A Construction of Solutions of the Ernst Equations

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In this article, we give a prescription for constructing formal solutions of the Ernst equations which are derived from the stationary axially symmetric Einstein-Maxwell equations. This is based on the treatment of [1].

0. Preliminaries

Let $ds^2 = g_{ij}dx^idx^j$ be a metric and $A = A_idx^i$ a electro-magnetic potential on $\mathbb{R}^{1+3}$. Then the Einstein-Maxwell field equations are given by

$$R_{ij} = 8\pi T_{ij}, \quad \nabla_k F^{ik} = 0 \quad (i,j,k = 0,1,2,3),$$

where $R_{ij}$ is Ricci curvature and

$$F_{ij} = \partial_i A_j - \partial_j A_i,$$

$$T_{ij} = \frac{1}{8\pi}(F_{ik}F_{j}^{k} - \frac{1}{4}g_{ij}F_{k1}F^{kl}).$$

Since we are concerned with stationary axisymmetric solutions, we choose coordinates $(x^0,x^1,x^2,x^3) = (\tau,\phi,z,\rho)$ on $\mathbb{R}^{1+3}$ where $\tau$ is time and $(\phi,z,\rho)$ are the cylindrical coordinates on $\mathbb{R}^3$.

We assume that the metric $ds^2$ takes the form

$$ds^2 = \sum_{i=0}^{1}h_{ij}dx^idx^j - \lambda^2((dx^1)^2 + (dx^2)^2) \quad (\lambda > 0)$$

and $h = (h_{ij}), \lambda$ and $A_i$ depend only on $z$ and $\rho$. Moreover, we assume that $h_{00} \neq 0, \det h = -\rho^2$ and $A_2 = A_3 = 0$, which are physically reasonable.

Then the stationary axisymmetric Einstein-Maxwell field equations are given, in matrix form, as follows:

$$d(\rho^{-1}h \epsilon * dA) = 0 \quad (1)$$

$$d \{ \rho^{-1}h \epsilon * dh - 2(\rho^{-1}h \epsilon * dA)^t A - 2A^t(\rho^{-1}h \epsilon * dA) \} = 0, \quad (2)$$

$$\frac{\partial z \lambda}{\lambda} = \frac{\rho}{4} \text{tr}(h^{-1} \partial_{\rho}hh^{-1} \partial_z h) - 2\rho \partial_{\rho}^t Ah^{-1} \partial_z A, \quad (3.a)$$
\[
\frac{\partial_{\rho}\lambda}{\lambda} = -\frac{1}{2\rho} + \frac{\rho}{8} \mathrm{tr}\{(h^{-1}\partial_{\rho}h)^2 - (h^{-1}\partial_{z}h)^2\} - \rho(\partial_{\rho}^{t}Ah^{-1}\text{\^{a}}_{\rho}A - \partial_{z}^{t}Ah^{-1}\partial_{z}A),
\]

where \( A = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}, \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \(* = \)Hodge operator for the metric \( dz^2 + d\rho^2 \). Since \( h_{00} \neq 0 \) and \( \det h = -\rho^2 \), we can parametrize \( h \) as

\[
h = \begin{pmatrix} f & f\omega \\ f\omega & f\omega^2 - \rho^2/f \end{pmatrix}.
\]

It is known that (3.a) and (3.b) are integrable, so we shall be concerned with (1) and (2) in what follows.

Next we introduce the so-called Ernst potential. Note that every closed form is exact since we consider it locally.

From (1), there exists a \( 2 \times 1 \)-matrix valued function \( B = \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} \) such that

\[
*dB = \rho^{-1}h\epsilon dA.
\]

Substituting (4) into (2),

\[
d(\rho^{-1}h\epsilon *dh + 2dB^tA + 2Ad^tB) = 0.
\]

The \((1,1)\)-th entry reads

\[
d(\rho^{-1}f^2 *d\omega + 2A_0dB_0 - 2B_0dA_0) = 0.
\]

Therefore, there exists \( \psi \) such that

\[
\rho^{-1}f^2d\omega = *d\psi + 2(A_0 * dB_0 - B_0 * dA_0) = 0.
\]

Using \( f, A_0, b_0 \) and \( \psi \), we put

\[
u = A_0 + iB_0, \quad u = f - |v|^2 + i\psi.
\]

The pair \((u, v)\) is called the Ernst potential. Then the following fact is well known.

**PROPOSITION 1.** \((h, A)\) is a solution of (1) and (2) if and only if \((u, v)\) is a solution of the following equations:

\[
f(d * du + \rho^{-1}d\rho \wedge *du) = (du + 2\overline{v}dv) \wedge *du, \quad (5)
\]

\[
f(d * dv + \rho^{-1}d\rho \wedge *dv) = (du + 2\overline{v}dv) \wedge *dv. \quad (6)
\]

But we change the definition of \( u \) into the following one:

\[
u = A_0 + iB_0, \quad u = f + |v|^2 + i\psi,
\]

so that our Ernst equations become

\[
f(d * du + \rho^{-1}d\rho \wedge *du) = (du - 2\overline{v}dv) \wedge *du, \quad (5')
\]

\[
f(d * dv + \rho^{-1}d\rho \wedge *dv) = (du - 2\overline{v}dv) \wedge *dv. \quad (6')
\]
1. Ernst Potential

Next we rewrite the equations (5') and (6') in terms of matrix.

