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An inverse problem for 1-dimensional heat equations

Tetsuya Hattori

1 Introduction

In this note we study the uniqueness in an inverse problem for 1-dimensional heat equations.

For $p \in C^1[0,1]$ and $a \in L^2(0,1)$, both of which are real-valued, let $(E_{p, a})$ be the heat equation

\begin{equation}
\frac{\partial u}{\partial t} + (p(x) - \frac{\partial^2}{\partial x^2})u = 0 \quad (0 < x < 1, 0 < t < \infty),
\end{equation}

with the Dirichlet boundary condition

\begin{equation}
|_{x=0} = |_{x=1} = 0 \quad (0 < t < \infty),
\end{equation}

and the initial condition

\begin{equation}
|_{t=0} = a(x) \quad (0 < x < 1).
\end{equation}

Let $u = u(t, x)$ be a unique solution of $(E_{p, a})$. Fix $x_0 \in (0, 1]$ and $T_1, T_2$ such that $0 \leq T_1 < T_2 < \infty$. Our problem is to study to what extent the "observation" $\{(u_x(t, 0), u_x(t, x_0)); T_1 \leq t \leq T_2\}$ determines the potential $p$ and the initial data $a$. To formulate this problem, we define the map $\chi_{x_0}$ by
\[(1.4)\quad \chi_{x_{0}}: (p, a) \mapsto \{(u_{x}(t, 0), u_{x}(t, x_{0})); T_{1} \leq t \leq T_{2}\},\]

and the set \(M_{p, a, x_{0}}\) by

\[(1.5)\quad M_{p, a, x_{0}} = \{(q, b) \in C^{1}[0, 1] \times L^{2}(0, 1); \chi_{x_{0}}(q, b) = \chi_{x_{0}}(p, a)\}.

Then the observation determines uniquely \((p, a)\) if and only if

\[(1.6)\quad M_{p, a, x_{0}} = \{(p, a)\}.

Remark 1.1. We can replace the time interval \([T_{1}, T_{2}]\) by \((0, \infty)\) in \(1.4\) because of the analyticity of \(u(t, x)\) with respect to \(t \in (0, \infty)\).

Let \(A_{p}\) denote the self-adjoint realization in \(L^{2}(0, 1)\) of \(p(x) - \partial^{2}/\partial x^{2}\) with the Dirichlet boundary condition. The eigenvalues and the eigenfunctions of \(A_{p}\) are denoted by \(\{\lambda_{n}\}\) and \(\{\varphi_{n}\}\), respectively, the latter being normalized as \(\|\varphi_{n}\|_{L^{2}(0, 1)} = 1\).

Definition 1.1. For \(a \in L^{2}(0, 1)\), the number

\[(1.7)\quad N_{p, a} = \#\{n; (a, \varphi_{n})_{L^{2}(0, 1)} = 0\}\]

is called the degenerate number of \(a\) with respect to \(A_{p}\).

The problem of uniqueness \(1.6\) is closely related to the degenerate number. In fact, Murayama [1] obtained the following result.

Theorem 0.1. (Murayama) If \(x_{0} = 1\), the observation determines \((p, a)\)
uniquely if and only if $N_{p,a} = 0$. 

One can also study the inverse problem for (1.1) with the Robin boundary condition:

$$\frac{\partial u}{\partial x} - hu|_{x=0} = \frac{\partial u}{\partial x} + Hu|_{x=1} = 0.$$ 

In this case, we aim at determining $p, h, H$ and $a$ through the observation $\{u(t,0), u(t,x_0); T_1 \leq t \leq T_2\}$. Then Suzuki [4] obtained the following result. 

**Theorem 0.2. (Suzuki)** In the case of the Robin boundary condition, the observation determines $p, h, H$ and $a$ uniquely if and only if $x_0 = 1$ and the degenerate number is equal to 0.

The above two theorems suggest that the uniqueness depends on not only $N_{p,a}$ but also the position of $x_0$. The aim of this paper is to show that, in the case of the Dirichlet boundary condition, generically, the uniqueness does not hold if $0 < x_0 < 1$.

