(Restricted) Quantized Enveloping Algebras
of Simple Lie superalgebras
and Universal R-Matrices

Hiroyuki Yamane
(Osaka University)

In this note, we define a (Jimbo type) quantized enveloping superalgebras
$U_q(G)$ of complex simple Lie superalgebras $G$ of types $A$, $B$, $C$, $D$ (all types) and
types $F_4$ and $G_3$
(distinguished types). We can get a defining relations of $U_q(G)$, which are consist
of q-Serre relations and additional relations. They were unknown even if $q = l$.
Moreover we define a restricted quantum groups $u_{\zeta}(G)$ at a root of unity $\zeta$.
Finally, we consider a Hopf algebraization of the Hopf superalgebra $u_{\zeta}(G)$ , and
construct the universal R-matrix of $u_{\zeta}(G)$. Our construction is due to Drinfeld's
quantum double construction. By using quantum double construction, we can also show a Poincaré-Birkhoff-Witt type theorem for $U_q(G)$ and $u_{\zeta}(G)$.

In [Y1-2], we introduced the (Drinfeld type) quantized enveloping superalgebras $U_h(G)$, showed $U_h(G)$ is an h-adic topologically free $\mathbb{C}[h]$-Hopf
algebra, and gave an explicit formula of universal R-matrix of $U_h(G)$. The
arguments used in this note are the essentially same arguments as we used in [Y2].

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§1. Quantum double construction.

Let $K$ be a field. Suppose char $(K) = 0$. Let $(A,\Delta, S, \epsilon)$ is $K$-Hopf algebras
with coproduct $\Delta : A \to A \otimes A$, antipode, $S : A \to A$ and counit $\epsilon : A \to K$.
Moreover we assume that there is a symmetric Hopf-pairing $< , > : A \otimes A \to K$
, namely $< , >$ is a symmetric $K$-bilinear form such that

1. $<\Delta(x), y \otimes z > = <x, y z >$,
2. $<S(x), y > = <x , S(y) >$,
3. $<1, x > = \epsilon(x)$
where \( x, y, z \in A \).

We call a Hopf-algebra \( A^{\text{op}} = (A, \Delta^{\text{op}}, S, \epsilon) \) the opposite Hopf-algebra of \( A \) where \( \Delta^{\text{op}} = \tau \circ \Delta \) and \( \tau(x \otimes y) = y \otimes x \).

Proposition 1.1. (Quantum double) There is a unique \( K \)-Hopf algebra \( (D = D(A), \Delta_D, S_D, \epsilon_D) \) satisfying:

1. As \( K \)-vector spaces, \( D \cong A \otimes A \).
2. The \( K \)-linear maps \( A \rightarrow A \otimes A \) (\( x \rightarrow x \otimes 1 \)) and \( A^{\text{op}} \rightarrow A \otimes A \) (\( x \rightarrow 1 \otimes x \)) are homomorphisms of Hopf-algebras.
3. The product of \( D \) is defined as follows; if \( x, y \in A \) and \( \Delta^{(2)}(x) = \sum i x_i^{(1)} \otimes x_i^{(2)} \otimes x_i^{(3)} \) and \( \Delta^{(2)}(y) = \sum j y_j^{(1)} \otimes y_j^{(2)} \otimes y_j^{(3)} \), then

\[
(v \otimes x) \cdot (y \otimes w) = \sum_{ij} <x_i^{(1)}, y_j^{(3)}> <x_i^{(3)}, S(y_j^{(1)})>(v y_j^{(2)} \otimes x_i^{(2)}w).
\]

Proposition 1.2. (Universal \( R \)-matrix of \( D(A) \)) Assume that \( \dim A < \infty \) and \( <, > \) is non-degenerate. Let \( \{ e_i \} \) and \( \{ e^i \} \) are two bases of \( A \) such that \( <e_i, e^j> = \delta_{ij} \). Then \( R = \sum_i (e_i \otimes 1) \otimes (1 \otimes e^i) \in D \otimes D \) satisfies:

0. \( R^{-1} = (1 \otimes S^{-1})(R) \).
1. \( R \Delta_D(a) R^{-1} = \Delta_D^{\text{op}}(a) \quad (a \in D) \).
2. \( (1 \otimes \Delta_D)(R) = R_{13} R_{12} \), \( (\Delta_D \otimes 1)(R) = R_{23} R_{13} \).

