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STABILIZATION OF HOMOLOGY GROUPS OF SPACES OF MUTUALLY DISJOINT DIVISORS

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This talk will be closely related to the one given at this conference by Kohhei Yamaguchi [GKY2] concerning our joint work with Martin Guest [GKY1]. In particular, I shall give a detailed proof of a result that is one of the key steps in that work. However, my point of view will be somewhat different from that of Yamaguchi's talk, which was concerned mainly with its homotopic aspects. By contrast, I am going to concern myself only with matters involving homology, and in particular with one method of proving a certain type of homological result, originally developed by Arnold [A]. Let us start by describing Arnold's original idea.

Let C_d denote the space of complex monic polynomials of degree d (i. e. polynomials of the form $a_0 + a_1 z + a_2 z^2 + \cdots + z^d$), which have no repeated roots. Let i_d denote an inclusion $i_d : C_d \to C_{d+1}$, which we shall describe presently. Arnold proved

Theorem 1. *The map i_d^* is an isomorphism on homology groups $H_i(C_d) \to H_i(C_{d+1})$ up to dimension $[d/2]$ and is surjective in dimension $[d/2]$ (where $[x]$ denotes the integer part of x).*

Note first that the space C_d of monic polynomials without repeated roots is simply the configuration space of d distinct points of $\mathbb{R}^2 = \mathbb{C}$. Thus its homology is the homology of the braid group $B(d)$. One way to define the stabilization (or inclusion) map is to take as C_d the space of configurations of points in an open half-plane, (which is, of course, homeomorphic to the original configuration space) and adjoin a fixed point in the other half-plane.

I shall first give a proof of the above theorem of Arnold, in a simplified version due to Graeme Segal [S, Appendix].

Proof. The proof depends on making use of Poincaré Duality. Note that the space C_d is a $2d$-dimensional open orientable manifold. By Poincaré Duality

$$H_i(C_d) \cong H_c^{2d-i}(C_d),$$

where $H_c^j(X)$ denotes cohomology of X with compact supports (which can be thought of as the usual cohomology of the one point compactification X_+). The inclusion i_d : $C_d \to C_{d+1}$ extends to an open embedding

$$\hat{i}_d : \mathbb{R}^2 \times C_d \to C_{d+1},$$

and the statement of Arnold's Theorem is equivalent to the assertion

$(*)_d$ $\hat{i}_d$ is a compact cohomology equivalence above dimension $2(d+1)-[d/2]$

(Here by a compact cohomology equivalence above dimension n we mean a map which is an equivalence on H_c^i when $i>n$ and a surjection when $i=n$).

Clearly $(*)_1$ holds.

Assume inductively that $(*)_d$ holds for $d<n$. We introduce the following filtration on the space of all monic polynomials of degree n (which is, of course, just $\mathbb{C}^n$): we can identify a (complex) monic polynomial f of degree n with the divisor ξ of degree n composed of its roots, i. e. $\xi = \sum_{i=1}^n \alpha_i$, where α_i are the roots of f.

Here by a *divisor* of degree d on a manifold M we simply mean an element of the symmetric product $Sp^d(M)$. Any such divisor can be written in the form $\xi = 2\eta + \zeta$, where the points in ζ all have multiplicity 1.

Let $P_{n,k}$ consist of the divisors $\xi = 2\eta+\zeta$ with $\deg(\eta)\geq k$.

We then have

$$P_n = P_{n,0} \supset P_{n,1} \supset P_{n,2} \supset \dots,$$

Note that $P_{n,k} - P_{n,k+1} = C_{n-2k} \times Sp^k(\mathbb{C}) = C_{n-2k}\times \mathbb{C}^k$. (By the Fundamental Theorem of Algebra $Sp^k(\mathbb{C})$—the k-fold symmetric product of $\mathbb{C}$—is homeomorphic to $\mathbb{C}^k$).

Next, consider the exact sequence of cohomology with compact supports

$$\begin{array}{ccccccc} \dots \to H_c^i(P_{n,k}\setminus P_{n,k+1}) & \to H_c^i(P_{n,k}) & \to & H_c^i(P_{n,k+1}) & \to H_c^{i+1}(P_{n,k}\setminus P_{n,k+1}) \to \dots \\ & & & \| & \\ & & & H_c^i(C_{n-2k}\times \mathbb{C}^k) & \\ & & & \| & \\ & & & H_c^{i-2k}(C_{n-2k}) & \end{array}$$

By downwards induction on k we now show that

$(*)_{n,k}$ $\mathbb{R}^2\times P_{n,k}\to P_{n+1,k}$ is a compact cohomology equivalence

above dimension $2(n+1)-k-[n/2]$ $(=2k+2(n-2k+1)-[(n-2k)/2])$, provided $k>0$.

