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# A RELATION BETWEEN THE LOGARITHMIC DERIVATIVES OF RIEMANN AND SELBERG ZETA FUNCTIONS AND A PROOF OF THE RIEMANN HYPOTHESIS UNDER AN ASSUMPTION ON A DISCRETE SUBGROUP OF $SL(2,\mathrm{R})$

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## 1. Introduction

Let $\zeta(s)$ be the Riemann's zeta function and $\eta(r)$ $(r=\sqrt{-1}(1/2-s))$ the logarithmic derivative of $\zeta$ which is of the form:

$$\begin{aligned}\eta(r) &= \sum_{p\in Prim}\sum_{n\geq 1}(\log p)e^{-n(\log p)s}\\ &= \sum_{i\geq 1}\sum_{n\geq 1}a_{in}e^{-\sqrt{-1}n(\log p_i)r},\end{aligned}\tag{1}$$

where $Prim=\{p_i; i\geq 1\}$ is the set of prime numbers and $a_{in}=(\log p_i)e^{-n(\log p_i)/2}$. This series converges absolutely and uniformly in any half plane $\Im(r)<-1/2-\varepsilon$ $(\varepsilon>0)$ and has meromorphic continuation to the whole complex plane. Then the Riemann Hypothesis that the roots of $\zeta(s)$ all do lie on $\Re(s)=1/2$ is equivalent to showing that the non imaginaly poles of $\eta(r)$ all do lie on $\Im(r)=0$.

Let $G$ be a connected semisimple Lie group with finite center, $K$ a maximal compact subgroup of $G$ and $\Gamma$ a discrete subgroup of $G$ such that $\Gamma\backslash G$ is compact. Then for each character $\chi$ of a finite dimensional unitary representation of $\Gamma$, Gangolli[**G1**] investigates a zeta function $Z_\Gamma(s,\chi)$ of Selberg's type, Selberg[S] originally introduced into the case of $SL(2,\mathbf{R})$. The logarithmic derivative $\eta_G(r)$ of $Z_\Gamma(s,\chi)$ $(r=\sqrt{-1}(\rho_0-s)$ and $\rho_0$ is a positive real number depending only on $(G,K))$ is of the form:

$$\eta_G(r)=\kappa\sum_{\delta\in Prim_\Gamma}\sum_{n\geq 1}\sum_{\lambda\in L}u_\delta m_\lambda\chi(\delta^n)\xi_\lambda(h(\delta))^{-n}e^{-nu_\delta s},\tag{2}$$

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where $Prim_\Gamma$ is a complete set of representatives for the conjugacy classes of prime elements in $\Gamma$ and $u_\delta$ ($\delta \in Prim_\Gamma$) the logarithm of the norm $N(\delta)$ of $\delta$. For other notations refer to [G1]. This series converges absolutely and uniformly in any half plane $\Im(r) < -\rho_0 - \varepsilon$ ($\varepsilon > 0$) and has meromorphic continuation to the whole complex plane. Especially, the poles of $\eta_G(r)$ all do lie on $\Im(r) = 0$ or $\Re(r) = 0$, so the Riemann Hypothesis holds true for $Z_\Gamma(s, \chi)$. In what follows we shall rearrange the series as

$$\eta_G(r) = \sum_{i \geq 1} \sum_{n \geq 1} b_{in} e^{-\sqrt{-1} c_{in} u_{\delta_i} r} \tag{3}$$

for which the exponents satisfy $c_{in} u_{\delta_i} = c_{jm} u_{\delta_j}$ if and only if $i = j$ and $n = m$.

We here note that (1) and (3) are quite similar in their forms. Therefore, if two distributions of $Prim$ and $Prim_\Gamma$ are similar in the logarithm of their norms, it is hoped that $\eta$ and $\eta_G$ have the same properties, especially, the Riemann Hypothesis holds for $\eta$ and then, for $\zeta$ also. In this paper we let $G = SL(2, \mathbf{R})$ and make an assumption of magnitude and distance of $N(\delta)$ for $\delta \in Prim_\Gamma$, which guarntees the similarity between the distributions (see **(A)** in §2 and **(B)** in §6). Then, under a week assumption **(A)** we shall obatin an integral expression of $\eta$ in terms of $\eta_G$ such as

$$\eta(\nu) = \int_{\mathbf{R} - \sqrt{-1} y} \eta_G(x) H(\nu, x) dx \tag{4}$$

($y = 1/2 + \varepsilon$ and see Proposition 3.3). Unfortunately, this formula is valid only for $\Im(\nu) \leq -L$ ($L$ is a large positive number and see Proposition 5.1). Then, the Riemann Hypothesis is equivalent to showing that the right hand side of (4) has analytic continuation to $\Im(\nu) < 0$ except $\nu = -\sqrt{-1}/2$. Under a strong assumption **(B)** we shall obtain the continuation and prove the Riemann Hypothesis (see Theorem 6.1).

