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Author(s)
KAWAZOE, TAKESHI

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TAKESHI KAWAZOE

Department of Mathematics,
Faculty of Science and Technology,
Keio University

1. Introduction

Let $\zeta(s)$ be the Riemann’s zeta function and $\eta(r) \ (r = \sqrt{-1}(1/2 - s))$ the logarithmic derivative of $\zeta$ which is of the form:

$$
\eta(r) = \sum_{p \in \text{Prim}} \sum_{n \geq 1} (\log p) e^{-n(\log p)s} - \sum_{i \geq 1} \sum_{n \geq 1} a_{in} e^{-\sqrt{-1}n(\log p_{i})r},
$$

(1)

where $\text{Prim} = \{p_{i}; i \geq 1\}$ is the set of prime numbers and $a_{in} = (\log p_{i}) e^{-n(\log p_{i})/2}$. This series converges absolutely and uniformly in any half plane $\Im(r) < -1/2 - \varepsilon \ (\varepsilon > 0)$ and has meromorphic continuation to the whole complex plane. Then the Riemann Hypothesis that the roots of $\zeta(s)$ all do lie on $\Re(s) = 1/2$ is equivalent to showing that the non imaginary poles of $\eta(r)$ all do lie on $\Im(r) = 0$.

Let $G$ be a connected semisimple Lie group with finite center, $K$ a maximal compact subgroup of $G$ and $\Gamma$ a discrete subgroup of $G$ such that $\Gamma \backslash G$ is compact. Then for each character $\chi$ of a finite dimensional unitary representation of $\Gamma$, Gangolli[Gl] investigates a zeta function $Z_{\Gamma}(s, \chi)$ of Selberg’s type, Selberg[S] originally introduced into the case of $SL(2, \mathbb{R})$. The logarithmic derivative $\eta_{G}(r)$ of $Z_{\Gamma}(s, \chi) \ (r = \sqrt{-1}(\rho_{0} - s))$ and $\rho_{0}$ is a positive real number depending only on $(G, K)$ is of the form:

$$
\eta_{G}(r) = \kappa \sum_{\delta \in \text{Prim}_{\Gamma}} \sum_{n \geq 1} \sum_{\lambda \in L} u_{\delta} m_{\lambda} \chi(\delta^{n}) \xi_{\lambda}(h(\delta))^{-n} e^{-nu_{\delta}s},
$$

(2)

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where $\text{Prim}_\Gamma$ is a complete set of representatives for the conjugacy classes of prime elements in $\Gamma$ and $u_\delta$ ($\delta \in \text{Prim}_\Gamma$) the logarithm of the norm $N(\delta)$ of $\delta$. For other notations refer to [G1]. This series converges absolutely and uniformly in any half plane $\Im(v) < -\rho_0 - \epsilon$ ($\epsilon > 0$) and has meromorphic continuation to the whole complex plane. Especially, the poles of $\eta_G(r)$ all do lie on $\Im(v) = 0$ or $\Re(v) = 0$, so the Riemann Hypothesis holds true for $Z_\Gamma(s, \chi)$. In what follows we shall rearrange the series as

$$\eta_G(r) = \sum_{i \geq 1} \sum_{n \geq 1} b_{in} e^{-\sqrt{-1} \log N(\omega_i(n)) r}$$

for which the exponents satisfy $c_{in} u_{\delta_i} = c_{jm} u_{\delta_j}$ if and only if $i = j$ and $n = m$.

We here note that (1) and (3) are quite similar in their forms. Therefore, if two distributions of $\text{Prim}$ and $\text{Prim}_\Gamma$ are similar in the logarithm of their norms, it is hoped that $\eta$ and $\eta_G$ have the same properties, especially, the Riemann Hypothesis holds for $\eta$ and then, for $\zeta$ also. In this paper we let $G = SL(2, \mathbb{R})$ and make an assumption of magnitude and distance of $N(\delta)$ for $\delta \in \text{Prim}_\Gamma$, which guarantees the similarity between the distributions (see (A) in §2 and (B) in §6). Then, under a week assumption (A) we shall obtain an integral expression of $\eta$ in terms of $\eta_G$ such as

$$\eta(v) = \int_{R - \sqrt{-1} y} \eta_G(x) H(v, x) dx$$

($y = 1/2 + \epsilon$ and see Proposition 3.3). Unfortunately, this formula is valid only for $\Im(v) \leq -L$ ($L$ is a large positive number and see Proposition 5.1). Then, the Riemann Hypothesis is equivalent to showing that the right hand side of (4) has analytic continuation to $\Im(v) < 0$ except $\nu = -\sqrt{-1}/2$. Under a strong assumption (B) we shall obtain the continuation and prove the Riemann Hypothesis (see Theorem 6.1).