A reduction is necessary before going into the details. By the same argument as in Suzuki [4], one can show that, if $(q, b) \in M_{p,a,x_0}$, $b$ is uniquely determined by $q$. So, if we let

(1.8) $\tilde{M}_{p,a,x_0} = \{q \in C^1[0,1]; \text{there exists some } b \in L^2(0,1) \text{ such that } (q, b) \in M_{p,a,x_0}\}$

(1.6) is equivalent to

(1.9) $\tilde{M}_{p,a,x_0} = \{p\}$. 

2 Main results

Our results are summarized in the following two theorems.

**Theorem 1.** For each $x_0 \in (0, 1)$, there exists an open dense set $U_{x_0} \subseteq C^1[0,1]$ such that $p \in U_{x_0}$ implies $\tilde{M}_{p,a,x_0} \neq \{p\}$ for any $a \in L^2(0,1)$. In particular, when $x_0 \in (0, \frac{1}{2})$, we can choose $U_{x_0} = C^1[0,1]$.

**Remark 2.1.** Let $H = \{\frac{2k}{2k+1}; k \in \mathbb{N}\}$. For $x_0 \in (0,1) \setminus H$, $U_{x_0}$ contains all the constant functions. In other words, if $x_0 \in (0,1) \setminus H$ and $p$ is a constant function, then $\tilde{M}_{p,a,x_0} \neq \{p\}$ for any $a \in L^2(0,1)$.

**Theorem 2.** Let $p$ be constant and $N_{p,a} = 0$.

(i) In the case of $x_0 \in (\frac{1}{2}, 1)$, let

\[ R_1 = \{ q \in C^1[0,1]; q'(x_0) + q'(1) \leq 0 \}. \]

Then $R_1 \cap \tilde{M}_{p,a,x_0} = \{p\}$.

(ii) In the case of $x_0 = \frac{1}{2}$, let

\[ R_2 = \{ q \in C^1[0,1]; q'(x_0) + q'(0) \geq 0 \}. \]

Then $R_2 \cap \tilde{M}_{p,a,x_0} = \{p\}$.

(iii) In the case of $x_0 \in (0, \frac{1}{2})$, let

\[ R_3 = R_2 \cap \{ \text{the real analytic functions on } (0,1) \}. \]

Then $R_3 \cap \tilde{M}_{p,a,x_0} = \{p\}$. 

By Theorem 1, the uniqueness does not hold generically if $0 < x_0 < 1$.
And, by the above theorems, it follows that there exists a potential which has the same observation in $C^1[0,1] \setminus R_1$ if $p$ is constant, $N_{p,a} = 0$, and $x_0 \in (\frac{1}{2}, 1) \setminus H$. In the case of $x_0 = \frac{1}{2}$ or $x_0 \in (0, \frac{1}{2})$, the above statement holds for $R_2$ or $R_3$ instead of $R_1$, respectively.

3 A hyperbolic equation

The following propositions, which arise from Suzuki's deformation formula ([3] or [4]), are the key points of the proof of Theorems 1 and 2.

Let $D = \{(x,y) \in \mathbb{R}^2; 0 < y < x < 1\}$, and consider the following equations:

\[
\begin{align*}
(E) & \quad \begin{cases}
(3.1) & K_{xx} - K_{yy} + (p(y) - q(x))K = 0 \quad \text{on } D, \\
(3.2) & K(x,x) = \frac{1}{2} \int_0^x (q(s) - p(s))ds \quad (0 \leq x \leq 1), \\
(3.3) & K(x,0) = 0 \quad (0 \leq x \leq 1), \\
(3.4) & K(1,y) = 0 \quad (0 \leq y \leq 1), \\
(3.5) & K_x(x_0,y) = 0 \quad (0 \leq y \leq x_0), \\
(3.6) & K(x_0,x_0) = 0.
\end{cases}
\end{align*}
\]

Proposition 1. If there exist $q \in C^1[0,1]$ and $K \in C^2(\bar{D})$ such that $K$ does not vanish identically on $\bar{D}$ and satisfies the equation $(E)$, then $\tilde{M}_{p,a,x_0} \neq \{p\}$ for any $a \in L^2(0,1)$.

Remark 3.1. For $q \in C^1[0,1]$ in Proposition 1, $q \in \tilde{M}_{p,a,x_0}$ and $q \neq p$ holds.
Proposition 2. When $N_{p,a} = 0$, $q \in \tilde{M}_{p,a,x_0}$ if and only if there exists $K \in C^2(\bar{D})$ satisfying $(E)$.

