D$, $\eta_D$ becomes a convex function piecewise linear with respect to $\Delta'_D$, hence we have $D \in \widetilde{\mathrm{CPL}}(\Delta'_D)$. q.e.d.

**Remark.** (1) As we have seen in the proof, for any

$$D \in M_{\mathbf{R}} + \sum_{\rho \in \Delta_0(1)} \mathbf{R}_{\geq 0} V(\rho) \quad \subset \quad \bigoplus_{\rho \in \Delta_0(1)} \mathbf{R} V(\rho),$$

we can obtain a function $\eta_D$ and a simplicial and admissible fan $\Delta'_D$ such that $D \in \widetilde{\mathrm{CPL}}(\Delta'_D)$. This $\Delta'_D$ is not a subdivision of $\Delta_0$ in general. We can obtain, however, a simplicial and admissible fan $\Delta'_0$, which is full, by subdividing $\Delta'_D$. We can regard $D$ as an element in $\bigoplus_{\rho \in \Delta'_0(1)} \mathbf{R} V'(\rho) \cong \bigoplus_{\rho \in \Delta_0(1)} \mathbf{R} V(\rho)$. Let $P' := \sum_{\rho \in \Delta'_0(1)} \eta_D(n(\rho)) V'(\rho)$. Then $P'$ is an element in $\widetilde{\mathrm{CPL}}(\Delta'_0)$. By the definition of $\eta_D$, $D-P'$ belongs to $\sum_{\rho \in \Delta'_0(1)} \mathbf{R}_{\geq 0} V'(\rho)$ with the smallest number of positive coefficients. The terms with positive coefficients correspond to $\rho \in \Delta'_0(1) \setminus \Delta'_D(1)$, that is,

$$D = P' + \sum_{\rho \in \Delta'_0(1) \setminus \Delta'_D(1)} a_\rho V'(\rho) \in \widetilde{\mathrm{CPL}}(\Delta'_0) + \sum_{\rho \in \Delta'_0(1)} \mathbf{R}_{\geq 0} V'(\rho)$$

for some $a_\rho > 0$.

As we will see later, the dual cone $(\mathrm{cpl}(\Delta'_0))^\vee$ of the image $\mathrm{cpl}(\Delta'_0)$ of $\widetilde{\mathrm{CPL}}(\Delta'_0)$ is equal to the Mori cone $NE(X'_0)$ of $X'_0 := T_N\mathrm{emb}(\Delta'_0)$.

(2) In fact, the collection of all faces of $\mathrm{cpl}(\Delta)$'s for simplicial and admissible fans becomes a cone decomposition with support equal to $G_{\geq 0}$. We call this decomposition the GKZ-*decomposition* for $(N_{\mathbf{R}}, \Xi)$ and call $\mathrm{cpl}(\Delta)$ the GKZ-*cone*. Furthermore, we can describe all the elements in this collection explicitly. Indeed, by defining the GKZ-cones for any admissible convex polyhedral cone decompositions, we see that GKZ-cones corresponding to nonsimplicial fans become faces of GKZ-cones corresponding to some simplicial fans.

**Definition.** Suppose $\Delta$ and $\Delta'$ are simplicial fans admissible for $(N, \Xi)$.

$\Delta'$ is called the *star subdivision* of $\Delta$ with respect to a $\xi_1 \in \Xi \setminus \Xi(\Delta)$ if $\Delta'$ consists of the faces of the cones belonging to the union $A \cup B$, where $A$ and $B$ are defined as follows:

Let $\alpha \in \Delta$ be the unique cone containing $\xi_1$ in its relative interior and let $\beta_1, \ldots, \beta_s$ be the facets of $\alpha$ with $s := \dim \alpha$. Then

$$A := \Delta(r) \setminus \{\sigma \in \Delta(r) \mid \sigma \succ \alpha\}$$

and

$$B := \{\gamma + \beta_j + \mathbf{R}_{\geq 0}\xi_1 \mid 1 \leq j \leq s, \gamma \in \Delta(r-s) \text{ with } \gamma + \alpha \in \Delta(r) \text{ and } \gamma \cap \alpha = \{0\}\}.$$

Let $\Delta$ be a simplicial fan in $N$. Suppose that $\tau = \sigma_1 \cap \sigma_2 \in \Delta(r-1)$ for some $\sigma_1$ and $\sigma_2$ in $\Delta(r)$. Let us denote

$$\begin{aligned}
\rho_i &:= \mathbf{R}_{\geq 0}\xi_i \in \Delta(1) \qquad \text{for} \qquad i = 1, \ldots, r+1 \\
\sigma_1 &:= \tau + \rho_1 \\
\sigma_2 &:= \tau + \rho_2 \\
\tau &:= \rho_3 + \rho_4 + \cdots + \rho_{r+1}.
\end{aligned}$$

By renumbering the indices if necessary, we may assume that

$$\sum_{i=1}^{p} a_i \xi_i = \sum_{j=1}^{q} a_{p+j}\xi_{p+j} \qquad \text{for some} \qquad a_1, \ldots, a_{p+q} > 0.$$

Note that $p \geq 2$, $q \geq 0$ and $p + q \leq r + 1$.