Since $\eta(r)$ and $\eta_G(r)$ have a different growth order as $r \rightarrow \infty$ (cf. [E], Chap.9 and [H], Chap.6), we see that the distribution of $\text{Prim}$ and the one of norms of $\text{Prim}_\Gamma$ does not coincide. On the other hand we know that the prime number theorem that gives an approximation of the number of primes less than a given magnitude holds in an exactly same form for both $\text{Prim}$ and $\text{Prim}_\Gamma$ (cf. [E], Chap.4 and [H], Chap.2). Therefore, according to these facts we can believe that two distributions of $\text{Prim}$ and $\text{Prim}_\Gamma$ are similar in their norms. Actually, our strong assumption (B) expresses a similarity in the following fashion: there exists an injective map

$$\omega : \text{Prim} \rightarrow \text{Prim}_\Gamma$$

for which $\log N(\omega(p)) \leq 1/4 \log p$ or $\log N(\omega(p)) \leq \log p$ and the distance $\delta(p)$ between $\log N(\omega(p))$ and the nearest element being of the form $\log N(\omega(q))$ ($q \in \text{Prim}$) is bounded below by $\sigma (\log N(\omega(p)))^{-\theta}$ for positive constants $\sigma$ and $\theta$, roughly speaking, $\log N(\omega(p)) \leq \log p$ for almost all $p \in \text{Prim}$, but, if $\delta(p)$ is sufficiently small like in the case of twin prime elements, it must be $\log N(\omega(p)) \leq 1/4 \log p$. At present we have no idea to find a discrete subgroup $\Gamma$ of $SL(2, \mathbb{R})$ satisfying this property, however, we have enough reason to believe that a similarity between $\text{Prim}$ and $\text{Prim}_\Gamma$ deduces the Riemann Hypothesis.
2. Notations

Let $G = SL(2, \mathbb{R})$ and let $\chi$ be the trivial character of $\Gamma$. Then $\rho_0 = 1/2$ and the explicit form of $\eta_G$ is given by

$$\eta_G(r) = \sum_{i \geq 1} \frac{u_i/2}{\sinh(nu_i/2)} e^{-\sqrt{-1}nu_i r},$$

where $u_i = u_{\delta_i}$, and in (3) $c_{in} = n$ and

$$b_{in}^{-1} = 2u_{\delta_i}^{-1} \sinh(nu_i/2) \leq ce^{nu_i/2}. \quad (7)$$

For general references to the basic properties of $\eta_G$ see [G1], [H] and [S]. We denote the increasing sequence of prime numbers as $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ and the one of the norms of elements in $Prim_{\Gamma}$ as $N(\delta_1), N(\delta_2), N(\delta_3), \ldots$ respectively. We define $u_i = \log N(S_i)$ and

$$\delta_{in} = \frac{1}{2} \inf_{(m,j) \in \mathbb{N}^2} |nu_i - mu_j|. \quad (8)$$

for $i \geq 1$ and $n \geq 1$. Then, each $\delta_{in}$ is positive, because $\{u_i; i \geq 1\}$ does not have a finite point of accumulation (see [G2], p.415). Moreover, it is easy to see that there exists a positive constant $C$ such that for each $\alpha \geq 0$ and $\beta \geq 1$

$$\epsilon_{in} = \epsilon_{in}(\alpha, \beta, C) = Ce^{-\alpha n \log p_i} e^{-\beta nu_i} \leq \delta_{in} \quad (9)$$

for all $i$ and $n \geq 1$. We fix such a pair of $\alpha$ and $\beta$ till the end of §4.

As said in §1, the Riemann Hypothesis holds for $\eta_G$. Actually, the poles of $\eta_G$ are all simple and are as

$$\{\nu_j; j \in \mathbb{Z}\} \cup \{r_j; 1 \leq j \leq 2M\}, \quad (10)$$

where $\nu_j \in \mathbb{R}$ and $r_j \in \sqrt{-1}\mathbb{R}$ (cf. [G1], Proposition 2.7 and [H], p.68). Then it is known that $\nu_{-j} = -\nu_j$ and the poles of $\eta_G$ which concentrate along $[-\sqrt{-1}/2, \sqrt{-1}/2]$ can be denoted as

$$\{\nu_0, r_j, \overline{r}_j; 1 \leq j \leq M\}, \quad (11)$$

where we let $r_1, r_2, \ldots, r_M$ be the poles of $\eta_G$ which concentrate along $[-\sqrt{-1}/2, 0)$ and $\overline{r}_j = -r_j = r_{j+M}$. We denote the residues of $\eta_G$ at $\nu_j$ and $r_j$ by $n_j$ and $m_j$ respectively. Then, $n_{-j} = n_j$ and $m_j = m_{j+M} = 1$ for $1 \leq j \leq M$ (cf. [H], Chap.2).

We fix sufficiently small (resp. large) positive numbers $\epsilon$ and $\delta$ (resp. $E$), and a positive number $y$ such that $1/2 < y \leq 1/2 + \epsilon$. 

3. Transition from $\eta_G$ to $\eta$

Let $\phi$ be a $C^\infty$ compactly supported function on $\mathbb{R}$ satisfying

\begin{align*}
(i) \quad \text{supp}(\phi) &\subset (-1,1), \\
(ii) \quad \phi(0) &\equiv 1, \\
(iii) \quad \phi^{(k)}(0) &\equiv 0 \quad (1 \leq k \leq 2M)
\end{align*}

and let

\[ h_{in}(t) = \frac{a_{in}}{b_{in}} \phi\left(\frac{t-n(\log p_i)}{\epsilon_{in}}\right) \quad (t \in \mathbb{R}) \]

for $i \geq 1$ and $n \geq 1$. Then it is easy to see that $h_{in}$ satisfies the following conditions.

\begin{align*}
(i) \quad \text{supp}(h_{in}) &\subset (n(\log p_i) - \epsilon_{in}, n(\log p_i) + \epsilon_{in}), \\
(ii) \quad h_{in}(n(\log p_i)) &\equiv \frac{a_{in}}{b_{in}}, \\
(iii) \quad h_{in}^{(k)}(n(\log p_i)) &\equiv 0 \quad (1 \leq k \leq 2M).
\end{align*}