Remark. From (1) and (2), we can easily see that \( R \) satisfies the \( \text{Yang-Baxter equation} \):

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
\]

Therefore \( R \) is called the universal \( R \)-matrix of \( D \).

§2. Quantized enveloping (super)algebras.

Here we give an abstract definition of \( \text{Quantized enveloping (super)algebras} \) by using the \( \text{Quantum double construction} \).
Let $E$ be an $N$-dimensional $K$-vector space. Assume that there is a non-degenerate bi-linear form $(,): E \times E \to K$ with a basis $\{e_i \mid 1 \leq i \leq N \}$ such that $(e_i, e_j) = 0$ if $i \neq j$, $(e_i, e_i) \in \mathbb{Z} - \{0\}$. Let $\Pi = \{\alpha_i \in E \mid 1 \leq i \leq n\}$ be the set of linearly independent elements. Suppose that $(\alpha_i, \alpha_j) \in (1/4)\mathbb{Z}$. Let $\Pi \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ be the function. Write $p(i)$ for $p(\alpha_i)$. We call $p$ the \textit{parity function}. Put $P_+ = \mathbb{Z} e_1 \oplus \ldots \oplus \mathbb{Z} e_N$.

Let $q \in K^\times$. Let $U_{q\sigma}^+ b_+^\sigma$ be a $K$-algebra with generators $\{E_i (1 \leq i \leq n), K_\lambda (\lambda \in P_+) , \sigma\}$ and defining relations:

\[(U^1) \sigma^2 = 1, \sigma E_i \sigma = (-1)^{p(i)} E_i, \sigma K_\lambda \sigma = K_\lambda ,\]
\[(U^2) K_0 = 1, K_\lambda K_\mu = K_{\lambda + \mu} (\lambda, \mu \in P_+) ,\]
\[(U^3) K_\lambda E_i K^{-1}_\lambda = q^{(\alpha_i, \lambda)} E_i .\]

Moreover $U_{q\sigma}^+ b_+^\sigma$ has a $K$-Hopf algebra such that

\[(U^4) \Delta(\sigma) = \sigma \otimes \sigma , S(\sigma) = \sigma , \epsilon(\sigma) = 1 ,\]
\[(U^5) \Delta(K_\lambda) = K_\lambda \otimes K_\lambda , S(K_\lambda) = K^{-1}_\lambda , \epsilon(K_\lambda) = 1 ,\]
\[(U^6) \Delta(E_i) = E_i \otimes 1 + K_\alpha \sigma^{p(i)} \otimes E_i , S(E_i) = -K_\alpha^{-1} \sigma^{p(i)} E_i , \epsilon(E_i) = 0 .\]

Let $U_{q\sigma}^+ b_+ \ (\text{resp. } U_{q\sigma}^+ n_+, \ T)$ be an unital subalgebra generated by the elements $\{E_i (1 \leq i \leq n), K_\lambda (\lambda \in P_+)\}$ (resp. $\{E_i (1 \leq i \leq n)\}, \{K_\lambda (\lambda \in P_+)\}$).

Let $\Pi$ be the set of finite sequences of $(1, \ldots, n)$. Put $E_I = E_{i_1} E_{i_2} \ldots E_{i_p}$ for $I = (i_1, i_2, \ldots, i_p) \in \Pi$ and put $E_{\phi} = 1$.