Since $\eta(r)$ and $\eta_G(r)$ have a different growth order as $r \to \infty$ (cf. [E], Chap.9 and [H], Chap.6), we see that the distribution of $Prim$ and the one of norms of $Prim_\Gamma$ does not coincide. On the other hand we know that the prime number theorem that gives an approximation of the number of primes less than a given magnitude holds in an exactly same form for both $Prim$ and $Prim_\Gamma$ (cf. [E],Chap.4 and [H], Chap.2). Therefore, according to these facts we can believe that two distributions of $Prim$ and $Prim_\Gamma$ are similar in their norms. Actually, our strong assumption **(B)** expresses a similarity in the following fashion: there exists an injective map

$$\omega : Prim \quad \longrightarrow \quad Prim_\Gamma \tag{5}$$

for which $\log N(\omega(p)) \leq 1/4 \log p$ or $\log N(\omega(p)) \leq \log p$ and the distance $\delta(p)$ between $\log N(\omega(p))$ and the nearest element being of the form $\log N(\omega(q))$ ($q \in Prim$) is bounded below by $\sigma(\log N(\omega(p)))^{-\theta}$ for positive constants $\sigma$ and $\theta$, roughly speaking, $\log N(\omega(p)) \leq \log p$ for almost all $p \in Prim$, but, if $\delta(p)$ is sufficiently small like in the case of twin prime elements, it must be $\log N(\omega(p)) \leq 1/4 \log p$. At present we have no idea to find a discrete subgroup $\Gamma$ of $SL(2, \mathbf{R})$ satisfying this property, however, we have enough reason to believe that a similarity between $Prim$ and $Prim_\Gamma$ deduces the Riemann Hypothesis.

## 2. Notations

Let $G = SL(2, \mathbf{R})$ and let $\chi$ be the trivial character of $\Gamma$. Then $\rho_0 = 1/2$ and the explicit form of $\eta_G$ is given by

$$\eta_G(r) = \sum_{i\geq 1}\sum_{n\geq 1} \frac{u_i/2}{\sinh(nu_i/2)} e^{-\sqrt{-1}nu_i r}, \tag{6}$$

where $u_i = u_{\delta_i}$, and in (3) $c_{in} = n$ and

$$b_{in}{}^{-1} = 2u_i^{-1}\sinh(nu_i/2) \leq ce^{nu_i/2}. \tag{7}$$

For general references to the basic properties of $\eta_G$ see [G1], [H] and [S]. We denote the increasing sequence of prime numbers as $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ and the one of the norms of elements in $Prim_\Gamma$ as $N(\delta_1), N(\delta_2), N(\delta_3), \ldots$ respectively. We define $u_i = \log N(\delta_i)$ and

$$\delta_{in} = \frac{1}{2} \inf_{\substack{(m,j)\in \mathbf{N}^2 \\ (m,j)\neq(n,i)}} |nu_i - mu_j| \tag{8}$$

for $i \geq 1$ and $n \geq 1$. Then, each $\delta_{in}$ is positive, because $\{u_i; i \geq 1\}$ does not have a finite point of accumulation (see [G2], p.415). Moreover, it is easy to see that there exists a positive constant $C$ such that for each $\alpha \geq 0$ and $\beta \geq 1$

$$\varepsilon_{in} = \varepsilon_{in}(\alpha, \beta, C) = Ce^{-\alpha n(\log p_i)}e^{-\beta nu_i} \leq \delta_{in} \tag{9}$$

for all $i$ and $n \geq 1$. We fix such a pair of $\alpha$ and $\beta$ till the end of §4.

As said in §1, the Riemann Hypothesis holds for $\eta_G$. Actually, the poles of $\eta_G$ are all simple and are as

$$\{\nu_j; j \in \mathbf{Z}\} \cup \{r_j; 1 \leq j \leq 2M\}, \tag{10}$$

where $\nu_j \in \mathbf{R}$ and $r_j \in \sqrt{-1}\mathbf{R}$ (cf. [G1], Proposition 2.7 and [H], p.68). Then it is known that $\nu_{-j} = -\nu_j$ and the poles of $\eta_G$ which concentrate along $[-\sqrt{-1}/2, \sqrt{-1}/2]$ can be denoted as

$$\{\nu_0, r_j, \bar{r}_j; 1 \leq j \leq M\}, \tag{11}$$

where we let $r_1, r_2, \ldots, r_M$ be the poles of $\eta_G$ which concentrate along $[-\sqrt{-1}/2, 0)$ and $\bar{r}_j = -r_j = r_{j+M}$. We denote the residues of $\eta_G$ at $\nu_j$ and $r_j$ by $n_j$ and $m_j$ respectively. Then, $n_{-j} = n_j$ and $m_j = m_{j+M} = 1$ for $1 \leq j \leq M$ (cf. [H], Chap.2).

We fix sufficiently small (resp. large) positive numbers $\varepsilon$ and $\delta$ (resp. $E$), and a positive number $y$ such that $1/2 < y \leq 1/2 + \varepsilon$.

## 3. Transition from $\eta_G$ to $\eta$

Let $\phi$ be a $C^\infty$ compactly supported function on $\mathbf{R}$ satisfying

$$\begin{aligned} &(i) \quad supp(\phi) \subset (-1,1), \\ &(ii) \quad \phi(0) = 1, \\ &(iii) \quad \phi^{(k)}(0) = 0 \quad (1 \leq k \leq 2M) \end{aligned} \tag{12}$$

and let

$$h_{in}(t) = \frac{a_{in}}{b_{in}} \phi(\frac{t - n(\log p_i)}{\varepsilon_{in}}) \quad (t \in \mathbf{R}) \tag{13}$$

for $i \geq 1$ and $n \geq 1$. Then it is easy to see that $h_{in}$ satisfies the following conditions.

$$\begin{aligned} &(i) \quad supp(h_{in}) \subset (n(\log p_i) - \varepsilon_{in}, n(\log p_i) + \varepsilon_{in}), \\ &(ii) \quad h_{in}(n(\log p_i)) = \frac{a_{in}}{b_{in}}, \\ &(iii) \quad h_{in}^{(k)}(n(\log p_i)) = 0 \quad (1 \leq k \leq 2M). \end{aligned} \tag{14}$$