Let us further denote

$$\begin{aligned}
\varepsilon' &:= \rho_1 + \cdots + \rho_p \\
\varepsilon &:= \rho_{p+1} + \cdots + \rho_{p+q} \\
\varepsilon'_i &:= \rho_1 + \cdots + \overset{i}{\vee} + \cdots + \rho_p \qquad \text{for} \quad i = 1, \ldots, p \\
\varepsilon_j &:= \rho_{p+1} + \cdots + \overset{j}{\vee} + \cdots + \rho_{p+q} \qquad \text{for} \quad j = 1, \ldots, q.
\end{aligned}$$

Then it is clear that $\varepsilon \in \Delta(q)$ and $\varepsilon' \notin \Delta(p)$, because two cones cannot have a common relative interior point.

**Definition.** Suppose $\Delta$ and $\Delta'$ are simplicial fans admissible for $(N, \Xi)$.

$\Delta'$ is called the *flop* of $\Delta$ if there exists a $\tau = \sigma_1 \cap \sigma_2 \in \Delta(r-1)$ with some $\sigma_1, \sigma_2 \in \Delta(r)$ which satisfies the following: In the same notation as above,

(i) $q \geq 2$

(ii) $\varepsilon + \varepsilon'_i \in \Delta(p+q-1)$ for any $i = 1, \ldots, p$

(iii) Let

$$\Lambda := \{\lambda \in \Delta(r-p-q+1) \mid \lambda + \varepsilon + \varepsilon'_i \in \Delta(r),\ 1 \leq i \leq p,\ \lambda \cap (\varepsilon + \varepsilon') = \{0\}\}$$

satisfy the following property: For any $\lambda \in \Lambda$ and for any one-dimensional face $\rho_0$ of $\lambda$, if there exists a $\rho'_0 \in \Delta(1) \setminus (\{\rho_1, ..., \rho_{p+q}\} \cup \{\rho \in \Delta(1) \mid \rho \prec \lambda\})$ such that

$$\rho'_0 + \sum_{\rho \prec \lambda, \rho \neq \rho_0} \rho + \sum_{i=3}^{p+q} \rho_i \in \Delta(r-1),$$

then it is unique and $\rho'_0 + \sum_{\rho \prec \lambda, \rho \neq \rho_0} \rho \in \Lambda$.

Then the flop $\Delta'$ of $\Delta$ consists of the faces of the cones in the set

$$\Delta'(r) := (\Delta(r) \setminus \{\lambda + \varepsilon + \varepsilon'_i \mid \lambda \in \Lambda,\ 1 \leq i \leq p\}) \cup \{\lambda + \varepsilon_j + \varepsilon' \mid \lambda \in \Lambda,\ 1 \leq j \leq q\}.$$

Note that if $\Delta'$ is the flop of $\Delta$, then we see that

(1) $\Delta(1) = \Delta'(1)$.

(2) $\Delta$ is the flop of $\Delta'$.

(3) $\mathrm{cpl}(\Delta) \cap \mathrm{cpl}(\Delta')$ is a facet of both $\mathrm{cpl}(\Delta)$ and $\mathrm{cpl}(\Delta')$.

(4) There exists a nonsimplicial and admissible fan $\bar{\Delta}$ such that both $\Delta$ and $\Delta'$ are subdivisions of $\bar{\Delta}$ with $\bar{\Delta}(1) = \Delta(1)$. Indeed, $\bar{\Delta}$ consists of the faces of the cones in the set

$$\bar{\Delta}(r) := (\Delta(r) \setminus \{\lambda + \varepsilon + \varepsilon'_i \mid \lambda \in \Lambda,\ 1 \leq i \leq p\}) \cup \{\lambda + \varepsilon + \varepsilon' \mid \lambda \in \Lambda\}.$$

By [12, Theorem 3.12], we can describe a relation among the GKZ-cones in the GKZ-decomposition as a relation among the corresponding fans. Namely, the cone $\mathrm{cpl}(\Delta) \cap \mathrm{cpl}(\Delta')$ is a facet of both $\mathrm{cpl}(\Delta)$ and $\mathrm{cpl}(\Delta')$ if and only if one fan is a star subdivision or the flop of the other.

If $\Delta$ is simplicial, we have

$$\widetilde{\mathrm{CPL}}(\Delta) = \bigcap_{\sigma \in \Delta(r)} \left( M_{\mathbf{R}} + \sum_{\xi \in \Xi \setminus (\Xi(\Delta) \cap \sigma)} \mathbf{R}_{\geq 0} e^*_\xi \right),$$

and

$$\mathrm{cpl}(\Delta) = \bigcap_{\sigma \in \Delta(r)} \left( \sum_{\xi \in \Xi \setminus (\Xi(\Delta) \cap \sigma)} \mathbf{R}_{\geq 0} g(\xi) \right).$$