Lemma 2.1. As a $K$-vector space, $U_{q\sigma}^+ b_+^\sigma$ has a basis elements such that

\[E_I K_\lambda \sigma^c (I \in \Pi , \lambda \in P_+, c \in \{0, 1\}).\] In particular, we have

\[U_{q\sigma}^+ b_+^\sigma \simeq U_{q\sigma}^+ n_+ \otimes T \otimes K<\sigma> \text{ as } K\text{-vector spaces.}\]
Proposition 2.2. There is a symmetric Hopf-pairing

\[ \langle , \rangle : U_{q}^{+} \sigma \otimes U_{q}^{+} \sigma \rightarrow K \]

such that

(P.1) \[ \langle \sigma, E_{I}K_{\lambda} \sigma^{c} \rangle = \delta_{I \phi} (-1)^{c}, \]

(P.2) \[ \langle K_{\mu}, E_{I}K_{\lambda} \sigma^{c} \rangle = \delta_{I \phi} q^{(\mu, \lambda)}, \]

(P.3) \[ \langle E_{i}, E_{I}K_{\lambda} \sigma^{c} \rangle = \delta_{I (i)}. \]

We put \( I_{b+}^{O} = \text{Ker} \langle , \rangle \) and put \( U_{q}^{+} \sigma = U_{q}^{+} \sigma / I_{b+}^{O} \).

Let \( D (u_{q}^{+} \sigma) \) be the quantum double of \( u_{q}^{+} \sigma \) with respect to \( \langle , \rangle \). For \( X \in u_{q}^{+} \sigma \), we write \( X, X^{op} \) for \( X \otimes 1, 1 \otimes X \in D (u_{q}^{+} \sigma) \) respectively.

Lemma 2.3. In \( D (u_{q}^{+} \sigma) \), the following equations hold:

\( D \sim 1 \) \[ \sigma \cdot \sigma^{op} = \sigma^{op} \cdot \sigma, \quad \sigma K_{\lambda} \sigma = K_{\lambda} \sigma, \quad \sigma E_{i} \sigma = (-1)^{p(i)}E_{i}^{op}, \]

\[ \sigma^{op}K_{\lambda} \sigma^{op} = K_{\lambda}, \quad \sigma^{op}E_{i} \sigma^{op} = (-1)^{p(i)}E_{i}, \]

\( D \sim 2 \) \[ K_{\lambda} \cdot K_{\mu}^{op} = K_{\mu}^{op} \cdot K_{\lambda}, \]

\[ K_{\lambda} E_{i}^{op} K_{\lambda}^{-1} = q^{-(\alpha_{i}, \lambda)}E_{i}^{op}, \quad K_{\lambda}^{op} E_{i} K_{\lambda}^{op-1} = q^{-(\alpha_{i}, \lambda)}E_{i}, \]

\( D \sim 3 \) \[ E_{i} E_{j}^{op} - E_{j}^{op} E_{i} = \delta_{ij} (K_{\alpha} E_{i}^{op} - E_{i}^{op} - K_{\alpha} E_{i}). \]

Let \( L \) be \( \sigma^{op} \) ideal of \( K \)-algebra \( D (u_{q}^{+} \sigma) \) generated by \( \sigma \cdot \sigma^{op} - \sigma^{op} \cdot \sigma \) and \( K_{\lambda} \cdot K_{\mu}^{op} - K_{\mu}^{op} \cdot K_{\lambda} \) (\( \lambda \in P_{+} \)). It is clear that \( L \) is a Hopf-ideal. Put

\[ u_{q}^{\sigma} = u_{q}^{\sigma}(E, \Pi, p) = D (u_{q}^{+} \sigma) / L. \]

Put \( u_{q}^{n+} = U_{q}^{+} / (I_{b+}^{\sigma} \cap U_{q}^{+}), \quad t = T / (I_{b+}^{\sigma} \cap T). \)
Lemma 2.4.  (1) As $K$-vector spaces, 
\[ u_q^\sigma \simeq u_q n_+ \otimes t \otimes K^{<\sigma> \otimes u_q n_+} (Xt \sigma^c Y^{op} \leftarrow X \otimes t \otimes \sigma^c \otimes Y), \]
\[ (c = c, 1) \]

(2) For $1 \leq i \leq N$, let $\gamma_i = \min \{\gamma \mid K_{\gamma_i} \neq 1\} \in \mathbb{Z}_+ \cup (+ \infty)$. Then the elements $K_{\delta_1} \cdots K_{\delta_N} (0 \leq \delta_i < \gamma_i)$ form a $K$-basis of $t$.