Without loss of generality we may assume that $\epsilon_{11} \leq 1/2 \log 2$ and thus, $\text{supp}(h_{in}) \subset [1/2 \log 2, \infty)$ for all $i$ and $n \geq 1$. Here we put $\hat{h}_{in}(x) = (2\pi)^{-1} \int_{\mathbb{R}} h_{in}(z)e^{-\sqrt{-1}xz}dz$ and

\begin{align*}
H(\nu, x) &\equiv \sum_{i,n \geq 1} e^{\sqrt{-1}(nu-n(\log p_i)x)} \hat{h}_{in}(\nu-x) \\
&\equiv \sum_{i,n \geq 1} e^{\sqrt{-1}(n(\log p_i)\nu-n\nu;x)} \frac{a_{in}}{b_{in}} \epsilon_{in} \hat{\phi}(\epsilon_{in}(\nu-x)).
\end{align*}

We now consider a condition for which the series (15) converges. For $\theta \geq 0$ and $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$ we suppose that $\nu$ and $x$ satisfy

\begin{align*}
(a_E) \quad -E \leq \Im(\nu), \Im(x) \leq E, \\
(b^{p,q}_\theta) \quad \left\{ \begin{array}{l} \Im(\nu) - 1/2 - (1-\theta)\alpha \leq -1/p - \delta \\
-\Im(x) + 1/2 - (1-\theta)\beta \leq -1/q - \delta, \end{array} \right.
\end{align*}

where $\delta$ is a fixed sufficiently small positive number (see §2). Then, substituting the definition of $a_{in}$ and $b_{in}$ (see (1) and (7)) for (15b), we see that $|\nu - x|^\theta |H(\nu, x)|$ is dominated by

\[ c \sum_{i,n \geq 1} \log p_i e^{(\Im(\nu)-1/2)n(\log p_i)e^{(-\Im(x)+1/2)n\nu_i}} e^{1-\theta}(\epsilon_{in}(\nu-x))^{\theta} \hat{\phi}(\epsilon_{in}(\nu-x)). \]

(16)
Since $\hat{\phi}$ is rapidly decreasing and is holomorphic of exponential type $\leq 1$ (cf. [Su], p.146), for each $N \in \mathbb{N}$ there exists $C_N > 0$ for which
\[
|\hat{\phi}(x)| \leq C_N (1 + |x|)^{-N} e^{\Im(x)} \quad (x \in \mathbb{C}).
\]
(17)

Therefore, it follows from (9) and $(a_E)$ that $|\nu - x|^\sigma |H(\nu, x)|$ is dominated by
\[
cC^{1-\theta} C_{[\theta]+1} e^{2EC} \sum_{i, n \geq 1} \log p_i e^{(\Im(\nu) - 1/2 - (1-\theta)\alpha) n (\log p_i) - (\Im(x) + 1/2 - (1-\theta)\beta) n u_i},
\]
where $[\theta]$ is the greatest integer not exceeding $\theta$. Then, this series converges absolutely and uniformly by $(b_\theta^{p,q})$ and the Hölder’s inequality.

**Lemma 3.1.** If $\nu$ and $x$ satisfy $(a_E)$ and $(b_\theta^{p,q})$, then the series $H(\nu, x)$ converges absolutely and uniformly, and is holomorphic of $\nu$ and $x$. Moreover, if $(b_\theta^{p,q})(\theta \geq 0)$ is satisfied, there exists a positive constant $C$ such that
\[
|H(\nu, x)| \leq C |\nu - x|^{-\theta}.
\]

Throughout this paper we assume the following condition:

(A) There exists a positive constant $A$ such that
\[
u_i \leq A \log p_i \quad \text{for all } i \geq 1.
\]

Then we can replace $(b_\theta^{p,q})$ with

\[
(b_{\theta, \gamma}^{p,q}) \left\{
\begin{array}{l}
\Re(\nu) - 1/2 - (1-\theta)\alpha + \gamma \leq -1/p - \delta,
\\Re(x) + 1/2 - (1-\theta)\beta - \gamma/A \leq -1/q - \delta.
\end{array}
\right.
\]

where $\gamma \geq 0$. We fix such a $\gamma$.

We next let $-y \leq -y_0 \leq E$ and

\[
(c_{\theta, \gamma, y_0}^{p,q}) \left\{
\begin{array}{l}
\Re(\nu) - 1/2 - (1-\theta)\alpha + \gamma \leq -1/p - \delta,
\\gamma_0 + 1/2 - (1-\theta)\beta - \gamma/A \leq -1/q - \delta.
\end{array}
\right.
\]