By the property of the linear Gale transform, the set $\Lambda \subset \Xi$ is an $\mathbf{R}$-basis of $N_{\mathbf{R}}$ if and only if $g(\Xi \setminus \Lambda) := \{g(\xi) \mid \xi \in \Xi \setminus \Lambda\}$ is an $\mathbf{R}$-basis of $G$. Hence we see that every GKZ-cone $\mathrm{cpl}(\Delta)$ can be written as an intersection of cones which are generated by some $\mathbf{R}$-bases for $G$. Moreover, we get the converse correspondence as follows:

**Theorem 3.4** (cf. [2]) *For an* $\mathbf{R}$*-basis* $\Omega \subset g(\Xi)$ *for* $G$, *we denote*

$$C_\Omega := \sum_{g(\xi)\in\Omega} \mathbf{R}_{\geq 0} g(\xi),$$

*which is a maximal dimensional cone, that is,* $\dim C_\Omega = \#\Xi - r$.

*Let* $A$ *be a* $(\#\Xi - r)$*-dimensional cone in* $G_{\geq 0}$ *of the form* $A = \bigcap_\Omega C_\Omega$, *where* $\Omega \subset g(\Xi)$ *runs through some* $\mathbf{R}$*-bases for* $G$. *Suppose that for any* $\mathbf{R}$*-basis* $\Omega' \subset g(\Xi)$ *for* $G$, $C_{\Omega'}$ *contains* $A$ *whenever* $C_{\Omega'}$ *meets the interior of* $A$. *Then there exists a unique simplicial and admissible fan* $\Delta$ *satisfying* $\mathrm{cpl}(\Delta) = A$.

Proof. Let $\Theta$ be the set of all $\mathbf{R}$-bases $\Omega \subset g(\Xi)$ for $G$ satisfying $C_\Omega \supset A$. Choose an element $y$ from the interior of $A$. Let $x$ be the preimage of $y$ in $Z^*_{\mathbf{R}}$ under the map $Z^*_{\mathbf{R}} \to G$. Then $x$ is contained in the set

$$M_{\mathbf{R}} + \sum_{\xi\in\Xi} \mathbf{R}_{\geq 0} e^*_\xi = \bigcup \{\widetilde{\mathrm{CPL}}(\Delta) \mid \Delta : \text{simplicial and admissible}\}.$$

Thus there exists a simplical and admissible fan $\Delta$ satisfying $x \in \widetilde{\mathrm{CPL}}(\Delta)$. Namely, there exists an $m_\sigma \in M_{\mathbf{R}}$ for any $\sigma \in \Delta(r)$ such that $x_\xi \geq \langle m_\sigma, \xi\rangle$ for $\xi \in \Xi$ and that $x_\xi = \langle m_\sigma, \xi\rangle$ for $\xi \in \Xi(\Delta)\cap\sigma$. We claim that $x$ is contained in the interior of $\widetilde{\mathrm{CPL}}(\Delta)$. To show this, let us assume that $x$ is contained in the boundary of $\widetilde{\mathrm{CPL}}(\Delta)$. Then there exist $\sigma_0 \in \Delta(r)$ and $\xi_0 \in \Xi\backslash(\Xi(\Delta)\cap\sigma_0)$ such that $x_{\xi_0} = \langle m_{\sigma_0}, \xi_0\rangle$. Let $\sigma_0 = \mathbf{R}_{\geq 0}\xi_1 + \cdots + \mathbf{R}_{\geq 0}\xi_r$ for an $\mathbf{R}$-basis $\{\xi_1, \dots, \xi_r\} \subset \Xi(\Delta)$. Then the set $\Omega := \{g(\xi) \mid \xi \in \Xi,\ \xi \neq \xi_1, \dots, \xi_r\} \subset g(\Xi)$ becomes an $\mathbf{R}$-basis for $G$. We have

$$y \in \sum_{\substack{\xi\in\Xi \\ \xi\neq\xi_0,\xi_1,\dots,\xi_r}} \mathbf{R}_{\geq 0} g(\xi) \quad \subset \quad \sum_{\substack{\xi\in\Xi \\ \xi\neq\xi_1,\dots,\xi_r}} \mathbf{R}_{\geq 0} g(\xi) = C_\Omega.$$

By assumption, we have $C_\Omega \supset A$. Hence $y$ is contained in the interior of $C_\Omega$, a contradiction to the assumption $x_{\xi_0} = \langle m_{\sigma_0}, \xi_0\rangle$. Hence $x$ is contained in the interior of $\widetilde{\mathrm{CPL}}(\Delta)$. Hence $\Delta$ is the unique fan satisfying $x \in \widetilde{\mathrm{CPL}}(\Delta)$.

As we have seen above, any $r$-dimensional cone $\sigma \in \Delta(r)$ gives rise to an $\mathbf{R}$-basis

$$\Omega := \{g(\xi) \mid \xi \in \Xi,\ \mathbf{R}_{\geq 0}\xi \not\prec \sigma\} \subset g(\Xi)$$

for $G$, satisfying $C_\Omega \supset A$. Conversely, for any $\Omega \in \Theta$, the set

$$\sigma := \sum_{\substack{\xi\in\Xi \\ g(\xi)\notin\Omega}} \mathbf{R}_{\geq 0}\xi$$

becomes an $r$-dimensional cone in $\Delta$. Consequently, we have

$$\mathrm{cpl}(\Delta) = \bigcap_{\sigma\in\Delta(r)} \left(\sum_{\xi\in\Xi\backslash(\Xi(\Delta)\cap\sigma)} \mathbf{R}_{\geq 0} g(\xi)\right) = \bigcap_{\Omega\in\Theta} \left(\sum_{g(\xi)\in\Omega} \mathbf{R}_{\geq 0} g(\xi)\right) = A.$$

q.e.d.