(3) Let $u_q$ be an unital subalgebra of $u_q^\sigma$ generated by the elements 
\[ \{E_i, F_i = \sigma^{\alpha_i}(1 \leq i \leq n), K_{\lambda} \mid \lambda \in P_+\} \]
Then there is a Hopf-superalgebra structure on $u_q$ with coproduct $\hat{\Delta}$ defined by
\[ \hat{\Delta}(K_{\lambda}) = K_{\lambda} \otimes K_{\lambda}, \quad \hat{\Delta}(E_i) = E_i \otimes 1 + K_{\alpha_i} \otimes E_i, \quad \hat{\Delta}(F_i) = F_i \otimes K_{\alpha_i^{-1}} + 1 \otimes F_i. \]

Theorem 2.5. Assume that $q$ is an indeterminate and $K = C(q)$. Suppose that $(\alpha_i, \alpha_j) > 0$, $(\alpha_i, \alpha_j) \leq 0$ and $2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \in \mathbb{Z}$. Let $G$ be the Kac-Moody Lie algebra defined for $(\cdot, \cdot) : \mathbb{E} \times \mathbb{E} \rightarrow K$ and $\Pi$. Then $u_q$ is isomorphic to the Drinfeld-Jimbo quantized enveloping algebra $U_q(G)$.

Theorem 2.6. Let $G$ be the simple C-Lie algebra. Suppose that $\Pi$ is the set of the simple roots of $G$. Assume that $K = C$. Let $\zeta$ be an $m$-th root of unity such that $m \gg 1$. Then $u_\zeta$ is isomorphic to the Lusztig's quantum group at root of unity $u_\zeta(G)$.

Theorem 2.5 can be immediately proved by Proposition 2.4.1 in [T]. Theorem 2.6 also seems to be well-known. For example, see [R].

§3. Root Systems of Simple Lie Superalgebras.
Let $G$ be simple Lie superalgebras of types $A_{N-1}, B_N, C_N, D_N, F_4, G_3$. Let $(E, \Pi, p)$ be a triple related to a root system of $G$. From now on, we only
treat triples \((E, \Pi, p)\) following Dynkin diagrams.

In the following diagrams, the element under the i-th dot denotes the i-th simple root \(\alpha_i \in \Pi\). The i-th dot \(\times\) stands for \(\circ\) (resp. \(\odot\)) if \((\alpha_i, \alpha_i) \neq 0\) (resp. = 0). If i-th dot is \(\circ\), \(\odot\) or \(\bullet\), then we define \(p(\alpha_i) = 0, 1\) respectively. We also define a diagonal matrix \(D = (d_1, \ldots, d_n)\) such that \(A = D^{-1}(\alpha_i, \alpha_j)\) is a Cartan matrix of \(G\).

\[
\begin{array}{ccccccc}
1 & 2 & N-1 \\
\end{array}
\]

\(A_{N-1}\)
\[
\begin{array}{cccccccc}
\times & \cdots & \cdots & \times \\
\end{array}
\]

\((E_1, E_2, E_3, \ldots, E_N)\), \(\langle E_1, E_1 \rangle = \pm 1\), \(D = \text{diag}(1, \ldots, 1)\),

\[
\begin{array}{ccccccc}
1 & 2 & N-1 & N & 1 & 2 & N-1 & N \\
\end{array}
\]