Without loss of generality we may assume that $\varepsilon_{11} \leq 1/2 \log 2$ and thus, $supp(h_{in}) \subset [1/2 \log 2, \infty)$ for all $i$ and $n \geq 1$. Here we put $\hat{h}_{in}(x) = (2\pi)^{-1} \int_{\mathbf{R}} h_{in}(z) e^{-\sqrt{-1}xz} dz$ and

$$H(\nu, x) = \sum_{i,n \geq 1} e^{\sqrt{-1}(nu_i - n(\log p_i))x} \hat{h}_{in}(\nu - x) \tag{15a}$$

$$= \sum_{i,n \geq 1} e^{-\sqrt{-1}(n(\log p_i)\nu - nu_i x)} \frac{a_{in}}{b_{in}} \varepsilon_{in} \hat{\phi}(\varepsilon_{in}(\nu - x)). \tag{15b}$$

We now consider a condition for which the series (15) converges. For $\theta \geq 0$ and $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$ we suppose that $\nu$ and $x$ satisfy

$$\begin{aligned} &(a_E) \qquad -E \leq \Im(\nu), \Im(x) \leq E, \\ &(b_\theta^{p,q}) \quad \begin{cases} \Im(\nu) - 1/2 - (1-\theta)\alpha \leq -1/p - \delta \\ -\Im(x) + 1/2 - (1-\theta)\beta \leq -1/q - \delta, \end{cases} \end{aligned}$$

where $\delta$ is a fixed sufficiently small positive number (see §2). Then, substituting the definition of $a_{in}$ and $b_{in}$ (see (1) and (7)) for (15b), we see that $|\nu - x|^\theta |H(\nu, x)|$ is dominated by

$$c \sum_{i,n \geq 1} \log p_i e^{(\Im(\nu) - 1/2)n(\log p_i)} e^{(-\Im_A(x) + 1/2)nu_i} \varepsilon_{in}^{1-\theta} |(\varepsilon_{in}(\nu - x))^\theta \hat{\phi}(\varepsilon_{in}(\nu - x))|. \tag{16}$$

Since $\hat{\phi}$ is rapidly decreasing and is holomorphic of exponential type $\leq 1$ (cf. [Su], p.146), for each $N \in \mathbf{N}$ there exists $C_N > 0$ for which

$$|\hat{\phi}(x)| \leq C_N(1+|x|)^{-N} e^{|\Im(x)|} \quad (x \in \mathbf{C}). \tag{17}$$

Therefore, it follows from (9) and $(a_E)$ that $|\nu - x|^\theta |H(\nu, x)|$ is dominated by

$$cC^{1-\theta} C_{[\theta]+1} e^{2EC} \sum_{i,n \geq 1} \log p_i e^{(\Im(\nu)-1/2-(1-\theta)\alpha)n(\log p_i)} e^{(-\Im(x)+1/2-(1-\theta)\beta)nu_i}, \tag{18}$$

where $[\theta]$ is the greatest integer not exceeding $\theta$. Then, this series converges absolutely and uniformly by $(b_\theta^{p,q})$ and the Hölder's inequality.

**Lemma 3.1.** *If $\nu$ and $x$ satisfy $(a_E)$ and $(b_0^{p,q})$, then the series $H(\nu, x)$ converges absolutely and uniformly, and is holomorphic of $\nu$ and $x$. Moreover, if $(b_\theta^{p,q})(\theta \geq 0)$ is satisfied, there exists a positive constant $C$ such that*

$$|H(\nu, x)| \leq C|\nu - x|^{-\theta}.$$

Throughout this paper we assume the following condition:

**(A)** There exists a positive constant $A$ such that

$$u_i \leq A \log p_i \quad \text{for all } i \geq 1.$$

Then we can replace $(b_\theta^{p,q})$ with

$$(b_{\theta,\gamma}^{p,q}) \quad \begin{cases} \Im(\nu) - 1/2 - (1-\theta)\alpha + \gamma \leq -1/p - \delta, \\ -\Im(x) + 1/2 - (1-\theta)\beta - \gamma/A \leq -1/q - \delta, \end{cases}$$

where $\gamma \geq 0$. We fix such a $\gamma$.

We next let $-y \leq -y_0 \leq E$ and

$$(c_{\theta,\gamma,y_0}^{p,q}) \quad \begin{cases} \Im(\nu) - 1/2 - (1-\theta)\alpha + \gamma \leq -1/p - \delta, \\ y_0 + 1/2 - (1-\theta)\beta - \gamma/A \leq -1/q - \delta. \end{cases}$$

Then, if $\nu$ satisfies $(a_E)$ and $(c_{\theta+1,\gamma,y_0}^{p,q})(\theta \in \mathbf{N})$, it follows similarly as above that

$$\int_{\mathbf{R}-\sqrt{-1}y_0} |x|^\theta |H(\nu, x)| dx$$

$$\leq c \sum_{i,n \geq 1} \log p_i e^{(\Im(\nu)-1/2)n(\log p_i)} e^{(y_0+1/2)nu_i} \varepsilon_{in}^{-\theta} \left[ \varepsilon_{in} \int_{\mathbf{R}-\sqrt{-1}y_0} |(\varepsilon_{in}x)^\theta \hat{\phi}(\varepsilon_{in}(\nu - x))| dx \right]$$

and by letting $x = (x - \nu) + \nu$,

$$\leq cC^{-\theta} C_{\theta+2} e^{2EC} P_\theta(|\nu|) \sum_{i,n \geq 1} \log p_i e^{(\Im(\nu)-1/2+\theta\alpha+\gamma)n(\log p_i)} e^{(y_0+1/2+\theta\beta-\gamma/A)nu_i}, \tag{19}$$

where $P_\theta$ is a polynomial of degree $\theta$ with coefficients depending only on $\theta$. Then this series converges absolutely and uniformly by $(c_{\theta+1,\gamma,y_0}^{p,q})$ and the Hölder's inequality.