**Corollary 3.5** *There exists a one-to-one correspondence between the set of the simplicial and admissible fans and the set of maximal dimensional cones $\bigcap_{\Omega \in \Theta} C_\Omega$ which are not separated by $C_{\Omega'}$ for any $\mathbf{R}$-basis $\Omega' \subset g(\Xi)$ for $G$, where $\Theta$ runs through all the possible subsets of all the $\mathbf{R}$-bases $\Omega \subset g(\Xi)$ for $G$.*

Proof. By what we stated before Theorem 3.4, a simplicial and admissible fan gives rise to a cone of the form $\bigcap_\Omega C_\Omega$. We get the converse correspondence by Theorem 3.4.

q.e.d.

**Example.** Let $\Xi := \{n, n', -n, -n-n', n-n'\} \subset N \cong \mathbf{Z}^2$, where $\{n, n'\}$ is a $\mathbf{Z}$-basis for $N$. Then there exist eight different simplicial admissible fans. Among those fans, there is a unique fan $\Delta_0$ which is full. The corresponding toric variety $X_0 := T_N\mathrm{emb}(\Delta_0)$ is obtained from the weighted projective plane $\mathbf{P}(1,1,2) =: S$ by blowing-up at the following two $T_N$-fixed points of $S$:

$$\begin{aligned} p_1 &:= V(\mathbf{R}_{\geq 0}n' + \mathbf{R}_{\geq 0}(n-n')) \\ p_2 &:= V(\mathbf{R}_{\geq 0}n' + \mathbf{R}_{\geq 0}(-n-n')). \end{aligned}$$

From the defining relation, we get the relations

$$v(\mathbf{R}_{\geq 0}n) + v(\mathbf{R}_{\geq 0}(n-n')) = v(\mathbf{R}_{\geq 0}(-n)) + v(\mathbf{R}_{\geq 0}(-n-n'))$$

and

$$v(\mathbf{R}_{\geq 0}n') = v(\mathbf{R}_{\geq 0}(-n-n')) + v(\mathbf{R}_{\geq 0}(n-n'))$$

in $A^1(X_0)_\mathbf{Q}$. $G_{\geq 0}$ is a three-dimensional strongly convex cone spanned by the set

$$\{\ v(\mathbf{R}_{\geq 0}n), v(\mathbf{R}_{\geq 0}(n-n')), v(\mathbf{R}_{\geq 0}(-n)), v(\mathbf{R}_{\geq 0}(-n-n'))\ \}$$

in $A^1(X_0)_\mathbf{R} := A^1(X_0)_\mathbf{Q} \otimes_\mathbf{Q} \mathbf{R}$. By choosing all the $\mathbf{R}$-bases for $G = A^1(X_0)_\mathbf{R}$ from the set $g(\Xi) = \{v(\rho) \mid \rho \in \Delta_0(1)\}$, we get the GKZ-decomposition consisting of eight different three-dimensional cones. Using Theorem 3.4, we can express the corresponding fans immediately. The corresponding toric varieties are

(i) $S = \mathbf{P}(1,1,2)$,

(ii) (resp. (iii)) the equivariant blowing-up $X_1$ (resp. $X_2$) of $S$ at the $T_N$-fixed point $p_1$ (resp. $p_2$),

(iv) $X_0$,

(v) (resp. (vi)) the Hirzebruch surface $F_1 =: Y_1$ (resp. $Y_2$) obtained from $X_0$ by contracting $V(\mathbf{R}_{\geq 0}(n-n'))$ (resp. $V(\mathbf{R}_{\geq 0}(-n-n'))$), and

(vii) (resp. (viii)) the projective plane $\mathbf{P}_2(\mathbf{C}) =: Z_1$ (resp. $Z_2$) obtained from $Y_1$ (resp. $Y_2$) by contracting $V(\mathbf{R}_{\geq 0}(-n))$ (resp. $V(\mathbf{R}_{\geq 0}(-n))$) in $Y_1$ (resp. $Y_2$).

It is clear that the GKZ-decomposition of $G$ is uniquely determined by a given set $\Xi$. From this, we can obtain all *possible* fans and get information on the relations among these fans.

Suppose that $\Delta$ is a complete fan in $N$. Then by the property of the linear Gale transform, $G_{\geq 0}$ becomes a strongly convex cone. As we guess from the example above, the GKZ-decomposition of $G$ has some *core* which is a union of the GKZ-cones corresponding to fans which are full, simplicial and admissible. $\Delta$ becomes coaser as $\mathrm{cpl}(\Delta)$ goes to the boundary of $G_{\geq 0}$. In fact, the *core* in the above sense also becomes a cone in $G_{\geq 0}$, even if $\Delta$ is not complete, as we now show.