\(B_N\)
\[
\begin{array}{cccccccc}
\times & \cdots & \cdots & \times & \rightarrow \circ \\
\end{array}
\]

\((E_1, E_2, E_3, \ldots, E_N)\), \(\langle E_1, E_1 \rangle = \pm 1\), \(D = \text{diag}(1, \ldots, 1, 1/2)\),

\[
\begin{array}{ccccccc}
1 & 2 & N-1 & N \\
\end{array}
\]

\(C_N\)
\[
\begin{array}{cccccccc}
\times & \cdots & \cdots & \times & \leftarrow \circ \\
\end{array}
\]

\((E_1, E_2, E_3, \ldots, E_N)\), \(\langle E_1, E_1 \rangle = \pm 1\), \(D = \text{diag}(1, \ldots, 1, 2)\),

\[
\begin{array}{ccccccc}
1 & 2 & N-2 & O & N-1 & \cdots & N-1 \\
\end{array}
\]

\(D_N\)
\[
\begin{array}{cccccccc}
\times & \cdots & \cdots & \times & \rightarrow \circ \\
\end{array}
\]

\((E_1, E_2, E_3, \ldots, E_N)\), \(\langle E_1, E_1 \rangle = \pm 1\), \(D = \text{diag}(1, \ldots, 1)\),

\[
\begin{array}{ccccccc}
1 & 4 & 3 & 2 \\
\end{array}
\]

\(F_4\)
\[
\begin{array}{ccccccc}
\circ & \rightarrow & \rightarrow & \rightarrow \\
\end{array}
\]

\((E_1, E_2, E_3, E_4)\), \(\langle E_1, E_1 \rangle = 6\), \(\langle E_2, E_2 \rangle = \langle E_3, E_3 \rangle = \langle E_4, E_4 \rangle = -2\), \(D = \text{diag}(2, 1, 1, 2)\),
\[(G_3) \quad \otimes - \circ \equiv \equiv \equiv \circ \quad , \]

\[\varepsilon_1 - \varepsilon_2 \quad (\varepsilon_2 - \varepsilon_3)/2 \quad \varepsilon_3\]

\[(\varepsilon_1, \varepsilon_1) = -2, \quad (\varepsilon_2, \varepsilon_2) = 2, \quad (\varepsilon_3, \varepsilon_3) = -6, \quad \mathbb{D} = \text{diag}(1, 3, 1) .\]

§4. Defining relations of \( u_q^{\sigma(\mathbb{E}.\Pi.p)} \) of Simple Lie Superalgebras \( \mathbb{G} \).

Here we give defining relations of \( u_q^{n_+} \) of \( u_q^{\sigma(\mathbb{E}.\Pi.p)} \) (see Lemma 2.4) when \( q \) is not a root of unity.

Put \( P_+ = Z\alpha_1 \oplus \ldots \oplus Z\alpha_N \). We extend \( p \) to \( p : P_+ \rightarrow Z/2Z \) additively.

For \( \delta = m_1\alpha_1 + \ldots + m_N\alpha_N \in P_+ \), let \( (u_q^{n_+})_\delta \) be a \( K \)-subspace of \( u_q^{n_+} \) spaned by elements \( E_{i_1}E_{i_2} \ldots E_{i_p} \) \((\# \{i_a = i\} = m_i)\). Then we have \( u_q^{n_+} = \oplus_\delta \in P_+ (u_q^{n_+})_\delta \).