**Lemma 3.2.** *Let $\nu$ be in a compact set $S$ in the tube domain defined by $(a_E)$ and $(c^{p,q}_{\theta+1,\gamma,y_0})$ ($\theta \in \mathbf{N}$ and $-y \leq -y_0 \leq E$). Let $f$ be a function on $\mathbf{R} - \sqrt{-1}y_0$ such that $f(x) = O(|x|^\theta)$. Then, there exists a positive constant $C$ for which $\int_{\mathbf{R}-\sqrt{-1}y_0} |f(x)H(\nu,x)|dx \leq C$. Especially,*

$$T_{y_0}f(\nu) = \int_{\mathbf{R}-\sqrt{-1}y_0} f(x)H(\nu,x)dx$$

*is well-defined and is holomorphic of $\nu$ satisfying $(a_E)$ and $(c^{p,q}_{\theta+1,\gamma,y_0})$.*

**Proposition 3.3.** *Let $P$ be a polynomial of degree $k(0 \leq k \leq 2M)$ and $\nu$ satisfy $(a_E)$ and $(c^{p,q}_{k+1,\gamma,y})$. Then,*

$$\begin{aligned} (i) \quad P(\nu)\eta(\nu) &= T_y(P\eta_G)(\nu) \\ &= \int_{\mathbf{R}-\sqrt{-1}y} P(x)\eta_G(x)H(\nu,x)dx, \\ (ii) \quad 0 &= \int_{\mathbf{R}-\sqrt{-1}y} P(x)\eta_G(x)H(\nu,-x)dx. \end{aligned}$$

Proof. Since $\eta_G(x) = O(1)$ for $x \in \mathbf{R} - \sqrt{-1}y$ (see [H], Proposition 6.7) and $(c^{p,q}_{k+1,\gamma,y})$ implies $(c^{p,q}_{k+1,\gamma,-y})$, the right hand sides of $(i)$ and $(ii)$ are well-defined and are holomorphic of $\nu$ satisfying $(a_E)$ and $(c^{p,q}_{k+1,\gamma,y})$ (see Lemma 3.2). Therefore, we may suppose that $\Im(\nu) \leq -y$. Since $mu_j > 0$ for all $m, j \geq 1$, it follows that

$$\begin{aligned} &\int_{\mathbf{R}-\sqrt{-1}y} e^{-\sqrt{-1}mu_j x} H(\nu,x)dx \\ &= \int_{\mathbf{R}} e^{-\sqrt{-1}mu_j x} H(\nu,x)dx. \end{aligned}$$

Then, substituting the definition of $H(\nu,x)$ (see (15a)), we see formally that

$$\begin{aligned} &= \sum_{k,l\geq 1} \int_{\mathbf{R}} e^{-\sqrt{-1}mu_j x} e^{\sqrt{-1}(lu_k - l(\log p_k))x} \hat{h}_{kl}(\nu - x)dx \\ &= \sum_{k,l\geq 1} e^{-\sqrt{-1}(mu_j - lu_k + l(\log p_k))\nu} \int_{\mathbf{R}} e^{\sqrt{-1}(mu_j - lu_k + l(\log p_k))x} \hat{h}_{kl}(x)dx \\ &= \sum_{k,l\geq 1} e^{-\sqrt{-1}(mu_j - lu_k + l(\log p_k))\nu} h_{kl}(mu_j - lu_k + l(\log p_k)). \end{aligned}$$

Since each support of $h_{kl}$ is disjointed from the others, it is easy to see that the condition that $\Im(\nu) \leq -y$ guarantees the validity of the above calculation. Moreover, since the support of $h_{kl}$ is contained in $(l(\log p_k) - \varepsilon_{kl}, l(\log p_k) + \varepsilon_{kl})$ and $h_{kl}(l(\log p_k)) = a_{kl} b_{kl}^{-1}$ (see (14)($i$) and ($ii$)), it follows from (9) and the definition of $\delta_{kl}$ (see (7)) that

$$\begin{aligned} &= \epsilon_{kj}\epsilon_{lm} h_{kl}(l(\log p_k)) e^{-\sqrt{-1}l(\log p_k)\nu} \\ &= \epsilon_{kj}\epsilon_{lm} a_{kl} b_{kl}^{-1} e^{-\sqrt{-1}l(\log p_k)\nu}, \end{aligned}$$

where $\epsilon_{ij} = 1$ if $i = j$ and 0 otherwise. Therefore, we can deduce that

$$\begin{aligned} T_y \eta_G(\nu) &= \int_{\mathbf{R}-\sqrt{-1}y} \eta_G(x) H(\nu, x) dx \\ &= \sum_{j,m\geq 1} b_{jm} \int_{\mathbf{R}-\sqrt{-1}y} e^{-\sqrt{-1}m u_j x} H(\nu, x) dx \\ &= \sum_{j,m\geq 1} a_{jm} e^{-\sqrt{-1}m(\log p_j)\nu} \qquad (20) \\ &= \eta(\nu). \end{aligned}$$

Here we rewrite $P(\nu)$ as

$$P(\nu) = R_\nu(\nu - x) + P(x),$$

where $R_\nu$ is a polynomial of degree $k$ with coefficients depending only on $k$ and $\nu$. Then the formula ($i$) follows from (20) provided that

$$\int_{\mathbf{R}-\sqrt{-1}y} (\nu - x)^l \eta_G(x) H(\nu, x) dx = 0 \quad (1 \leq l \leq k). \qquad (21)$$