Let

$$G = \{ g \in SL_3(\mathbb{C}) ; g^*Jg = J \} \cong SU(1,2),$$

where $J = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix}$, and $K$ its maximal compact subgroup, i.e.,

$$K = \{ g \in G ; g^*g = 1 \}.$$

We define the Cartan involution $\Theta$ by $\Theta(g) = (g^*)^{-1}$ for $g \in G$.

Let $G = KAN$ be an Iwasawa decomposition with

$$A = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/a \end{pmatrix} ; a > 0 \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \psi + i|v|^2/2 & 1 & i\bar{v} \\ \psi - i|v|^2/2 & i\bar{v} & 1 \end{pmatrix} ; \psi \in \mathbb{R}, v \in \mathbb{C} \right\}.$$

Now we parametrize an element $P$ in $AN$ as follows [2]:

$$P = \begin{pmatrix} f^{1/2} & 0 & 0 \\ \sqrt{2i}v & 1 & 0 \\ (\psi + i|v|^2)/f^{1/2} & \sqrt{2i}v/f^{1/2} & 1/f^{1/2} \end{pmatrix}.$$

with $f$, $v$ and $\psi$ as above.

It is well known that $(u,v)$ is a solution of (5'), (6') if and only if $P$ is a solution of the following equation:

$$d(\rho * dMM^{-1}) = 0 \quad \text{with} \quad M = \Theta(P)^{-1}P. \quad (7)$$

Let $g$ the Lie algebra of $G$, i.e.,

$$g = \{ X \in sl_3(\mathbb{C}) ; X^*J + JX = 0 \},$$

where $J$ is as above. We denote by $\theta$ the involution of $g$ induced from the involution $\Theta$ of $G$.

DEFINITION. Let $A$ and $I$ be $g$-valued 1-forms defined by

$$A = \frac{1}{2}(dPP^{-1} + \theta(dPP^{-1})),$$

$$I = \frac{1}{2}(dPP^{-1} - \theta(dPP^{-1})).$$

We define a $g$-valued 1-form $\Omega$ with a spectral parameter to be

$$\Omega = \Omega(s) = A + \frac{1 - 2sz - 2z\rho^*}{\Lambda}I,$$

with $\Lambda = \{(1 - 2sz)^2 + 4s^2\rho^2\}^{1/2}$.

Note that $\Omega(0) = A + I = dPP^{-1}$. 
PROPOSITION 2. $\Omega$ satisfies the integrability condition, i.e.,
$$d\Omega - \Omega \wedge \Omega = 0$$
if and only if $P$ is a solution of (7).

For any solution $P$ of the equation (7), by Proposition 2, there exists $P = P(s; z, \rho) \in SL(3, \mathbb{C}[[z, \rho, s]])$ which satisfies
$$dP = \Omega P, \quad P|_{s=0} = P$$
where $\mathbb{C}[[z, \rho, s]]$ is a ring of formal power series in $z, \rho, s$ and $SL(3, \mathbb{C}[[z, \rho, s]])$ is a group consisting of all matrices of determinant 1 whose entries are the elements of $\mathbb{C}[[z, \rho, s]]$.

2. A Prescription for Constructing Solutions

Before giving a prescription for constructing solutions of the Ernst equations, we introduce a formal loop group and its subgroups, following [5].

Let $G^{(\infty)}$ be an infinite dimensional group
$$\{g(s) \in SL(3, \mathbb{C}[[s^{-1}]]); g(s)^*Jg(s) = J\},$$
where $\mathbb{C}[[s^{-1}]]$ is a ring of formal power series in $s^{-1}$ and $g(s)^* = \overline{g(\overline{s})}$.

Next we introduce a formal loop group $G_R$. Let $R$ be a ring of formal power series $\mathbb{C}[[z, \rho]]$ and $I$ an ideal of $R$ generated by $\rho$, i.e., $I = (\rho)$. We put
$$R_n = \begin{cases} I^n & \text{for } n > 0 \\ R & \text{for } n \leq 0. \end{cases}$$
Then we define
$$G_R = \{u = \sum_{n \in \mathbb{Z}} u_n t^n; u_n \in gl(3, R_n), u_0 \text{ is invertible}\},$$
and its subgroups
$$N_R = \{u = \sum_{n \in \mathbb{Z}} u_n t^n \in G_R; u_n = 0 (n > 0), u_0 = 1\},$$
$$P_R = \{u = \sum_{n \in \mathbb{Z}} u_n t^n \in G_R; u_n = 0 (n < 0)\}.$$

REMARK. If we define
$$G_R^{(0)} = \{u = \sum_{n \in \mathbb{Z}} u_n t^n; u_n \in gl(3, R_{-n}), u_0 \text{ is invertible}\},$$
then $G_R^{(0)}$ also forms a group. And for any $g(s) \in G^{(\infty)}$,
$$g((\frac{\rho}{t} + 2z - \rho t)^{-1}) \in G_R \cap G_R^{(0)}.$$

Our main theorem is:
THEOREM. For any $g(s) \in G^{(\infty)}$, there exists uniquely an element $k(t) \in G_{R}$ which satisfies the following conditions:

(i) $\Theta(k(-\frac{1}{t})) = k(t), \det k(t) = 1$;
(ii) $k(t)g((\frac{\rho}{t} + 2z - \rho t)^{-1})^{-1}$ is an element of $P_{R}$;

Putting $p(t) = k(t)g((\frac{\rho}{t} + 2z - \rho t)^{-1})^{-1} = \sum_{n \geq 0} p_{n} t^{n}$,

(iii) $p_{0}$ is an element of $AN$ and is a solution of the Ernst equation (7).