**Theorem 3.6** *Let $\Xi$ be a finite subset of primitive elements in $N$ such that $\Xi$ spans $N_{\mathbf{R}}$ over $\mathbf{R}$. We denote by $\tilde{C}$ the union of $\widetilde{\mathrm{CPL}}(\Delta)$'s corresponding to all fans which are full, simplicial and admissible for $(N, \Xi)$. Then $\tilde{C}$ is equal to the set of those elements*

$$x = \sum_{\xi\in\Xi} x_\xi e_\xi^* \in M_{\mathbf{R}} + \sum_{\xi\in\Xi} \mathbf{R}_{\geq 0} e_\xi^*$$

*which satisfy*

$$a_1 x_{\xi_1} + \cdots + a_p x_{\xi_p} \geq x_\xi,$$

*whenever*

$$\xi_1, ..., \xi_p, \xi \in \Xi \quad \textit{and} \quad a_1\xi_1 + \cdots + a_p\xi_p = \xi \quad \textit{for some} \quad a_1, ..., a_p \geq 0.$$

*So the image $C$ of $\tilde{C}$ in $G$ becomes a convex polyhedral cone in $G_{\geq 0}$. If both $\widetilde{\mathrm{CPL}}(\Delta)$ and $\widetilde{\mathrm{CPL}}(\Delta')$ are contained in $\tilde{C}$, then $\Delta$ can be obtained from $\Delta'$ by a finite succession of flops.*

Proof. Suppose that $x = \sum_{\xi\in\Xi} x_\xi e_\xi^*$ is contained in $\tilde{C}$. Clearly $x$ is an element in $M_{\mathbf{R}} + \sum_{\xi\in\Xi} \mathbf{R}_{\geq 0} e_\xi^*$. There exists a fan $\Delta$ which is full, simplicial, admissible, and satisfying $x \in \widetilde{\mathrm{CPL}}(\Delta)$. Hence, there exists an $\eta \in \mathrm{CPL}(\Delta)$ such that $x_\xi = \eta(\xi)$ for any $\xi \in \Xi$. If $a_1\xi_1 + \cdots + a_p\xi_p = \xi$ holds for $\xi_1, ..., \xi_p, \xi \in \Xi$ and for some $a_1, ..., a_p > 0$, then

$$x_\xi = \eta(\xi) = \eta(a_1\xi_1 + \cdots + a_p\xi_p) \leq a_1\eta(\xi_1) + \cdots + a_p\eta(\xi_p) = a_1 x_{\xi_1} + \cdots + a_p x_{\xi_p},$$

because $\eta$ is convex.

Conversely, suppose that $x = \sum_{\xi\in\Xi} x_\xi e_\xi^* \in M_{\mathbf{R}} + \sum_{\xi\in\Xi} \mathbf{R}_{\geq 0} e_\xi^*$ satisfies the assumption. Recall that

$$M_{\mathbf{R}} + \sum_{\xi\in\Xi} \mathbf{R}_{\geq 0} e_\xi^* = \bigcup_{\Delta} \widetilde{\mathrm{CPL}}(\Delta),$$

where $\Delta$ runs through all the simplicial and admissible fans. Thus there exists a simplicial and admissible fan $\Delta$ satisfying $x \in \widetilde{\mathrm{CPL}}(\Delta)$. Namely, there exists an $\eta \in \mathrm{CPL}(\Delta)$ such that $x_\xi \geq \eta(\xi)$ for any $\xi \in \Xi$, where the equality holds if $\xi \in \Xi(\Delta)$. For any $\xi \in \Xi\setminus\Xi(\Delta)$, we can find an $r$-dimensional cone $\sigma := \mathbf{R}_{\geq 0}\xi_1 + \cdots + \mathbf{R}_{\geq 0}\xi_r \in \Delta(r)$ containing $\xi$. Thus,

$$\xi = a_1\xi_1 + \cdots + a_r\xi_r \quad \text{for some} \quad a_1, ..., a_r \geq 0.$$

Hence we have

$$x_\xi \geq \eta(\xi) = \eta(a_1\xi_1 + \cdots + a_r\xi_r) = a_1\eta(\xi_1) + \cdots + a_r\eta(\xi_r) = a_1x_{\xi_1} + \cdots + a_rx_{\xi_r} \geq x_\xi,$$

by assumption. This implies that $x_\xi = \eta(\xi)$ for all $\xi \in \Xi$. We can find a subdivision $\Delta'$ of $\Delta$ such that $\Delta'$ is full, simplicial and admissible as in Theorem 3.1. It is clear that $x \in \widetilde{\mathrm{CPL}}(\Delta')$.

As for the last statement of the theorem, we just note that $\Delta$ and $\Delta'$ are full. So $\Delta(1) = \Delta'(1)$ and the case of a star subdivision cannot occur in $\mathcal{C}$. q.e.d.