For \( \delta, \nu \in P_+ \) and \( X_\delta \in (u_q^{n_+})_\delta, \quad X_\nu \in (u_q^{n_+})_\nu \), put

\[\text{ad}_{\iota}, \quad 1^{E_{\delta}}(X_\nu) = [X_\delta, X_\nu] = X_\delta X_\nu - (-1)^{p(\delta)p(\nu)} q^{-(\delta,\nu)} X_\nu X_\delta.\]

Theorem 4.1. Let \( (\mathbb{E}.\Pi.p) \) be a triple introduced in §3. Assume that \( q \) is not a root of unity. Let \( u_q^{n_+} \) be of \( u_q^{\sigma(\mathbb{E}.\Pi.p)} \) (see Lemma 2.4). Then, as \( K \)-algebra, \( u_q^{n_+} \) is defined with the generators \( E_i \) \((1 \leq i \leq n)\) and the relations:

\[(r1) \quad [E_i, E_j] = 0 \quad \text{if} \quad (\alpha_i, \alpha_j) = 0 , \]
\[(r2) \quad \text{ad}_{\iota} \cdot jE_i \cdot m_{ij} = 0 \quad \text{if} \quad (\alpha_i, \alpha_j) \neq 0 \quad \text{and} \quad m_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \in Z \]
\[(r3) \quad \text{ad}_{\iota} \cdot jE_N \cdot m_{N-1}N = 0 \quad \text{if} \quad \cross \Rightarrow \bullet \]
\[(r4) \quad E_i E_j E_k = 0 \quad \text{if} \quad \cross \quad \text{or} \quad \cross \Rightarrow \bullet \]
(r5) $[ [E_{N-2} , E_{N-1} ], E_N ] = [ [E_{N-2} , E_N ], E_{N-1} ]$

if

N-1

N

(r6) $[ [ [E_{N-2} , E_{N-1} ], [E_{N-2} , E_{N-1} ], E_N ] ], E_{N-1} ] = 0$

N-2 N-1 N

if

(r7) $[ [ [ [E_{N-3} , E_{N-2} ], E_{N-1} ], E_N ], E_{N-1}, E_{N-2}, E_{N-1} ] ] = 0$

N-3 N-2 N-1 N

if

§5. Root vectors of $u_q^{\sigma(B.\Pi.p)}$ of Simple Lie Superalgebras $G$.

Here we assume that there is $m >> 1$ satisfying $q^m \neq 1$ for $1 \leq m \leq m$.

Assume that $(E.\Pi.p)$ is the triple in §3. Let $\Phi$ be the set of roots of $G$ and $\Phi_+$ the set of positive roots with respect to $\Pi$. Let $\Phi_+^{\text{red}}$ be the set of positive roots defined by

$\Phi_+^{\text{red}} = \{ \beta \in \Phi_+ | \beta / 2 \notin \Phi_+ \}$. For $\beta = c_1 \alpha_1 + \ldots + c_N \alpha_N \in \mathbb{P}_+$, put

$\text{ht}(\beta) = c_1 + \ldots + c_N$, $g(\beta) = \min\{i | i \neq 0\}$ and $c_\beta = c_{g(\beta)}$.

Define a half integer $\text{ht}^\beta$ by $\text{ht}(\beta) = \text{ht}(\beta)/c_\beta$. For $\alpha, \beta \in \mathbb{P}_+$, we say that $\alpha < \beta$ if they satisfy one of the following $\epsilon \frac{1}{2} \mathbb{Z}$

(1) $g(\alpha) < g(\beta)$,

(2) $g(\alpha) = g(\beta)$ and $\text{ht}(\alpha) < \text{ht}(\beta)$,

(3) $\Pi$ is of type $D_N$, $p(e_i - e_N) = 0$ and $\alpha = e_i - e_N$, $\beta = 2e_i$ or
\[ \alpha = 2g_i, \beta = g_i + g_N \quad \text{or} \quad \alpha = g_i - g_N, \beta = g_i + g_N. \]

We define q-root vectors \( E_\beta \) \((\beta \in \Phi_{+}^{\text{red}})\) of \( u_q n_+ \) of \( u_q^{\sigma(E, \Pi, p)} \) as follows.