We can show these propositions in the same way as in [4].

4 Proof of theorems

Sketch of proof of Theorem 2.

If $x_0 \in (\frac{1}{2}, 1)$, we see that $q \in \tilde{M}_{p,a,x_0}$ implies $q'(x_0) + q'(1) = \int_{x_0}^1 (q-p)^2 dx$ by Proposition 2 and a straightforward calculation. Therefore, $q \in R_1 \cap \tilde{M}_{p,a,x_0}$ implies $q \equiv p$ on $[x_0, 1]$, i.e. $K(x, x) = 0$ for $x \in [x_0, 1]$. By solving $(E)$, we get $K \equiv 0$ on $\bar{D}$, so $K(x, x) = 0$ for $x \in [0, 1]$. From (3.2), $q \equiv p$ on $[0, 1]$.

If $x_0 \in (0, \frac{1}{2}]$, by Proposition 2 we see that $q \in \tilde{M}_{p,a,x_0}$ implies $q'(x_0) + q'(0) = -\int_{0}^{x_0} (q-p)^2 dx$. We then proceed in the same way as above.

Proof of Theorem 1.

(I) The case of $x_0 \in [\frac{1}{2}, 1)$.

Let $G = \{g \in C^1[x_0, 1]; g'(x_0) = g(1) = 0\}$.

< Step 1 > For $p, q \in C^1[0, 1]$ and $g \in G$, we construct $K \in C^2(\bar{D})$ satisfying (3.1), (3.3), (3.4), (3.5) and

\begin{equation}
K_y(x, 0) = g \quad (x_0 \leq x \leq 1).
\end{equation}

This $K$ is constructed as follows. We divide $D$ into the pieces $D_0$, $D_1$, ..., $D_{2m+2}$, $\bar{D}$ (Figure 1) and solve the equation successively. Here, $g'(x_0) = g(1) = 0$ serves
as a compatibility condition for the $C^2$-regularity of $K$. ([4])

Notation. $K$ in Step 1 is denoted by $K_g(x, y; q, p)$. In particular, when $p$ is fixed, $K$ is denoted by $K_g(x, y; q)$.

Remark 4.1.

(1) $K_g$ is a $C^2(\overline{D})$-valued analytic function of $q, g$ and $p$.

(2) $K$ is linear with respect to $g$.

(3) There exists a monotone increasing continuous function

\[ \tau : [0, \infty) \to (0, \infty) \]

such that

\[ \| K_g(\cdot, \cdot; p, q) \|_{C^2(\overline{D})} \leq \tau(\| p \|_{C^1[0,1]} + \| q \|_{C^1[0,1]} + \| g \|_{C^1[x_0,1]} \]

\[ \| K_g(\cdot, \cdot; p_1, q_1) - K_g(\cdot, \cdot; p_2, q_2) \|_{C^2(\overline{D})} \]

\[ \leq \tau(\| p \|_{C^1[0,1]} + \| q \|_{C^1[0,1]})(\| p_1 - p_2 \|_{C^1[0,1]} + \| q_1 - q_2 \|_{C^1[0,1]} + \| g \|_{C^1[x_0,1]} \]

for any $p, q \in C^1[0,1]$ and any $g \in G$. ([4])
For fixed $p$, we consider the map

$$T_g : C^1[0,1] \rightarrow C^1[0,1]$$

$$q \mapsto 2\frac{d}{dx}K_g(x,x;q) + p.$$ 

By Remark 4.1 (3), there exists $\delta > 0$ such that, if $\|g\| < \delta$, $T_g$ is a contraction map on some ball $U_B \subset C^1[0,1]$. So, $T_g$ has a unique fixed point on $U_B$, denoted by $q(g)$. $K_g(x,y;q(g))$ satisfies (3.2).

**Remark 4.2.** $q(g)$ is analytic in $g$, so $K_g(x,y;q(g))$ is also analytic in $g$.

**Proposition 3.** If there exists $\tilde{g} \in G$ such that $K_{\tilde{g}}(x_0,x_0;p,p) \neq 0$, then $\tilde{M}_{p,a,x_0} \neq \{p\}$ for any $a \in L^2(0,1)$.