# 4 Applications

In this section we deal with two applications. We first describe the dual cone of a GKZ-cone corresponding to a fan which is full, simplicial and admissible for a fixed $(N, \Xi)$. Secondly, we consider a fan which consists of all the faces of a strongly convex cone all of whose proper faces are simplicial.

Recall that $\dim N_{\mathbf{R}} = r$. An $(r-1)$-dimensional cone $\tau \in \Delta(r-1)$ is called an *internal wall* if there exist $\sigma$ and $\sigma'$ in $\Delta(r)$ such that $\tau = \sigma \cap \sigma'$. It is clear that every $(r-1)$-dimensional cone is an internal wall when $\Delta$ is complete.

**Theorem 4.1** (cf. [12, Theorem 2.3]) *Let $\Delta$ be an $r$-dimensional full and simplicial fan in $N$ with convex support. Then for each internal wall $\tau \in \Delta(r-1)$, there exists a nonzero element $l_\tau \in G^*$ uniquely determined up to positive scalar multiple such that*

$$\mathrm{cpl}(\Delta)^\vee = \sum_{\tau:\ \text{internal wall}} \mathbf{R}_{\geq 0} l_\tau.$$

Proof. Let $\tau \in \Delta(r-1)$ be an internal wall. Then there exist $\sigma_1(\tau)$ and $\sigma_2(\tau)$ in $\Delta(r)$ such that $\tau = \sigma_1(\tau) \cap \sigma_2(\tau)$. Let

$$\begin{aligned} \rho_i(\tau) &:= \mathbf{R}_{\geq 0}\xi_i(\tau) \in \Delta(1) \qquad \text{for} \qquad i = 1, ..., r+1 \\ \sigma_1(\tau) &:= \tau + \rho_1(\tau) \\ \sigma_2(\tau) &:= \tau + \rho_2(\tau) \\ \tau &:= \rho_3(\tau) + \rho_4(\tau) + \cdots + \rho_{r+1}(\tau). \end{aligned}$$

By the definition of $\widetilde{\mathrm{CPL}}(\Delta)$, we see that

$$\widetilde{\mathrm{CPL}}(\Delta) = \bigcap_{\tau:\text{ internal wall}} \left( M_{\mathbf{R}} + \mathbf{R}_{\geq 0} e^*_{\xi_1(\tau)} + \sum_{\substack{\xi \in \Xi \\ \xi \neq \xi_1(\tau), \dots, \xi_{r+1}(\tau)}} \mathbf{R} e^*_{\xi} \right),$$

hence

$$\mathrm{cpl}(\Delta) = \bigcap_{\tau:\text{ internal wall}} \left( \mathbf{R}_{\geq 0} g(\xi_1(\tau)) + \sum_{\substack{\xi \in \Xi \\ \xi \neq \xi_1(\tau), \dots, \xi_{r+1}(\tau)}} \mathbf{R} g(\xi) \right)$$

and

$$\mathrm{cpl}(\Delta)^{\vee} = \sum_{\tau:\text{ internal wall}} \left( \mathbf{R}_{\geq 0} g(\xi_1(\tau)) + \sum_{\substack{\xi \in \Xi \\ \xi \neq \xi_1(\tau), \dots, \xi_{r+1}(\tau)}} \mathbf{R} g(\xi) \right)^{\vee}$$

Hence we have

$$\begin{aligned}
\mathrm{cpl}(\Delta)^{\vee}_{\tau} &:= \left( \mathbf{R}_{\geq 0} g(\xi_1(\tau)) + \sum_{\substack{\xi \in \Xi \\ \xi \neq \xi_1(\tau), \dots, \xi_{r+1}(\tau)}} \mathbf{R} g(\xi) \right)^{\vee} \\
&= \left\{ \sum_{\xi \in \Xi} a_{\xi} e_{\xi} \mid \sum_{\xi \in \Xi} a_{\xi} \xi = 0,\ a_{\xi_1(\tau)} \geq 0,\ a_{\xi} = 0 \text{ for } \xi \in \Xi \setminus \{\xi_1(\tau), \dots, \xi_{r+1}(\tau)\} \right\} \\
&= \left\{ \sum_{i=1}^{r+1} a_{\xi_i(\tau)} e_{\xi_i(\tau)} \mid \sum_{i=1}^{r+1} a_{\xi_i(\tau)} \xi_i(\tau) = 0,\ a_{\xi_1(\tau)} \geq 0 \right\} \\
&= \left\{ a \cdot \sum_{i=1}^{r+1} a_i e_{\xi_i(\tau)} \mid \sum_{i=1}^{r+1} a_i \xi_i(\tau) = 0,\ a_1 > 0, a \geq 0 \right\}.
\end{aligned}$$