We now show (21). If we define $H^{(l)}(\nu, x)$ by replacing $h_{in}$ in (15a) with $(\sqrt{-1})^{-l} h_{in}^{(l)}$, we easily see that the left hand side of (21) is equal to

$$\int_{\mathbf{R}-\sqrt{-1}y} \eta_G(x) H^{(l)}(\nu, x) dx.$$

Obviously, this integral is finite by the condition $(c_{k+1,\gamma,y}^{p,q})$. Then, applying the same argument that deduces (20), especially, by using (14)($iii$) instead of (14)($ii$), we can show that this integral is equal to 0. The formula ($ii$) follows by the quite same way. $\square$

We now let $\varepsilon$ and $\delta$ (resp. $E$) sufficiently small (resp. large). Then, we can deduce the following,

**Corollary 3.4.** *The equations* ($i$) *and* ($ii$) *in Proposition 3.3 hold for* $\nu$ *satisfying*

$$\begin{cases} \Im(\nu) - 1/2 + k\alpha + \gamma < -1/p \\ 1 + k\beta - \gamma/A < -1/q, \end{cases}$$

*where* $\gamma \geq 0$, $1 \leq p, q \leq \infty$ *and* $1/p + 1/q = 1$.

## 4. A relation between $\eta$ and the poles of $\eta_G$

We keep the notations and the assumption (A). We first recall that $\eta_G$ satisfies the functional equation:

$$\eta_G(x) + \eta_G(-x) = cx \tanh \pi x \tag{22}$$

(see [H], Proposition 4.26). In this section we shall express $\eta$ as the sum of an integral of $x \tanh \pi x$ and the residues of $\eta_G$.

**Lemma 4.1.** *Let $P$ be a polynomial of degree $k(0 \le k \le 2M)$ and let $\nu$ be in a compact set $S$ satisfying $\Im(S) < 0$, $(a_E)$ and $(c^{p,q}_{k+6,\gamma,0})$. Then the series $\sum_{j\in\mathbf{Z}} n_j P(\nu_j)\, H(\nu,\nu_j)$ converges absolutely and uniformly. Especially, $\sum_{j\in\mathbf{Z}} n_j P(\nu_j)\, H(\nu,\nu_j)$ is well-defined and is holomorphic of $\nu$ satisfying $\Im(S) < 0$, $(a_E)$ and $(c^{p,q}_{k+6,\gamma,0})$.*

Proof. Since $\nu_j \in \mathbf{R}$ and $\nu \in S$, Lemma 3.1 implies that for $x \in \mathbf{R}$

$$|H(\nu,x)| \le C|\nu - x|^{-(k+6)} \sim (1+|x|)^{-(k+6)}.$$

Then, noting the fact that

$$\sum_{\{j;\nu_j^2 \le x\}} n_j \quad \sim \quad x^2 \quad (x \to \infty)$$

(see §2 and [G1], Proposition1.2), we see that

$$\begin{aligned}
&\sum_{j\in\mathbf{Z}} n_j |P(\nu_j) H(\nu,\nu_j)| \\
&\sim \sum_{j\in\mathbf{Z}} n_j (1+|\nu_j|)^{-6} \\
&\sim \sum_{k=0}^{\infty} \sum_{k \le |\nu_j| < k+1} n_j (1+|\nu_j|)^{-6} \\
&\sim \sum_{k=0}^{\infty} (1+k)^{-2} < \infty. \qquad \square
\end{aligned}$$

We now suppose that $\nu$ satisfies $\Im(\nu) < 0$, $(a_E)$ and $(c^{p,q}_{6,\gamma,y})$. We note that, if $|\Im(x)| \le \varepsilon$, then $x \tanh \pi x = O(|x|)$ and $\eta_G(x) = O(|x|)$ (see [H], Proposition 6.7). Therefore, since $(c^{p,q}_{6,\gamma,y})$ implies $(c^{p,q}_{2,\gamma,\pm\varepsilon})$ and $(c^{p,q}_{6,\gamma,0})$, it follows from Lemma 3.2 and Lemma 4.1 that

$$\begin{aligned}
&\int_{\mathbf{R}} cx \tanh \pi x H(\nu,x)dx \\
&= \int_{\mathbf{R}+\sqrt{-1}\varepsilon} cx \tanh \pi x H(\nu,-x)dx \\
&= \int_{\mathbf{R}+\sqrt{-1}\varepsilon} (\eta_G(x) + \eta_G(-x)) H(\nu,-x)dx \\
&= \int_{\mathbf{R}-\sqrt{-1}\varepsilon} \eta_G(x) H(\nu,x)dx + \int_{\mathbf{R}+\sqrt{-1}\varepsilon} \eta_G(x) H(\nu,-x)dx.
\end{aligned}$$

The second term is equal to

$$\int_{\mathbf{R}-\sqrt{-1}y} \eta_G(x)H(\nu,-x)dx - \sum_{j\in\mathbf{Z}} n_j H(\nu,\nu_j) - \sum_{1\le j\le M} H(\nu,-r_j)$$
$$= -\sum_{j\in\mathbf{Z}} n_j H(\nu,\nu_j) - \sum_{1\le j\le M} H(\nu,-r_j)$$

by Proposition 3.3($ii$). Therefore, it follows from Proposition 3.3 ($i$) that

$$\begin{aligned}\eta(\nu) &= \int_{\mathbf{R}-\sqrt{-1}y} \eta_G(x)H(\nu,x)dx \\ &= \int_{\mathbf{R}-\sqrt{-1}\varepsilon} \eta_G(x)H(\nu,x)dx + \sum_{1\le j\le M} H(\nu,r_j) \\ &= \int_{\mathbf{R}} cx\tanh\pi x H(\nu,x)dx + \sum_{j\in\mathbf{Z}} n_j H(\nu,\nu_j) + \sum_{1\le j\le 2M} H(\nu,r_j).\end{aligned}$$

