Rational points of bounded height on toric varieties

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Let $\Sigma$ be a complete $d$-dimensional regular fan in $N_\mathbb{R}$ defining a smooth compact $d$-dimensional toric variety over a number field $F$, $\Sigma^{(i)}$ the set of all $i$-dimensional cones in $\Sigma$. Let the elements of $\Sigma^{(i)}$ have integral generators $e_1, \ldots, e_n$. We define some rational function on $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$ associated with the combinatorial structure of the fan $\Sigma$

$$f_\Sigma(s) = \sum_{\sigma \in \Sigma^{(d)}} f_\sigma(s),$$

where $f_\sigma(s) = (s_{j_1} \cdots s_{j_d})^{-1}$, if $e_{j_1}, \ldots, e_{j_d}$ are generators of the cone $\sigma$.

For archimedean completions of $F$, we put

$$f_{\Sigma, \mathcal{P}}(s) = \frac{1}{\sqrt{\delta_{\mathcal{P}}}} P_\Sigma(||\mathcal{P}||^{-s_1}, \ldots, ||\mathcal{P}||^{-s_n}).$$

Denote by $P_\Sigma(t_1, \ldots, t_n)$ the rational function defined as the Gilbert-Poincare serie of the $\mathbb{Z}_{\geq 0}$-graded Stanley-Reisner ring $R(\Sigma)$ corresponding to $\Sigma$.

For any prime $\mathcal{P}$ ideal of $F$, we denote by $||\mathcal{P}||$ the cardinality of the residue field of $\mathcal{P}$, by $\delta_{\mathcal{P}}$ absolute different of the nonarchimedean local field $F_\mathcal{P}$, and put

$$f_{\Sigma, \mathcal{P}}(s) = \frac{1}{\sqrt{\delta_{\mathcal{P}}}} P_\Sigma(||\mathcal{P}||^{-s_1}, \ldots, ||\mathcal{P}||^{-s_n}).$$

Denote by $K_\Sigma(s)$ the following product

$$f_{\Sigma, \mathcal{P}}(s) f_{\Sigma, \mathcal{P}}^{r_2}(s) \prod_{\mathcal{P}} f_{\Sigma, \mathcal{P}}(s),$$

where $r_1$ is the number of real embeddings of $F$, $r_2$ is the number of complex embeddings of $F$.

Let $r_F$ the residue of the Dedekind zeta function $\zeta_F(z)$ at $z = 1$;

$$r_F = \frac{2^{r_1}(2\pi)^{r_2} h R}{\sqrt{|D_F| w}}.$$
Theorem. Let $D(s) = s_1D_1 + \cdots + s_nD_n$ ($s_i > 0$) be an effective divisor on toric variety $V_\Sigma$, $H_\Sigma(s, x)$ corresponding height function on $F$-rational points $x \in T(F) \cong F^*$. Let $T^1(A_F) = (I^1(F))^d$ where $I^1(F)$ is the group of idele with norm 1 of the field $F$, $d\mu$ the standard Haar measure on $T^1(A_F)$. Then

$$\int_{T^1(A_F)} H_\Sigma(s, x)^{-1} d\mu = (2\pi r_F)^{-d} \int_{M_R} K(s + im) dm.$$ 

This theorem can be applied to the problem of the asymptotic distribution of rational points of bounded height on toric varieties (cf. [1]).

Example. Let $\Sigma$ defines $\mathbb{P}^d$. Then

$$f_\Sigma(s) = \frac{s_1 + \cdots + s_{d+1}}{s_1 \cdots s_{d+1}}, \quad P_\Sigma(t_1, \ldots, t_{d+1}) = \frac{1 - t_1 \cdots t_{d+1}}{(1 - t_1) \cdots (1 - t_{n+1})},$$

$$K_\Sigma(s) = \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|D_F|}}\right)^d \left(\frac{s_1 + \cdots + s_{d+1}}{s_1 \cdots s_{d+1}}\right)^{r_1+r_2} \frac{\zeta_F(s_1) \cdots \zeta_F(s_{d+1})}{\zeta_F(s_1 + \cdots + s_{d+1})}.$$ 

Applying the residue formula to the $d$-dimensional integral, we get

$$\int_{T^1(A_F)} H_\Sigma(s, x)^{-1} = \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|D_F|}}\right)^d \left(\frac{s_1 + \cdots + s_{d+1}}{s_1 + \cdots + s_{d+1} - d}\right)^{r_1+r_2} \frac{\zeta_F(s_1 + \cdots s_{d+1} - d)}{\zeta_F(s_1 + \cdots + s_{d+1})}.$$ 

The residue of $\int_{T^1(A_F)} H_\Sigma(s, x)^{-1} d\mu$ at $s = (1, \ldots, 1)$ is

$$\left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|D_F|}}\right)^d (d+1)^{r_1+r_2-1} \left(\frac{2^{r_1}(2\pi)^{r_2} h R}{\sqrt{|D_F|} w}\right) \zeta_F^{-1}(d+1).$$

This number gives the coefficient in the asymptotic formula of Schanuel for the number of rational points in projective spaces [2].

References