For the proof we reduce the problem to Birkhoff decomposition (3.17) of formal loop groups established in [5]:

LEMMA. Any element $u$ of $G_{R}$ can be uniquely decomposed as

$u = w^{-1}v, \ w \in N_{R}, v \in P_{R}$. 

For the detail of the proof of the theorem, we refer to [3].

3. Examples of Solutions

In this section we shall see how the prescription given in the previous section works, giving some simple examples.

Note that $SL(2, \mathbb{R})$ can be embedded in $G$ by the mapping

$$
\begin{pmatrix}
 a & b \\
 c & d
\end{pmatrix} \mapsto
\begin{pmatrix}
 a & 1 \\
 c & d
\end{pmatrix}.
$$

We use this embedding whenever we treat a field without electro-magnetic potentials.

Example 1 For $g(s) = \begin{pmatrix} 1 & 0 \\ -s^{-1} & 1 \end{pmatrix}$ with $s^{-1}$ replaced by

$s^{-1} = \frac{\rho}{t} + 2z - \rho t$,

the element $k(t) \in G_{R}$ in the theorem is determined in the following way: By the condition (i) of the theorem, $k(t)$ is written as

$$
\begin{pmatrix}
 a(-\frac{1}{t}) & b(t) \\
 -b(-\frac{1}{t}) & a(t)
\end{pmatrix},
$$

so that

$$
p(t) = \begin{pmatrix}
 a(-\frac{1}{t}) & b(t) \\
 -b(-\frac{1}{t}) & a(t)
\end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\rho}{t} + 2z - \rho t & 1 \end{pmatrix} \in P_{R}. \quad (8)
$$

Then the $(1,2)$-th entry of the right hand side of (8) can be expanded as

$$
b(t) = b_{1}t + b_{2}t^{2} + \cdots,
$$
since \( p_0 \) is lower triangular.

In a similar way the (2.2)-th entry reads
\[ a(t) = a_0 + a_1 t + a_2 t^2 + \cdots. \]

Since the (1,1)-th entry
\[ (a_0 - \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots) + (b_1 t + b_2 t^2 + \cdots) \left( \frac{\rho}{t} + 2z - \rho t \right) \]
contains no negative-power-terms in \( t \), it follows that \( a(t) = a_0 \).

By the same reason for the (2,1)-th entry, it follows that
\[ b(t) = b_1 t, \quad \text{and} \quad b_1 + \rho a_0 = 0. \]

Since \( \det k(t) = 1 \), it follows that
\[ a_0 = \frac{1}{\sqrt{1 - \rho^2}}. \]

Therefore
\[ p_0 = \frac{1}{\sqrt{1 - \rho^2}} \begin{pmatrix} 1 - \rho^2 & 0 \\ 2z & 1 \end{pmatrix}, \]
and
\[ M = \Theta(p_0^{-1})p_0 = \frac{1}{1 - \rho^2} \begin{pmatrix} (1 - \rho^2)^2 + 4z^2 & 2z \\ 2z & 1 \end{pmatrix}. \]

This is the first example given in [4].

Next we give another example which has a non-trivial electro-magnetic potential.

**Example 2** For \( g(s) = \begin{pmatrix} 1 \\ cs^{-1} \\ i|c|^2 s^{-2}/2 \\ i\overline{c}s^{-1} \end{pmatrix} \) (where \( c \) is an arbitrary complex number), \( k(t) \) is given by
\[ k(t) = \begin{pmatrix} a \\ -2c\rho t^{-1}/(2 - |c|^2 \rho^2) \\ i|c|^2 \rho^2 at^{-2}/2 \\ -i\overline{c}\rho at^{-1} \end{pmatrix}, \]
and \( M = \Theta(p_0^{-1})p_0 \) is given by
\[ M = \begin{pmatrix} a^{-2} + 4|c|^2 z^2 + 4a^2 |c|^4 z^4 & 2\overline{c}z + 4a^2 \overline{c}|c|^2 z^3 & -2ia^2 |c|^2 z^2 \\ 2cz + 4a^2 c|c|^2 z^3 & 1 + 4a^2 |c|^2 z^2 & -2ia^2 cz \\ 2ia^2 |c|^2 z^2 & 2ia^2 \overline{c}z & a^2 \end{pmatrix}, \]
where
\[ a = \frac{2}{2 - |c|^2 \rho^2}. \]
REFERENCES