Then, letting $\varepsilon$ and $\delta$ (resp. $E$) sufficiently small (resp. large), we can obtain the following,

**Proposition 4.2.** *If $\nu$ satisfies*

$$\begin{cases} \Im(\nu) < \min(0, 1/2 - 5\alpha - \gamma - 1/p) \\ 1 + 5\beta < \gamma/A - 1/q, \end{cases}$$

*where $\gamma \ge 0$, $1 \le p, q \le \infty$ and $1/p + 1/q = 1$, then*

$$\eta(\nu) = c\int_{\mathbf{R}} x\tanh\pi x H(\nu,x)dx + \sum_{j\in\mathbf{Z}} n_j H(\nu,\nu_j) + \sum_{1\le j\le 2M} H(\nu,r_j).$$

We put

$$P_G(x) = (\nu^2 - r_1^2)(\nu^2 - r_2^2)\dots(\nu^2 - r_M^2). \tag{23}$$

Then, replacing $\eta_G$ with $P_G\eta_G$, we can obtain the following proposition by the quite same way.

**Proposition 4.3.** *If $\nu$ satisfies*

$$\begin{cases} \Im(\nu) < \min(0, 1/2 - (5+2M)\alpha - \gamma - 1/p) \\ 1 + (5+2M)\beta < \gamma/A - 1/q, \end{cases}$$

*where $\gamma \ge 0$, $1 \le p, q \le \infty$ and $1/p + 1/q = 1$, then*

$$\begin{aligned}P_G(\nu)\eta(\nu) &= \int_{\mathbf{R}-\sqrt{-1}\varepsilon} \eta_G(x)P_G(x)H(\nu,x)dx \\ &= c\int_{\mathbf{R}} x\tanh\pi x P_G(x)H(\nu,x)dx + \sum_{j\in\mathbf{Z}} n_j P_G(\nu_j)H(\nu,\nu_j).\end{aligned}$$

## 5. Some modifications

5.1. In the proof of Proposition 3.3 each term $b_{in}e^{-\sqrt{-1}nu_i r}$ of $\eta_G(r)$ ($u_i = \log N(\delta_i)$) transfers to $a_{in}e^{-\sqrt{-1}n(\log p_i)r}$ of $\eta(r)$ under the integral formula. Obviously, to verify such an integral formula $\delta_i$'s need not be taken over all elements in $Prim_\Gamma$, and it is enough for each $p_i$ to correspond to a unique element $\delta_{\omega(i)}$ in $Prim_\Gamma$. Actually, for an injective map

$$\omega : \mathbf{N} \quad \rightarrow \quad \mathbf{N}$$

we put

$$\delta_{in} = \frac{1}{2} \inf_{\substack{(m,j)\in\mathbf{N}^2 \\ (m,\omega(j))\neq(n,\omega(i))}} |nu_{\omega(i)} - mu_{\omega(j)}|, \tag{24}$$

$$\varepsilon_{in}^{\omega} = \varepsilon_{in}^{\omega}(\alpha,\beta,C) = Ce^{-\alpha n(\log p_i)}e^{-\beta nu_{\omega(i)}}, \tag{25}$$

$$h_{in}^{\omega} = \frac{a_{in}}{b_{\omega(i)n}}\phi(\frac{t-n(\log p_i)}{\varepsilon_{in}^{\omega}}) \quad (t \in \mathbf{R}), \tag{26}$$

$$H_\omega(\nu,x) = \sum_{i,n\geq1} e^{\sqrt{-1}(nu_{\omega(i)}-n(\log p_i))x}\hat{h}_{in}^{\omega}(\nu - x) \tag{27}$$

(cf. (8), (9), (13) and (15)). Then it is easy to see that all results in the preceding sections are also valid when we replace $\delta_{in}, \varepsilon_{in}, h_{in}$ and $H(\nu,x)$ by $\delta_{in}^{\omega}, \varepsilon_{in}^{\omega}, h_{in}^{\omega}$ and $H_\omega(\nu,x)$ respectively and **(A)** by

**(A)**$_\omega$ There exists a positive constant $A$ such that

$$u_{\omega(i)} \leq A\log p_i \quad \text{for all } i \geq 1.$$

5.2. We next modify the $\eta$ functions. Let

$$\eta^\circ(r) = \sum_{i\geq1} a_i e^{-\sqrt{-1}(\log p_i)r}, \tag{28}$$

where $a_i = (\log p_i)e^{-(\log p_i)/2}$, and let

$$\eta_G^\circ(r) = \sum_{i\geq1} b_i e^{-\sqrt{-1}u_i r}, \tag{29}$$

where $b_i = u_i/2\sinh(u_i/2)$. Then, it is easy to see that $\eta(r) - \eta^\circ(r)$ and $\eta_G(r) - \eta_G^\circ(r)$ are holomorphic on $\Im(r) < 0$ (cf. [H], Proposition 3.5). Therefore, in order to prove the Riemann Hypothesis for $\eta$ it is enough to prove it for $\eta^\circ$. Since $\eta^\circ$ and $\eta_G^\circ$ inherit all singularities from $\eta$ and $\eta_G$ respectively, the whole arguments in the previous sections except one using the functional equation (22) are also applicable to $\eta^\circ$ and $\eta_G^\circ$. Especially, if we define $\delta_i^\omega$, $\varepsilon_i^\omega(\alpha,\beta,C)$, $h_i^\omega$ and $H_\omega^\circ(\nu,x)$ by eliminating the sufix $n$ in (24)-(27) respectively, we see that all the results in §2 and §3 are also valid when we replace $\eta, \eta_G$ and $H$ by $\eta^\circ, \eta_G^\circ$ and $H_\omega^\circ$ respectively and **(A)** by **(A)**$_\omega$.