**Definition 5.1.** For \( \beta \in \Phi_{+}^{\text{red}} \), we define the element \( E_\beta \in u_q n_+ \) as follows. (For type \( F_4 \), (resp. \( G_3 \)), we write \( E_{abcd} \) and \( \dot{E}_{abcd} \) for \( Eaa_{1} + b\alpha_{4} + c\alpha_{3} + d\alpha_{2} \) and \( \dot{E}aa_{1} + ba_{4} + c\alpha_{3} + d\alpha_{2} \) (resp. \( Eaa_{1} + b\alpha_{3} + c\alpha_{2} \) and \( \dot{E}aa_{1} + ba_{3} + c\alpha_{2} \)).

(1) We put \( E\alpha_{i} = E_{i} \) \((1 \leq i \leq n)\).

(2) Let \( \alpha \in \Phi_{+}^{\text{red}} \) and \( 1 \leq i \leq n \) be such that \( g(\alpha) < i \) and \( \alpha + \alpha_i \in \Phi \). Put \( \dot{E}\alpha + \alpha_i = [E\alpha , E_{i}] \). If \( \Pi \) is of type \( B_N \), \( i = N \) and \( \alpha = g_j \) \((1 \leq j \leq N-1)\), let \( E\alpha + \alpha_N = (q^{1/2} + q^{-1/2})^{-1}E\alpha + \alpha_N \). If \( \Pi \) is of type \( D_N \), \( i = N \) and \( \alpha = \alpha_{N-1} \), let \( E\alpha + \alpha_N = (q+q^{-1})^{-1}E\alpha + \alpha_N \). If \( \Pi \) is of type \( F_4 \), let \( E_{1120} = (q + q^{-1})^{-1}E_{1120} \) and \( E_{1232} = (q^2 + 1 + q^{-2})^{-1}E_{1232} \). If \( \Pi \) is of type \( G_3 \), let \( E_{121} = (q + q^{-1})^{-1}E_{121} \), \( E_{021} = (q + q^{-1})^{-1}E_{021} \) and \( E_{031} = (q^2 + 1 + q^{-2})^{-1}E_{031} \). Otherwise, put \( E\alpha + \alpha_i = \dot{E}\alpha + \alpha_i \).

(3) Let \( \alpha, \beta \in \Phi_{+}^{\text{red}} \) such that \( g(\alpha) = g(\beta) \), \( \alpha < \beta \), \( h(\beta) - h(\alpha) \leq 1 \) and \( \alpha + \beta \in \Phi_{+}^{\text{red}} \). Put \( \dot{E}\alpha + \beta = [E\alpha , \dot{E}\beta ] \). If \( \Pi \) is of type \( C_N \) (resp. \( D_N \), \( F_4 \) or \( G_3 \)), then \( E\alpha + \beta \) is defined by \( (q + q^{-1})^{-1}E\alpha + \beta \) (resp. \( (q + q^{-1})^{-1}E\alpha + \beta \), \( (q^2 + q^{-2})^{-1}E\alpha + \beta \) or \( (q^2 + 1 + q^{-2})^{-1}E\alpha + \beta \)).

By using similar computations in [Y2], we have
Proposition 5.2. (1) As a K-vector space, $u_q n_+$ is spaned by the elements

\[ n_\alpha \Pi \in E_\alpha \\alpha \in \Phi_+^{\text{red}} \quad (n_\alpha \in \mathbb{Z}_+ \text{ if } (\alpha, \alpha) \neq 0, n_\alpha = 0, 1 \text{ if } (\alpha, \alpha) = 0). \]

Here $\Pi$ denotes a product taken with a total order on $\Phi_+^{\text{red}}$ compatible with the partial order $\lt$.

(2)

\[ \lt n_\alpha < m_\alpha \\lt \Pi E_\alpha, \Pi E_\alpha \\alpha \in \Phi_+^{\text{red}}, \alpha \in \Phi_+^{\text{red}} \]

\[ = \Pi \delta_{n_\alpha m_\alpha} \psi(n_\alpha; (-1)^{p(\alpha)} q^{(\alpha, \alpha)} ) <E_\alpha, E_\alpha> \quad \alpha \in \Phi_+^{\text{red}} \]

Here $\psi(n; t) = \Pi_{1 \leq i \leq n} \{(t^{i-1})/(t-1)\}$.