Note that the relation $\sum_{i=1}^{r+1} a_i \xi_i(\tau) = 0$ is nothing but a positive constant multiple of the relation among the primitive elements $\{\xi_1(\tau), \dots, \xi_{r+1}(\tau)\}$. Since we assume $a_1 > 0$, it is clear that $a_2 > 0$. By renumbering the indices if necessary, we have a relation

$$\sum_{i=1}^{p} a_i \xi_i(\tau) = \sum_{j=1}^{q} (-a_{p+j}) \xi_{p+j}(\tau) \quad \text{for some} \quad a_1, \dots, a_p, (-a_{p+1}), \dots, (-a_{p+q}) > 0$$

among the elements in a minimal linearly dependent subset of $\{\xi_1(\tau), \dots, \xi_{r+1}(\tau)\}$, where $p, q$ are intergers with $p \geq 2$ and $p + q \leq r + 1$. If we put

$$l_{\tau} := \sum_{i=1}^{p} a_i e_{\xi_i(\tau)} - \sum_{j=1}^{q} (-a_{p+j}) e_{\xi_{p+j}(\tau)},$$

then it is clear that $\mathrm{cpl}(\Delta)^\vee_\tau = \mathbf{R}_{\geq 0} l_\tau$. q.e.d.

By this theorem, we can explain all $\mathrm{cpl}(\Delta')$'s which have common facets with $\mathrm{cpl}(\Delta)$ in the GKZ-decomposition. Indeed, for a one-dimensional face $R := \mathbf{R}_{\geq 0} l_{\tau_0}$ of the dual cone $\mathrm{cpl}(\Delta)^\vee$, let us denote by $F_R$ the facet of $\mathrm{cpl}(\Delta)$ corresponding to $R$. In the same notation, we may assume that $l_{\tau_0} = \sum_{i=1}^{p} a_i e_{\xi_i} - \sum_{j=1}^{q} (-a_{p+j}) e_{\xi_{p+j}}$ for some $\xi_1, ..., \xi_{p+q} \in \Xi$ and $a_1, ..., a_p, (-a_{p+1}), ..., (-a_{p+q}) > 0$. Then by [12, Theorem 3.12], we get the following:

(1) If $q = 0$, then $F_R$ is contained in the boundary $\partial G_{\geq 0}$ of $G_{\geq 0}$.

(2) If $q = 1$, then there exists another simplicial fan $\Delta'$ which is a star subdivision of $\Delta$ with respect to $\xi_{p+1}$ such that $F_R = \mathrm{cpl}(\Delta) \cap \mathrm{cpl}(\Delta')$.

(3) If $q \geq 2$, then there exists another simplicial fan $\Delta'$ such that $\Delta$ and $\Delta'$ are flops of each other with $F_R = \mathrm{cpl}(\Delta) \cap \mathrm{cpl}(\Delta')$.

**Example.** Let $\Delta$ be a *complete* simplicial fan which is full, and let $X := T_N\mathrm{emb}(\Delta)$ be the corresponding toric variety. In this case, we have a perfect pairing

$$A^{r-1}(X)_{\mathbf{Q}} \times A^1(X)_{\mathbf{Q}} \longrightarrow A^r(X)_{\mathbf{Q}} \cong \bigwedge^r M_{\mathbf{Q}}$$

as in [11]. Thus we have the mutually dual short exact sequences

$$0 \leftarrow N_{\mathbf{R}} \leftarrow (T_N \mathrm{Div}(X))^*_{\mathbf{R}} = \bigoplus_{\rho \in \Delta(1)} \mathbf{R} e(\rho) \leftarrow A^{r-1}(X)_{\mathbf{R}} \otimes_{\mathbf{R}} (A^r(X)_{\mathbf{R}})^* \leftarrow 0$$

$$0 \rightarrow M_{\mathbf{R}} \rightarrow T_N \mathrm{Div}(X)_{\mathbf{R}} = \bigoplus_{\rho \in \Delta(1)} \mathbf{R} V(\rho) \rightarrow A^1(X)_{\mathbf{R}} \rightarrow 0.$$

We see that $G_{\geq 0} = (A^1(X)_{\mathbf{R}})_{\geq 0}$ is equal to the cone spanned by the linear equivalence classes of $T_N$-stable effective divisors, and $\mathrm{cpl}(\Delta) \subset (A^1(X)_{\mathbf{R}})_{\geq 0}$ becomes the cone spanned by the linear equivalence classes of numerically effective divisors. Also, the dual cone $\mathrm{cpl}(\Delta)^\vee \subset A^{r-1}(X)_{\mathbf{R}}$ becomes the cone of effective one-cycles modulo linear equivalence, that is, the Mori cone $NE(X) := \sum_{\tau \in \Delta(r-1)} \mathbf{R}_{\geq 0} v(\tau)$ (cf.[10] and [13]).

**Remark.** Batyrev [1, Theorem 2.15] expressed the Mori cone in a different way, when $\Delta$ is complete and nonsingular. He used a new concept of *primitive collections.* If $\Delta$ is complete and nonsingular, then $\mathrm{cpl}(\Delta)^\vee = \sum_{\tau \in \Delta(r-1)} \mathbf{R}_{\geq 0} l_\tau$. If $\mathbf{R}_{\geq 0} l_\tau$ is an *extremal ray* (i.e., a one-dimensional face) of $\mathrm{cpl}(\Delta)^\vee$, then $\tau$ gives rise to a primitive collection. We see that not all of the primitive collections come from the extremal rays of $\mathrm{cpl}(\Delta)^\vee$ in this way. The total cone $\mathrm{cpl}(\Delta)^\vee$ itself, however, is equal to the cone $\mathrm{Pr}(X)$ generated by the primitive relations for primitive collections, and coincides with the Mori cone $NE(X)$.