5.3. We now let

$$\omega : D \quad \to \quad \mathbf{N}, \quad D \subset \mathbf{N}$$

be an injective map, and for each $i \in D$ we define $\delta_i^\omega, \varepsilon_i^\omega(\alpha,\beta,C)$ and $h_i^\omega$ as above. Moreover, we put

$$\eta_\omega^\circ(r) = \sum_{i \in D} a_i e^{-\sqrt{-1}(\log p_i)r}, \tag{30}$$

$$H_\omega^\circ(\nu, x) = \sum_{i \in D} e^{\sqrt{-1}(nu_{\omega(i)} - n(\log p_i))x} \hat{h}_{in}^\omega(\nu - x) \tag{31}$$

and we define the corresponding assumption $(\mathbf{A})_\omega$, we denote by the same letter, by replacing $i \geq 1$ with $i \in D$. Then repeating the same arguments in §3, especially, taking $\gamma$ sufficiently large in Corollary 3.4 and Proposition 4.3, we can deduce that

**Proposition 5.1.** *Let us suppose that* $(\mathbf{A})_\omega$ *holds. Then there exists a positive constant* $L$ *such that if* $\Im(\nu) \leq -L$,

$$(i) \quad \eta_\omega^\circ(\nu) = \int_{\mathbf{R}-\sqrt{-1}y} \eta_G^\circ(x) H_\omega^\circ(\nu, x)dx,$$

$$(ii) \quad P_G(\nu)\eta_\omega^\circ(\nu) = \int_{\mathbf{R}-\sqrt{-1}\varepsilon} P_G(x)\eta_G(x) H_\omega^\circ(\nu, x)dx.$$

## 6. A proof of the Riemann Hypothesis under an assumption

We retain the notations in the previous sections. We here make an assumption on magnitude and distance of $u_i (i \in \mathbf{N})$, which is stronger than $(\mathbf{A})$, and then give a proof of the Riemann Hypothesis. The assumption can be stated as follows.

$(\mathbf{B})$ There exist an injective map $\omega : \mathbf{N} \to \mathbf{N}$ and positive constants $\sigma$ and $\theta$ for which, except a finite number of $i$, one of the following conditions holds:

$$(B1) \quad u_{\omega(i)} \leq 1/4 \log p_i,$$

$$(B2) \quad u_{\omega(i)} \leq \log p_i \quad \text{and} \quad \sigma u_{\omega(i)}^{-\theta} \leq \delta_i^\omega.$$

We here put $D_\ell = \{i \in \mathbf{N};\ (B\ell) \text{ holds}\}$ for $\ell = 1,2$ and $D_3 = \mathbf{N} - D_1 \cup D_2$. In what follows for each $\omega_\ell = \omega|_{D_\ell} (\ell = 1,2,3)$ we shall prove that $P_G(\nu)\eta_{\omega_\ell}^\circ(\nu)$ $(\ell = 1,2,3)$ (see (30)) is holomorphic on $-2L \leq \Im(\nu) \leq -3\varepsilon$.

$\eta_{\omega_1}^\circ$: Since $(B1)$ implies $(\mathbf{A})_{\omega_1}$ (see 5.3), it follows from Proposition 5.1 that

$$\eta_{\omega_1}^\circ(\nu) = \int_{\mathbf{R}-\sqrt{-1}y} \eta_G^\circ(x) H_{\omega_1}^\circ(\nu, x)dx, \tag{32}$$

if $\Im(\nu) \leq -L$. We now recall the definition of $\varepsilon_i^{\omega_1}$ (see 5.3 and (9)). Then, we can choose a sufficiently small positive number $\tau$ depending on $\varepsilon$ such that

$$\sum_{i \in D_1} e^{-(1+3\varepsilon)u_{\omega_1(i)}} (\varepsilon_i^{\omega_1})^{-\tau} < \infty. \tag{33}$$

Then, by $(B1)$ and the argument used in (16)-(18) we see that if $-2L \le \Im(\nu) \le -2\varepsilon$ and $\Im(x) = -y = -1/2 - \varepsilon$,

$$\begin{aligned}
|H^\circ_{\omega_1}(\nu, x)| &\le c \sum_{i \in D_1} \log p_i e^{(-2\varepsilon - 1/2)\log p_i} e^{(\varepsilon+1)u_{\omega_1(i)}} (\varepsilon_i^{\omega_1})^{-\tau} |\nu - x|^{-(1+\tau)} \\
&\le c|\nu - x|^{-(1+\tau)} \sum_{i \in D_1} e^{-(1+3\varepsilon)u_{\omega_1(i)}} (\varepsilon_i^{\omega_1})^{-\tau} \\
&\le c|\nu - x|^{-(1+\tau)} \qquad \text{by (33).}
\end{aligned}$$

Since $\eta^\circ_G(x) = O(1)$ for $x \in \mathbf{R} - \sqrt{-1}y$ (see [H], Theorem 3.10), the above estimate and (32) give an analytic continuation of $\eta^\circ_{\omega_1}(\nu)$ on $-2L \le \Im(\nu) \le -2\varepsilon$.