Then, if $\nu$ satisfies $(a_E)$ and $(c_{\theta+1, \gamma, y_0}^{p,q})(\theta \in \mathbb{N})$, it follows similarly as above that
\[
\int_{\mathbb{R} - \sqrt{-1}y_0} |x|^\theta |H(\nu, x)| dx
\]
\[
\leq c \sum_{i, n \geq 1} \log p_i e^{(\Im(\nu) - 1/2) n (\log p_i)} e^{(y_0 + 1/2) n u_i} \epsilon_i \ \epsilon_i \ \int_{\mathbb{R} - \sqrt{-1}y_0} [ (\epsilon_i x)^\theta \hat{\phi}(\epsilon_i (\nu - x)) ] dx
\]
and by letting $x = (x - \nu) + \nu$,
\[
\leq c C^{1-\theta} C_{[\theta]+2} e^{2EC} P_\theta(|\nu|) \sum_{i, n \geq 1} \log p_i e^{(\Im(\nu) - 1/2 + \theta \alpha + \gamma) n (\log p_i)} e^{(y_0 + 1/2 + \theta \beta - \gamma/A) n u_i},
\]
where $P_\theta$ is a polynomial of degree $\theta$ with coefficients depending only on $\theta$. Then this series converges absolutely and uniformly by $(c_{\theta+1, \gamma, y_0}^{p,q})$ and the Hölder's inequality.
Lemma 3.2. Let \( \nu \) be in a compact set \( S \) in the tube domain defined by \((a_E)\) and 
\((c_{k+1, \gamma, y_0}^{\rho q}) (\theta \in \mathbb{N} \text{ and } -y \leq -y_0 \leq E)\). Let \( f \) be a function on \( \mathbb{R} - \sqrt{-1}y \) such that \( f(x) = O(|x|^\theta) \). Then, there exists a positive constant \( C \) for which \( \int_{\mathbb{R} - \sqrt{-1}y} |f(x)H(\nu, x)|dx \leq C \). Especially, 
\[
T_{y_0}f(\nu) = \int_{\mathbb{R} - \sqrt{-1}y_0} f(x)H(\nu, x)dx
\]
is well-defined and is holomorphic of \( \nu \) satisfying \((a_E)\) and \((c_{k+1, \gamma, y_0}^{\rho q})\).

Proposition 3.3. Let \( P \) be a polynomial of degree \( k(0 \leq k \leq 2M) \) and \( \nu \) satisfy \((a_E)\) and 
\( (c_{k+1, \gamma, y}^{\rho q}) \). Then,

(i) \[
P(\nu)\eta(\nu) = T_{y}(P\eta_{G})(\nu) = \int_{\mathbb{R} - \sqrt{-1}y} P(x)\eta_{G}(x)H(\nu, x)dx,
\]

(ii) \[
0 = \int_{\mathbb{R} - \sqrt{-1}y} P(x)\eta_{G}(x)H(\nu, -x)dx.
\]

Proof. Since \( \eta_{G}(x) = O(1) \) for \( x \in \mathbb{R} - \sqrt{-1}y \) (see [H], Proposition 6.7) and \((c_{k+1, \gamma, y}^{\rho q})\) implies \((c_{k+1, -\gamma, y}^{\rho q})\), the right hand sides of (i) and (ii) are well-defined and are holomorphic of \( \nu \) satisfying \((a_E)\) and \((c_{k+1, \gamma, y}^{\rho q})\) (see Lemma 3.2). Therefore, we may suppose that \( \Im(\nu) \leq -y \). Since \( mu_j > 0 \) for all \( m, j \geq 1 \), it follows that

\[
\int_{\mathbb{R} - \sqrt{-1}y} e^{-\sqrt{-1}mu_j x}H(\nu, x)dx = \int_{\mathbb{R}} e^{-\sqrt{-1}mu_j x}H(\nu, x)dx.
\]

Then, substituting the definition of \( H(\nu, x) \) (see (15a)), we see formally that

\[
= \sum_{k,l \geq 1} \int_{\mathbb{R}} e^{-\sqrt{-1}l(u_k - l(\log p_k))x} \tilde{h}_{kl}(\nu - x)dx
= \sum_{k,l \geq 1} e^{-\sqrt{-1}(mu_j - l(u_k + l(\log p_k))\nu) \int_{\mathbb{R}} e^{\sqrt{-1}(mu_j - l(u_k + l(\log p_k))x} \tilde{h}_{kl}(x)dx
= \sum_{k,l \geq 1} e^{-\sqrt{-1}(mu_j - l(u_k + l(\log p_k))\nu} h_{kl}(mu_j - l(u_k + l(\log p_k))).
\]

Since each support of \( h_{kl} \) is disjointed from the others, it is easy to see that the condition that \( \Im(\nu) \leq -y \) guarantees the validity of the above calculation. Moreover, since the
support of $h_{kl}$ is contained in $(l(\log p_k) - \epsilon_{kl}, l(\log p_k) + \epsilon_{kl})$ and $h_{kl}(l(\log p_k)) = a_{kl}b_{kl}^{-1}$ (see (14)(i) and (ii)), it follows from (9) and the definition of $\delta_{kl}$ (see (7)) that

$$h_{kl}(l(\log p_k)) = a_{kl}b_{kl}^{-1}e^{-\sqrt{-1}l(\log p_k)\nu},$$

where $\epsilon_{ij} = 1$ if $i = j$ and 0 otherwise. Therefore, we can deduce that

$$T_y \eta_G(\nu) = \int_{\mathbb{R}-\sqrt{-1}y} \eta_G(x)H(\nu, x)dx$$

$$= \sum_{j,m \geq 1} b_{jm} \int_{\mathbb{R}-\sqrt{-1}y} e^{-\sqrt{-1}mu_{j}x}H(\nu, x)dx$$

$$= \sum_{j,m \geq 1} a_{jm} e^{-\sqrt{-1}m(\log p_j)\nu}$$

$$= \eta(\nu).$$

Here we rewrite $P(\nu)$ as

$$P(\nu) = R_\nu(\nu - x) + P(x),$$

where $R_\nu$ is a polynomial of degree $k$ with coefficients depending only on $k$ and $\nu$. Then the formula (i) follows from (20) provided that

$$\int_{\mathbb{R}-\sqrt{-1}y} (\nu - x)^l \eta c(x)H(\nu, x)dx = 0 \quad (1 \leq l \leq k).$$