§6. Poincaré-Birkhoff-Witt type Theorem $u_q \sigma(G, \Pi, q)$ of Simple Lie Superalgebras $G$.

Define $d_\alpha \in (1/2)\mathbb{Z}_+$ by $d_\alpha = |(\alpha, \alpha)|/2$ if $(\alpha, \alpha) \neq 0$, $d_\alpha = 2$ if $\Pi$ is of type $G_3$ and $\alpha = \alpha_1 + 2\alpha_3 + c\alpha_2$, $d_\alpha = 1$ otherwise.

For $\alpha = c_1 \alpha_1 + \cdots + c_N \alpha_N \in \mathbb{P}_+$, put
Lemma 6.1. \( b(\alpha) \) can be written as \((-1)^a q^b\) for some \( a, b \in \mathbb{Z}_+ \). (For the precise value of \( b(\alpha) \), see [Y2; Lemma 10.3.1]).

By Proposition 5.2 and Lemma 6.1, we have:

Theorem 6.2. \((PBW-type\; theorem)\) The elements

\[
\begin{array}{c}
\delta_{\alpha} \\
\Pi \alpha \in \Phi_+^{\text{red}}
\end{array}
\]

form a \( K \)-basis of \( u_q n_+ \).

Proposition 6.3. Let \( m > 10 \) and \( \zeta \) a primitive \( m \)-th root of unity. Then, as \( K \)-algebra, \( u_\zeta n_+ \) is defined with the generators \( E_i \) \((1 \leq i \leq n)\) and the relations \((r1-7)\) in Theorem 4.1 and relations

\[
\gamma_{\alpha} \\
E_{\alpha} = 0 \quad (\alpha \in \Phi_+^{\text{red}}).
\]

\((rr1)\)

§7. Universal R-matrix \( \underline{S7.} \) \( u_\zeta^\sigma(\mathcal{E}, \Pi, p) \) of Simple Lie Superalgebra \( \mathcal{G} \).

Keep notation in §3-6. For \( \alpha = c_1 \alpha_1 + \cdots + c_N \alpha_N \in P_+ \), put

\[
F_{\alpha} = (\Pi_{1 \leq i \leq n} (q^{-d_i} - q^{d_i}))^{-1}(E_{\alpha})^{\text{op}}\sigma^{p(\alpha)} \quad \text{(see Lemma 4.2)}
\]

and

\[
u(\alpha) = (-1)^{ht(\alpha)}/b(\alpha).
\]

Theorem 7.1. \((Universal\; R-matrix\; of\; u_\zeta^\sigma)\) Keep notation in Proposition 6.3.
The Universal R-matrix $R$ of $u_\zeta^\sigma = u_\zeta^\sigma(E.\Pi.p)$ is given by

$$R = \{ \Pi_{\alpha \in \Phi_+^{\text{red}}} \ (\Sigma_{0 \leq \delta_\alpha < \gamma_\alpha} ((q^{d_\alpha} - q^{-d_\alpha})u(\alpha)E_\alpha \otimes F_\alpha \sigma^{p(\alpha)})^\delta_{\alpha}) $$

$$\psi(n_\alpha ; (-1)^{p(\alpha)}q^{(\alpha,\alpha)})$$

$$\cdot \{1/2 \Sigma_{0 \leq c, d \leq 1} (-1)^{cd} \sigma^c \otimes \sigma^d \} \cdot \Pi_{1 \leq i \leq N} \{ (1/\gamma_i) \Sigma_{0 \leq \delta_i, \phi_i < \gamma_i} \zeta_{(\xi_i, \xi_i)} \delta_i \phi_i K_{\xi_i} \delta_i \otimes K_{\xi_i} \phi_i \}$$

References.