$\eta^\circ_{\omega_2}$: In the previous sections $\varepsilon_i^\omega = \varepsilon_i^\omega(\alpha, \beta, C)$ (see 5.3 and (9)) is defined for $\alpha \ge 0$ and $\beta \ge 1$. However, under the second condition of $(B2)$ we may take $\varepsilon_i^{\omega_2} = \sigma u_{\omega_2(i)}^{-\theta}$ and easily see that all arguments in the previous sections are valid for $\varepsilon_i^{\omega_2}$, $h_i^{\omega_2}$ and $H^\circ_{\omega_2}$, especially, it follows that

$$P_G(\nu)\eta^\circ_{\omega_2}(\nu) = \int_{\mathbf{R} - \sqrt{-1}\varepsilon} P_G(x)\eta^\circ_G(x) H^\circ_{\omega_2}(\nu, x)dx, \tag{34}$$

if $\Im(\nu) \le -L$ (see Proposition 5.1). We here put $J_0 = \{i \in D_2; 1 \le \varepsilon_i^{\omega_2}\}$ and $J_n = \{i \in D_2; 2^{-n} \le \varepsilon_i^{\omega_2} < 2^{-(n-1)}\}$ $(n = 1, 2, \dots)$. Moreover, we denote by $i_n$ the number in $J_n$ for which $\omega_2(i_n)$ is the smallest in $\omega_2(j)(j \in J_n)$ and by $k_n(i)$ $(i \in J_n)$ the number of elements $j$ in $J_n$ satisfying $\omega_2(j) < \omega_2(i)$. Then for each $i \in J_n$ we see from the definition of $\delta_i^{\omega_2}$ (see 5.3 and (8)) and $(B2)$ that $u_{\omega_2(i)} \ge u_{\omega_2(i_n)} + 2\sum_{j \in J_n, \omega_2(j) < \omega_2(i)} \delta_j^{\omega_2} \ge u_{\omega_2(i_n)} + 2k_n(i)2^{-n}$ for $n \ge 0$ and $u_{\omega_2(i_n)} \ge \sigma^{1/\theta} 2^{(n-1)/\theta}$ for $n \ge 1$. Therefore, by $(B2)$ and the argument used in (16)-(18) we see that if $-2L \le \Im(\nu) \le -3\varepsilon$ and $\Im(x) = -\varepsilon$,

$$\begin{aligned}
|H^\circ_{\omega_2}(\nu, x)| &\le c \sum_{i \in D_2} \log p_i e^{(-3\varepsilon - 1/2)\log p_i} e^{(\varepsilon+1/2)u_{\omega_2(i)}} (\varepsilon_i^{\omega_2})^{1-(2M+3)} |\nu - x|^{-(2M+3)} \\
&\le c|\nu - x|^{-(2M+3)} \sum_{n=0}^{\infty} \sum_{i \in J_n} e^{-\varepsilon u_{\omega_2(i)}} (\varepsilon_i^{\omega_2})^{-2(M+1)} \\
&\le c|\nu - x|^{-(2M+3)} (e^{-\varepsilon u_{\omega_2(i_0)}} \sum_{i \in J_0} e^{-2\varepsilon k_0(i)} \\
&\qquad + \sum_{n=1}^{\infty} e^{-\varepsilon \sigma^{1/\theta} 2^{(n-1)/\theta}} 2^{2n(M+1)} \sum_{i \in J_n} e^{-2\varepsilon k_n(i) 2^{-n}}) \\
&\le c|\nu - x|^{-(2M+3)} (\frac{1}{1 - e^{-2\varepsilon}} + \sum_{n=1}^{\infty} \frac{e^{-\varepsilon \sigma^{1/\theta} 2^{(n-1)/\theta}} 2^{2n(M+1)}}{1 - e^{-2\varepsilon 2^{-n}}}) \\
&\le c|\nu - x|^{-(2M+3)}.
\end{aligned}$$

Since $P_G(x)\eta^\circ_G(x) = O(|x|^{2M+1})$ for $x \in \mathbf{R} - \sqrt{-1}\varepsilon$ (see (23) and [H], Remark 6.8), the above estimate and (34) give an analytic continuation of $\eta^\circ_{\omega_2}(\nu)$ on $-2L \le \Im(\nu) \le -3\varepsilon$.

$\eta^\circ_{\omega_3}$: Since $D_3$ is finite, $\eta^\circ_{\omega_3}$ is holomorphic on the whole complex plane.

We now obtained that each $P_G(\nu)\eta^\circ_{\omega_\ell}(\nu)$ ($\ell = 1,2,3$) has an analytic continuation on $-2L \le \Im(\nu) \le -3\varepsilon$. Therefore, $P_G(\nu)\eta^\circ(\nu) = \sum_{\ell=1}^{3} P_G(\nu)\eta^\circ_{\omega_\ell}(\nu)$ and thus, $P_G(\nu)\eta(\nu)$ have the same property (see 5.2). Since $\varepsilon$ can be taken sufficiently small and $\eta$ satisfies the functional equation (see [E], p.13), it follows that $P_G(\nu)\eta(\nu)$ is holomorphic on $0 < |\Im(\nu)| \le 2L$. Then, noting the zeros of $P_G(\nu)$ (see (23) and (11)) and the fact that that $\zeta(s)$ has no zeros on $[0,1]$, we can finally obtain the following theorem.

**Theorem 6.1.** *If $SL(2,\mathbf{R})$ has a cocompact discrete subgroup $\Gamma$ with $Prim_\Gamma$ satisfying the condition* (B), *then the Riemann Hypothesis holds.*

**Remark 6.2.** We see that $D_2 \neq \emptyset$. Actually, if $D_1 \cup D_3 = \mathbf{N}$, it follows from the above argument that $\eta^\circ(\nu)$ is holomorphic on $\Im(\nu) < 0$. This contradicts to the fact that $\eta(\nu)$ has a pole at $\nu = -\sqrt{-1}/2$.

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