We now show (21). If we define $H^{(l)}(\nu, x)$ by replacing $h_{in}$ in (15a) with $(\sqrt{-1})^{-l}h_{in}^{(l)}$, we easily see that the left hand side of (21) is equal to

$$\int_{\mathbb{R}-\sqrt{-1}y} \eta_G(x)H^{(l)}(\nu, x)dx.$$
4. A relation between $\eta$ and the poles of $\eta_G$

We keep the notations and the assumption (A). We first recall that $\eta_G$ satisfies the functional equation:

$$\eta_G(x) + \eta_G(-x) = cx \tanh \pi x \quad (22)$$

(see [H], Proposition 4.26). In this section we shall express $\eta$ as the sum of an integral of $x \tanh \pi x$ and the residues of $\eta_G$.

**Lemma 4.1.** Let $P$ be a polynomial of degree $k (0 \leq k \leq 2M)$ and let $\nu$ be in a compact set $S$ satisfying $\Im(S) < 0$, $(a_E)$ and $(c^{p,q}_{6,\gamma,0})$. Then the series $\sum_{j \in \mathbb{Z}} n_j P(\nu_j) H(\nu, \nu_j)$ converges absolutely and uniformly. Especially, $\sum_{j \in \mathbb{Z}} n_j P(\nu_j) H(\nu, \nu_j)$ is well-defined and is holomorphic of $\nu$ satisfying $\Im(S) < 0$, $(a_E)$ and $(c^{p,q}_{6,\gamma,0})$.

**Proof.** Since $\nu_j \in \mathbb{R}$ and $\nu \in S$, Lemma 3.1 implies that for $x \in \mathbb{R}$

$$|H(\nu, x)| \leq C|\nu - x|^{-(k+6)} \sim (1 + |x|)^{-(k+6)}.$$ 

Then, noting the fact that

$$\sum_{\{j; \nu_j^2 \leq x\}} n_j \sim x^2 \quad (x \to \infty)$$

(see §2 and [G1], Proposition 1.2), we see that

$$\sum_{j \in \mathbb{Z}} n_j |P(\nu_j)H(\nu, \nu_j)| \sim \sum_{j \in \mathbb{Z}} n_j (1 + |\nu_j|)^{-6}$$

$$\sim \sum_{k=0}^{\infty} \sum_{k \leq |\nu_j| < k+1} n_j (1 + |\nu_j|)^{-6}$$

$$\sim \sum_{k=0}^{\infty} (1 + k)^{-2} < \infty. \quad \square$$

We now suppose that $\nu$ satisfies $\Im(\nu) < 0$, $(a_E)$ and $(c^{p,q}_{6,\gamma,0})$. We note that, if $|\Im(x)| \leq \epsilon$, then $x \tanh \pi x = O(|x|)$ and $\eta_G(x) = O(|x|)$ (see [H], Proposition 6.7). Therefore, since $(c^{p,q}_{6,\gamma,0})$ implies $(c^{p,q}_{5,\gamma,\pm\epsilon})$ and $(c^{p,q}_{6,\gamma,0})$, it follows from Lemma 3.2 and Lemma 4.1 that

$$\int_{\mathbb{R}}cx \tanh \pi x H(\nu, x)dx$$

$$= \int_{\mathbb{R} + \sqrt{-1} \mathbb{R}} cx \tanh \pi x H(\nu, -x)dx$$

$$= \int_{\mathbb{R} + \sqrt{-1} \mathbb{R}} (\eta_G(x) + \eta_G(-x)) H(\nu, -x)dx$$

$$= \int_{\mathbb{R} - \sqrt{-1} \mathbb{R}} \eta_G(x) H(\nu, x)dx + \int_{\mathbb{R} + \sqrt{-1} \mathbb{R}} \eta_G(x) H(\nu, -x)dx.$$
The second term is equal to
\[
\int_{\mathbb{R}-\sqrt{-1}y} \eta_G(x)H(\nu, -x)dx - \sum_{j\in \mathbb{Z}} n_jH(\nu, \nu_j) - \sum_{1 \leq j \leq M} H(\nu, -r_j)
\]
\[
= - \sum_{j\in \mathbb{Z}} n_jH(\nu, \nu_j) - \sum_{1 \leq j \leq M} H(\nu, -r_j)
\]
by Proposition 3.3(ii). Therefore, it follows from Proposition 3.3 (i) that
\[
\eta(\nu) = \int_{\mathbb{R}-\sqrt{-1}y} \eta c(x)H(\nu, x)dx
\]
\[
= \int_{\mathbb{R}-\sqrt{-1}\epsilon} \eta_G(x)H(\nu, x)dx + \sum_{1 \leq j \leq M} H(\nu, r_j)
\]
\[
= \int R \, x \, \tanh \pi xH(\nu, x)dx + \sum_{j\in \mathbb{Z}} n_jH(\nu, \nu_j) + \sum_{1 \leq j \leq 2M} H(\nu, r_j).
\]

Then, letting \( \epsilon \) and \( \delta \) (resp. \( E \)) sufficiently small (resp. large), we can obtain the following,

**Proposition 4.2.** If \( \nu \) satisfies
\[
\begin{cases} \Im(\nu) < \min(0, 1/2 - 5\alpha - \gamma - 1/p) \\
1 + 5\beta < \gamma/A - 1/q,
\end{cases}
\]
where \( \gamma \geq 0, 1 \leq p, q \leq \infty \) and \( 1/p + 1/q = 1 \), then
\[
\eta(\nu) = c \int \, x \, \tanh \pi xH(\nu, x)dx + \sum_{j\in \mathbb{Z}} n_jH(\nu, \nu_j) + \sum_{1 \leq j \leq 2M} H(\nu, r_j).
\]

We put
\[
P_G(x) = (\nu^2 - r_1^2)(\nu^2 - r_2^2)\ldots(\nu^2 - r_M^2).
\] (23)

Then, replacing \( \eta c \) with \( P_G \eta_G \), we can obtain the following proposition by the quite same way.