**Proof of Proposition 3.** Let $\tilde{g}$ be as above. By Remark 4.1 (2), we can choose $\|\tilde{g}\|_{C^1[x_0,1]}$ sufficiently small. We set

$$f(t) = K_{t\tilde{g}}(x_0,x_0;q(t\tilde{g})) = tK_{\tilde{g}}(x_0,x_0;q(t\tilde{g})).$$

We remark that $f(t)$ is an entire function and $q(0) = p$. From the assumption, we have $f(0) = 0$ and $f'(0) = K_{\tilde{g}}(x_0,x_0;p,p) \neq 0$. So, there exist $t_1, t_2 \in \mathbb{R}$, whose absolute values are very small, such that $f(t_1) > 0$ and $f(t_2) < 0$ by the inverse function theorem. $S(g) = K_g(x_0,x_0;q(g))$ is continuous with respect to $g$. So, there exists $g_1 \in G$ such that $\|t_1\tilde{g} - g_1\|_{C^1[x_0,1]}$ is very small and $g_1$ is linearly independent of $t_2\tilde{g}$ and that $S(g_1) > 0$. Since $S(g_1) > 0$ and $S(t_2\tilde{g}) < 0$, there exists $\hat{g} \in G$ such that $S(\hat{g}) = 0$, by the continuity of the function $S(\cdot)$. We remark that $\hat{g}$ does not vanish identically because $g_1$ is linearly independent of $t_2\tilde{g}$, and that $\|\hat{g}\|_{C^1[x_0,1]}$ is very small. Hence,
satisfies $(E)$, so $\tilde{M}_{p,a,x_0} \neq \{p\}$ for any $a \in L^2(0,1)$.

< Step 4 >

Lemma 1. If $x_0 \in [\frac{1}{2}, 1) \setminus H$ and $p$ is a constant function, the assumption of Proposition 3 holds.

Lemma 2. If $x_0 \in H$, there exists $p_0 \in C^1[0,1]$ such that the assumption of Proposition 3 holds.

Admitting these lemmas for the moment, we continue the proof of Theorem 1.

If $x_0 \in [\frac{1}{2}, 1) \setminus H$, there exists $\hat{g} \in G$ such that $K_{\hat{g}}(x_0, x_0; 0,0) \neq 0$ by Lemma 1. Let

$$U_{x_0} = \{p \in C^1[0,1]; K_{\hat{g}}(x_0, x_0; p, p) \neq 0\}.$$ 

Then $U_{x_0}$ is an open set. $F(t) = K_{\hat{g}}(x_0, x_0; tp_0, tp_0)$ is an entire function with respect to $t$ for any $p_0 \in C^1[0,1]$, so the zeros of $F$ are discrete. Therefore $U_{x_0}$ is dense in $C^1[0,1]$. And $p \in U_{x_0}$ implies that $\tilde{M}_{p,a,x_0} \neq \{p\}$ for any $a \in L^2(0,1)$ by Proposition 3 and Lemma 1.

If $x_0 \in H$, then we proceed in the same way as above. This completes the proof of Theorem 1 in the case of $x_0 \in [\frac{1}{2}, 1)$.

We next explain the proof of Lemma 1 and 2. Lemma 1 follows from a direct calculation, so we consider only Lemma 2.
Proof of Lemma 2. Let $x_0 = \frac{2k}{2k+1}$ and devide $D$ as in Figure 2.

We then have

\begin{equation}
K_g(x_0, x_0; p, p) = 2 \sum_{j=1}^{k} (-1)^{k+j-1} \int D_j R(p) K_g(p) dxdy,
\end{equation}

where $R(p)(x, y) = p(x) - p(y)$, $K_g(p) = K_g(x, y; p, p)$. Let $g = x^2 - 2x_0x + 2x_0 - 1 \in G$, and assume that $K_g(x_0, x_0; p, p) = 0$ for any $p \in C^1[0, 1]$. We differentiate (4.2) at $p = 0$, then we have

\begin{equation}
\sum_{j=1}^{k} (-1)^j \int D_j R(p) K_g(0) dxdy = 0
\end{equation}

for any $p \in C^1[0, 1]$. We now put $p(x) = x$ in the left-hand side of (4.3), then we have "the left-hand side of (4.3)" is not equal to zero. This is a contradiction, so there exists $p_0$ such that $K_g(x_0, x_0; p_0, p_0) \neq 0$.