From now on, let $\pi$ be an $r$-dimensional strongly convex rational polyhedral cone in $N_{\mathbf{R}}$ such that any proper face of it is simplicial.

Let $\Delta_0$ be the fan consisting of all the faces of $\pi$. Then the corresponding toric variety $X_0$ has one bad isolated singularity at one point $\mathrm{orb}(\pi)$ and $X_0 \setminus \mathrm{orb}(\pi)$ has at most quotient singularities.

Let $\Xi := \{n(\rho) \mid \rho \in \Delta_0(1)\}$ and consider the $\mathbf{Q}$-linear Gale transform of $(N, \Xi)$. Since $\pi$ is strongly convex, $G_{\geq 0}$ becomes the whole space $G$. For any GKZ-cone $\mathrm{cpl}(\Delta)$ in the GKZ-decomposition, the corresponding fan $\Delta$ is a quasi-projective simplicial subdivision of $\Delta_0$ with $\Delta(1) = \Delta_0(1)$. The corresponding toric variety $X = T_N\mathrm{emb}(\Delta)$ has at most quotient singularities. We also see that for any pair of GKZ-cones, the corresponding fans can be obtained from each other by a finite succession of flops.

**Definition.** A fan $\Delta$ is called a *small* simplicial subdivision of $\pi$ if it satisfies the following:

(i) $\Delta$ is simplicial.

(ii) $|\Delta| = \pi$ .

(iii) Any proper face of $\pi$ is contained in $\Delta$.

(iv) For any cone $\sigma \in \Delta$, $\dim \sigma$ is greater than $r/2$ whenever $\sigma$ meets the interior $\mathrm{int}(\pi)$ of $\pi$.

Such a small simplicial subdivision may not exist and may not be unique. In fact, we have some examples of $\pi$ which have no small simplicial subdivisions.

**Proposition 4.2** *Let $\pi$ be an* even-*dimensional strongly convex cone and $\Xi = \{n(\rho) \mid \rho \prec \pi, \dim \rho = 1\}$ . Suppose that there exists a small simplicial subdivision $\Delta$ of $\pi$. If* $\mathrm{cpl}(\Delta')$ *is a* GKZ-*cone such that $F := \mathrm{cpl}(\Delta) \cap \mathrm{cpl}(\Delta')$ is a facet of both* $\mathrm{cpl}(\Delta)$ *and* $\mathrm{cpl}(\Delta')$, *then $\Delta'$ cannot be small.*

Proof. Let $R := \mathbf{R}_{\geq 0} l_\tau$ be the extremal ray of $\mathrm{cpl}(\Delta)^\vee$ corresponding to the facet $F$. Then there exist a minimal linearly dependent set $\{n(\rho_1), ..., n(\rho_{p+q})\}$ and a relation

$$\sum_{i=1}^{p} a_i n(\rho_i) = \sum_{j=1}^{q} a_{p+j} n(\rho_{p+j}) \qquad \text{for some} \qquad a_i, a_{p+j} > 0,$$

where $\rho_i, \rho_{p+j} \in \Delta(1)$ are one-dimensional faces of $\sigma_1, \sigma_2 \in \Delta(r)$ satisfying $\sigma_1 \cap \sigma_2 = \tau$, while $p$ and $q$ are intergers such that $p + q \leq r + 1$ and $p, q \geq 2$.

Without loss of generality, we may assume that $\rho_1 + \tau = \sigma_1$ and $\rho_2 + \tau = \sigma_2$. Then, by the construction of the flop $\Delta'$ of $\Delta$, we see that

$$\rho_1 + \cdots + \rho_p \notin \Delta, \qquad \rho_{p+1} + \cdots + \rho_{p+q} \in \Delta,$$

$$\rho_1 + \cdots + \rho_p \in \Delta', \qquad \rho_{p+1} + \cdots + \rho_{p+q} \notin \Delta'.$$

Thus, $\rho_1 + \cdots + \rho_p$ and $\rho_{p+1} + \cdots + \rho_{p+q}$ are not proper faces of $\pi$, and these cones intersect the interior of $\pi$. Since $\Delta$ is small, $q > r/2$, hence $p + r/2 < p + q \leq r + 1$. We have $p \leq r/2$, which implies that $\Delta'$ cannot be small. q.e.d.

If we cut this cone $\pi$ by a hyperplane not passing through the orign, then the intersection becomes an $(r-1)$-dimensional simplicial convex polytope. Thus, by considering combinatorial types of simplicial convex polytopes, we have some information in lower dimensional cases.

**Proposition 4.3** (1) *If $r = 3$, then every* non-divisorial *simplicial subdivision of $\pi$, that is, simplicial subdivision without adding new one-dimensional cones, becomes small.*

(2) *If $\pi$ with $r = 4$ has a small simplicial subdivision, then it is unique.*

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