**Proposition 4.3.** If \( \nu \) satisfies
\[
\begin{cases} \Im(\nu) < \min(0, 1/2 - (5 + 2M)\alpha - \gamma - 1/p) \\
1 + (5 + 2M)\beta < \gamma/A - 1/q,
\end{cases}
\]
where \( \gamma \geq 0, 1 \leq p, q \leq \infty \) and \( 1/p + 1/q = 1 \), then
\[
P_G(\nu)\eta(\nu) = \int_{\mathbb{R}-\sqrt{-1}\epsilon} \eta_G(x)P_G(x)H(\nu, x)dx
\]
\[
= c \int \, x \, \tanh \pi xP_G(x)H(\nu, x)dx + \sum_{j\in \mathbb{Z}} n_jP_G(\nu_j)H(\nu, \nu_j).
\]
5. Some modifications

5.1. In the proof of Proposition 3.3 each term $b_{in}e^{-\sqrt{-1}nu_{r}}$ of $\eta_{G}(r)$ ($u_{i} = \log N(\delta_{i})$) transfers to $a_{in}e^{-\sqrt{-1}(\log p_{i})u_{r}}$ of $\eta(r)$ under the integral formula. Obviously, to verify such an integral formula $\delta_{i}$'s need not be taken over all elements in $\text{Prim}_{\Gamma}$, and it is enough for each $p_{i}$ to correspond to a unique element $\delta_{\omega(i)}$ in $\text{Prim}_{\Gamma}$. Actually, for an injective map

$$\omega : \mathbb{N} \rightarrow \mathbb{N}$$

we put

$$\delta_{in} = \frac{1}{2} \inf_{\substack{(m,j) \in \mathbb{N}^{2} \setminus (n, \omega(j)))}} |nu_{\omega(i)} - mu_{\omega(j)}|,$$

$$\epsilon_{in} = \epsilon_{in}(\alpha, \beta, C) = Ce^{-\alpha n(\log p_{i})}e^{-\beta nu_{\omega(i)}},$$

$$h_{in} = \frac{a_{in}}{b_{\omega(i)n}}\phi\left(\frac{t - n(\log p_{i})}{\epsilon_{in}}\right) (t \in \mathbb{R}),$$

$$H_{\omega}(\nu, x) = \sum_{i, n \geq 1} e^{\sqrt{-1}(nu_{\omega(i)} - n(\log p_{i}))x} h_{in}(\nu - x).$$

(cf. (8), (9), (13) and (15)). Then it is easy to see that all results in the preceding sections are also valid when we replace $\delta_{in}, \epsilon_{in}, h_{in}$ and $H(\nu, x)$ by $\delta_{in}^{\omega}, \epsilon_{in}^{\omega}, h_{in}^{\omega}$ and $H_{\omega}(\nu, x)$ respectively and (A) by

\[(A)_{\omega}\text{ There exists a positive constant } A \text{ such that}\]

$$u_{\omega(i)} \leq A \log p_{i} \text{ for all } i \geq 1.$$

5.2. We next modify the $\eta$ functions. Let

$$\eta^{o}(r) = \sum_{i \geq 1} a_{i}e^{-\sqrt{-1}(\log p_{i})r},$$

where $a_{i} = (\log p_{i})e^{-(\log p_{i})/2}$, and let

$$\eta_{G}^{o}(r) = \sum_{i \geq 1} b_{i}e^{-\sqrt{-1}u_{i}r},$$

where $b_{i} = u_{i}/2 \sinh(u_{i}/2)$. Then, it is easy to see that $\eta(r) - \eta^{o}(r)$ and $\eta_{G}(r) - \eta_{G}^{o}(r)$ are holomorphic on $\mathbb{S}(r) < 0$ (cf. [H], Proposition 3.5). Therefore, in order to prove the Riemann Hypothesis for $\eta$ it is enough to prove it for $\eta^{o}$. Since $\eta^{o}$ and $\eta_{G}^{o}$ inherit all singularities from $\eta$ and $\eta_{G}$ respectively, the whole arguments in the previous sections except one using the functional equation (22) are also applicable to $\eta^{o}$ and $\eta_{G}^{o}$. Especially, if we define $\delta_{i}^{\omega}, \epsilon_{i}^{\omega}(\alpha, \beta, C), h_{i}^{\omega}$ and $H_{\omega}^{o}(\nu, x)$ by eliminating the sufix $n$ in (24)-(27) respectively, we see that all the results in §2 and §3 are also valid when we replace $\eta, \eta_{G}$ and $H$ by $\eta^{o}, \eta_{G}^{o}$ and $H_{\omega}^{o}$ respectively and (A) by (A)$_{\omega}$.
5.3. We now let
\[ \omega : D \to \mathbb{N}, \quad D \subset \mathbb{N} \]
be an injective map, and for each \( i \in D \) we define \( \delta^\omega_i, \epsilon^\omega_i(\alpha, \beta, C) \) and \( h^\omega_i \) as above. Moreover, we put
\[ \eta^{\omega}(r) = \sum_{i \in D} a_i e^{-\sqrt{-1}(\log p_i) r}, \]
\[ H^{\omega}(\nu, x) = \sum_{i \in D} e^{\sqrt{-1}(\nu \omega(i) - n(\log p_i)) x} h^{\omega}_i(\nu - x) \]
and we define the corresponding assumption \((A)_{\omega}\), we denote by the same letter, by replacing \( i \geq 1 \) with \( i \in D \). Then repeating the same arguments in §3, especially, taking \( \gamma \) sufficiently large in Corollary 3.4 and Proposition 4.3, we can deduce that