(II) The case of $x_0 \in (0, \frac{1}{2})$. 
Let \( f \in C^1[0,1], f(1) = 0, f = 0 \) on \([0,2x_0]\) and \( f \) does not vanish identically on \([0,1]\). For \( p, q \in C^1[0,1] \) and \( f \), there exists \( K \in C^2(\bar{D}) \) satisfying \((3.1), (3.3), (3.4)\) and \( K_y(x,0) = f \ (0 \leq x \leq 1) \). \( K \) is uniquely determined. We remark that \( K \) satisfies \((3.5)\) and \((3.6)\) by the assumptions on \( f \). We now consider the map

\[
T_f : q \mapsto 2 \frac{d}{dx} K(x, x) + p.
\]

If \( \| f \|_{C^1[0,1]} \) is sufficiently small, then \( T_f \) is a contraction map on some ball in \( C^1[0,1] \). We can then argue as before.

5 Other observations and stability

We briefly explain what occurs when we take different observations. We first consider:

(1) \[ \{u_x(t,0), u(t,x_0); T_1 \leq t \leq T_2\} \ (x_0 \in (0,1]). \]

For this observation, we define \( M'_{p,a,x_0}, \tilde{M}'_{p,a,x_0} \) in the same way as \( M_{p,a,x_0}, \tilde{M}_{p,a,x_0} \), respectively. In this case, we have

**Theorem 3.** For each \( x_0 \in (0,1], \)

\[ \{p \in C^1[0,1]; \tilde{M}'_{p,a,x_0} \neq \{p\} \text{ for any } a \in L^2(0,1)\} = C^1[0,1]. \]

We next consider:

(2) \[ \{u_x(t,0), u_x(t,x_0), u(t,x_0); T_1 \leq t \leq T_2\} \ (x_0 \in (0,1]). \]
We define $M_{p,a,x_{0}}, \tilde{M}_{p,a,x_{0}}^{*}$ in the same way as above. Then we have

**Theorem 4.**

(i) If $x_{0} = 1, \tilde{M}_{p,a,x_{0}}^{*} = \{p\}$ holds if and only if $N_{p,a} = 0$.

(ii) If $x_{0} \in (\frac{1}{2}, 1)$ and $N_{p,a} < +\infty$, then $\tilde{M}_{p,a,x_{0}}^{*} = \{p\}$.

(iii) If $x_{0} = \frac{1}{2}, \tilde{M}_{p,a,x_{0}}^{*} = \{p\}$ holds if and only if $N_{p,a} \leq 1$.

(iv) If $x_{0} \in (0, \frac{1}{2})$, for any $p \in C^{1}[0,1]$ and any $a \in L^{2}(0,1)$, we have $\tilde{M}_{p,a,x_{0}}^{*} \neq \{p\}$.

For $q \in C^{1}[0,1]$, we consider a bounded operator

$$
\Lambda_{q} : L^{2}(0,1) \longrightarrow C^{0}(I) \times C^{0}(I)
$$

$$a \longmapsto (u_{x}(t,0), u_{x}(t,1)),
$$

where $u = u(t,x)$ is the solution of $(E_{q,a})$ and $I = [T_{1}, T_{2}], T_{1} > 0$. By Theorem 0.1, it is easy to see that $\Lambda_{q_{0}} = \Lambda_{q_{1}}$ implies $q_{0} = q_{1}$. So, the map $q \mapsto \Lambda_{q}$ is injective. To study the continuity of the inverse map is an interesting problem. Using the result of [2], we obtain:

**Theorem 5.** Let $\{q_{j}\}_{j=1}^{\infty} \subset C^{1}[0,1]$ and $\sup_{j} \|q_{j}\|_{L^{2}(0,1)} < +\infty$, then $\Lambda_{q_{j}} \rightarrow \Lambda_{q_{0}}$ in $B(L^{2}(0,1), C^{0}(I) \times C^{0}(I))$ if and only if $q_{j} \rightarrow q_{0}$ in $L^{2}(0,1)$ weakly.

**References**