**Proposition 5.1.** Let us suppose that \((A)_{\omega}\) holds. Then there exists a positive constant \( L \) such that if \( \Im(\nu) \leq -L \),

(i) \[ \eta^{\omega}(\nu) = \int_{R - \sqrt{-1}y} \eta^{\omega}_G(x) H^{\omega}(\nu, x) dx, \]

(ii) \[ P_G(\nu) \eta^{\omega}(\nu) = \int_{R - \sqrt{-1} \epsilon} P_G(x) \eta^{\omega}_G(x) H^{\omega}(\nu, x) dx. \]

6. A proof of the Riemann Hypothesis under an assumption

We retain the notations in the previous sections. We here make an assumption on magnitude and distance of \( u_i (i \in \mathbb{N}) \), which is stronger than \((A)\), and then give a proof of the Riemann Hypothesis. The assumption can be stated as follows.

**\(B\)** There exist an injective map \( \omega : \mathbb{N} \to \mathbb{N} \) and positive constants \( \sigma \) and \( \theta \) for which, except a finite number of \( i \), one of the following conditions holds:

(B1) \[ u_{\omega(i)} \leq 1/4 \log p_i, \]

(B2) \[ u_{\omega(i)} \leq \log p_i \quad \text{and} \quad \sigma u_{\omega(i)}^{-\theta} \leq \delta^\omega_i. \]

We here put
\[ D_{\ell} = \{ i \in \mathbb{N} ; (B \ell) \text{ holds} \} \quad \text{for} \quad \ell = 1, 2 \quad \text{and} \quad D_3 = \mathbb{N} - D_1 \cup D_2. \]

In what follows for each \( \omega = \omega_{|D_\ell} (\ell = 1, 2, 3) \) we shall prove that \( P_G(\nu) \eta^\omega_\ell(\nu) \) \((\ell = 1, 2, 3) \) (see (30)) is holomorphic on \(-2L \leq \Im(\nu) \leq -3 \epsilon\).

\( \eta^\omega_1 \): Since \((B1)\) implies \((A)_{\omega_1}\) (see 5.3), it follows from Proposition 5.1 that

\[ \eta^\omega_1(\nu) = \int_{R - \sqrt{-1}y} \eta^\omega_1(x) H^\omega_1(\nu, x) dx, \]

if \( \Im(\nu) \leq -L \). We now recall the definition of \( \epsilon^\omega_{\ell} \) (see 5.3 and (9)). Then, we can choose a sufficiently small positive number \( \tau \) depending on \( \epsilon \) such that

\[ \sum_{i \in D_1} e^{-(1+3\epsilon)u_{\ell}(i)}(\epsilon^\omega_{\ell})^{-\tau} < \infty. \]
Then, by (B1) and the argument used in (16)-(18) we see that if $-2L \leq \Im(\nu) \leq -2\varepsilon$ and $\Im(x) = -y = -1/2 - \varepsilon$,

$$|H_{\omega_{1}}^{o}(\nu, x)| \leq c \sum_{i \in D_{1}} \log p_{i} \log p_{i} e^{\varepsilon \omega_{1}(i)} (e_{i}^{\omega_{1}})^{-r} |\nu - x|^{-(1+r)}$$

$$\leq c |\nu - x|^{-(1+r)} \sum_{i \in D_{1}} e^{-3\varepsilon \omega_{1}(i)} (e_{i}^{\omega_{1}})^{-r}$$

$$\leq c |\nu - x|^{-(1+r)} \quad \text{by (33)}.$$

Since $\eta_{G}^{o}(x) = O(1)$ for $x \in \mathbb{R} - \sqrt{-1} y$ (see [H], Theorem 3.10), the above estimate and (32) give an analytic continuation of $\eta_{\omega_{1}}^{o}(\nu)$ on $-2L \leq \Im(\nu) \leq -2\varepsilon$.

$\eta_{\omega_{2}}^{o}$: In the previous sections $\epsilon_{i}^{\omega} = \epsilon_{1}^{\omega}(\alpha, \beta, C)$ (see 5.3 and (9)) is defined for $\alpha \geq 0$ and $\beta \geq 1$. However, under the second condition of (B2) we may take $\epsilon_{2}^{\omega_{2}} = \sigma u_{\omega_{2}(i)}^{-\theta}$ and easily see that all arguments in the previous sections are valid for $\epsilon_{i}^{\omega_{2}}$, $h_{1}^{\omega_{2}}$ and $H_{\omega_{2}}^{o}$, especially, it follows that

$$P_{G}(\nu)\eta_{\omega_{2}}^{o}(\nu) = \int_{\mathbb{R} - \sqrt{-1} y} P_{G}(x) \eta_{G}^{o}(x) H_{\omega_{2}}^{o}(\nu, x) dx,$$

(34)

if $\Im(\nu) \leq -L$ (see Proposition 5.1). We here put $J_{0} = \{i \in D_{2}; 1 \leq \epsilon_{i}^{\omega_{2}}\}$ and $J_{n} = \{i \in D_{2}; 2^{-n} \leq \epsilon_{i}^{\omega_{2}} < 2^{-(n-1)}\} (n = 1, 2, \ldots)$. Moreover, we denote by $i_{n}$ the number in $J_{n}$ for which $\omega_{2}(i_{n})$ is the smallest in $\omega_{2}(j) (j \in J_{n})$. Then for each $i \in J_{n}$ we see from the definition of $\delta_{1}^{\omega_{2}}$ (see 5.3 and (8)) and (B2) that $u_{\omega_{2}(i_{n})} \geq u_{\omega_{2}(i)} + 2\sum_{j \in J_{n}, \omega_{2}(j) < \omega_{2}(i)} \delta_{j}^{\omega_{2}} \geq 2k_{n}(i)2^{-n}$ for $n \geq 0$ and $u_{\omega_{2}(i_{n})} \geq \sigma^{1/\theta}2^{(n-1)/\theta}$ for $n \geq 1$. Therefore, by (B2) and the argument used in (16)-(18) we see that if $-2L \leq \Im(\nu) \leq -3\varepsilon$ and $\Im(x) = -\varepsilon$,

$$|H_{\omega_{2}}^{o}(\nu, x)| \leq c \sum_{i \in D_{2}} \log p_{i} e^{\varepsilon \omega_{2}(i)} \sum_{n=0}^{\infty} \sum_{i \in J_{n}} e^{-\epsilon u_{\omega_{2}(i)}(e_{i}^{\omega_{2}})^{-2\varepsilon} - 2(M+3)} |\nu - x|^{-2(M+3)}$$

$$\leq c |\nu - x|^{-2(M+3)} \sum_{n=0}^{\infty} \sum_{i \in J_{n}} e^{-\epsilon u_{\omega_{2}(i)}(e_{i}^{\omega_{2}})^{-2\varepsilon}}$$

$$\leq c |\nu - x|^{-2(M+3)} \left( e^{-\epsilon u_{\omega_{2}(i)}(e_{i}^{\omega_{2}})^{-2\varepsilon}} + \sum_{n=1}^{\infty} e^{-\epsilon u_{\omega_{2}(i)}(e_{i}^{\omega_{2}})^{-2\varepsilon}} 2n(M+1) \sum_{i \in J_{n}} e^{-2\epsilon k_{n}(i)2^{-n}} \right)$$

$$\leq c |\nu - x|^{-2(M+3)} \left( \frac{1}{1 - e^{-2\epsilon}} + \sum_{n=1}^{\infty} \frac{e^{-\epsilon u_{\omega_{2}(i)}(e_{i}^{\omega_{2}})^{-2\varepsilon}} 2n(M+1) \sum_{i \in J_{n}} e^{-2\epsilon k_{n}(i)2^{-n}}}{1 - e^{-2\epsilon}2^{-n}} \right)$$

$$\leq c |\nu - x|^{-2(M+3)}.$$

Since $P_{G}(\nu)\eta_{G}^{o}(x) = O(|x|^{2M+1})$ for $x \in \mathbb{R} - \sqrt{-1} \varepsilon$ (see (23) and [H], Remark 6.8), the above estimate and (34) give an analytic continuation of $\eta_{\omega_{2}}^{o}(\nu)$ on $-2L \leq \Im(\nu) \leq -3\varepsilon$.

$\eta_{\omega_{3}}$: Since $D_{3}$ is finite, $\eta_{\omega_{3}}^{o}$ is holomorphic on the whole complex plane.
We now obtained that each $P_G(\nu)\eta^o(\nu) (\ell = 1, 2, 3)$ has an analytic continuation on $-2L \leq \Im(\nu) \leq -3\epsilon$. Therefore, $P_G(\nu)\eta^o(\nu) = \sum_{\ell=1}^{3} P_G(\nu)\eta^o_{\omega_\ell}(\nu)$ and thus, $P_G(\nu)\eta(\nu)$ have the same property (see 5.2). Since $\epsilon$ can be taken sufficiently small and $\eta$ satisfies the functional equation (see [E], p.13), it follows that $P_G(\nu)\eta(\nu)$ is holomorphic on $0 < |\Im(\nu)| \leq 2L$. Then, noting the zeros of $P_G(\nu)$ (see (23) and (11)) and the fact that that $\zeta(s)$ has no zeros on $[0,1]$, we can finally obtain the following theorem.

**Theorem 6.1.** If $SL(2, \mathbb{R})$ has a cocompact discrete subgroup $\Gamma$ with $Prim_\Gamma$ satisfying the condition (B), then the Riemann Hypothesis holds.

**Remark 6.2.** We see that $D_2 \neq \emptyset$. Actually, if $D_1 \cup D_2 = \mathbb{N}$, it follows from the above argument that $\eta^o(\nu)$ is holomorphic on $\Im(\nu) < 0$. This contradicts to the fact that $\eta(\nu)$ has a pole at $\nu = -\sqrt{-1}/2$.

**References**