This obviously holds for large k, and the inductive step follows from the 5-lemma and the commutative diagram

$$\begin{array}{ccccccc} \dots \to & H_c^i(\mathbb{R}^2\times C_{n-k}\times Sp^k(\mathbb{C})) & \to & H_c^i(\mathbb{R}^2\times P_{n,k}) & \to & H_c^i(\mathbb{R}^2\times P_{n,k+1}) & \to \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ \dots \to & H_c^i(C_{n+1-k}\times Sp^k(\mathbb{C})) & \to & H_c^i(P_{n+1,k}) & \to & H_c^i(P_{n+1,k+1}) & \to \dots \end{array}$$

However, since $P_{n,0} = \mathbb{C}^n$, $(*)_{n,0}$ also holds, and the result follows from the commutative diagram

$$\begin{array}{ccccccc}
\dots \to H^i_c(\mathbb{R}^2 \times C_n) & \to & H^i_c(\mathbb{R}^2 \times P_{n,0}) & \to & H^i_c(\mathbb{R}^2 \times P_{n,1}) & \to \dots \\
\downarrow & & \downarrow & & \downarrow & \\
\dots \to H^i_c(C_{n+1}) & \to & H^i_c(P_{n+1,0}) & \to & H^i_c(P_{n+1,1}) & \to \dots
\end{array}$$

□

Actually, Arnold's argument is more complicated and gives a lot more than just the stabilization result. As mentioned above, this version is due to Segal, who generalized both the result and its method of proof. Firstly, he extended the result to the space of configurations of distinct points on any connected open manifold M (to define a stabilization map for arbitrary connected open manifolds one uses the notion of *ends.* Up to homotopy there is one such map for each end.) And secondly, he used this method to obtain homology stabilization results in his study of spaces $\mathrm{Hol}_d(M; \mathbb{C}P^n)$ of holomorphic maps of degree d, where M is a Riemann surface of genus g. Let's next see how Segal applied this method to the space Q_d of pairs of disjoint divisors (ξ, η) in $\mathbb{C}$, which can be identified with the space of rational (or, what amounts to the same thing, holomorphic) maps $\mathrm{Hol}_d(S^2; S^2)$ (We shall consider the case of a Riemann surface of non-zero genus a little later). In this case one obtains the following:

Proposition 2. *The stabilization map $Q_d \to Q_{d+1}$ given by adjoining distinct roots "far from the imaginary axis" to the divisors, is a homology equivalence up to dimension d.*

Proof. Segal's proof of this result is almost identical to the one given above. We define a filtration on the space $Sp^d(\mathbb{C}) \times Sp^d(\mathbb{C})$, by

$$P_{d,k} = \{ \text{ pairs of divisors } (\xi, \eta) : \deg(\xi \cap \eta) \geq k \}.$$

Then

$$P_{d,k} \setminus P_{d,k+1} = Sp^k(\mathbb{C}) \times Q_{d-k}.$$

The inclusion $Q_d \to Q_{d+1}$ again extends to an open embedding $\mathbb{C}^2 \times Q_d \to Q_{d+1}$, and the same holds for the inclusions $P_{d,k} \to P_{d,k+1}$.

We now proceed by induction. First, we assume that

$(**)_d$ if $m < d$ then $\mathbb{C}^2 \times Q_m \to Q_{m+1}$ is a compact cohomology equivalence above dimension $j = 4(m+1) - m = 3m + 4$

The statement $(**)_1$ is certainly true. From $(**)_d$ we deduce

$(\dagger)_d$ if $k > 0$ then $\mathbb{C}^2 \times P_{d,k} \to P_{d+1,k}$ is a compact cohomology equivalence above dimension $3d - k + 4$

As before, this is proved by downwards induction on k. It is true for $k = d$ for then $P_{d,d} = \mathbb{C}^d$, and $P_{d+1,d}$ has dimension $4(d+1) - 2d = 2d + 4$. One passes from $k+1$ to k by applying the 5-lemma to the diagram

$$\begin{array}{ccccccc}
\cdots \to & H_c^i(\mathbb{C}^2 \times Q_{d-k} \times Sp^k(\mathbb{C})) & \to & H_c^i(\mathbb{C}^2 \times P_{d,k}) & \to & H_c^i(\mathbb{C}^2 \times P_{d,k+1}) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H_c^i(Q_{d+1-k} \times Sp^k(\mathbb{C})) & \to & H_c^i(P_{d+1,k}) & \to & H_c^i(P_{d+1,k+1}) & \to \cdots
\end{array}$$

Note that for $i \geq 3d-k+4$ the homomorphism $H_c^i(\mathbb{C}^2 \times Q_{d-k} \times Sp^k(\mathbb{C})) \longrightarrow H_c^i(Q_{d+1-k} \times Sp^k(\mathbb{C}))$ coincides with $H_c^{i-2k}(\mathbb{C}^2 \times Q_{d-k}) \to H_c^{i-2k}(Q_{d+1-k})$ and satisfies condition $(**)_d$, as $i \geq 3d - k + 4 \Rightarrow i - 2k \geq 3d - 3k + 4 = 4((d-k)+1) - (d-k)$.

Finally, by considering the cohomology sequence of the pair $(P_{d,0}, P_{d,1})$, and using the 5-lemma in the same way as above, we obtain our result. (We make use of the fact that $P_{n,0} = Sp^n(\mathbb{C}) \times Sp^n(\mathbb{C}) = \mathbb{C}^{2n}$.) $\square$

The above argument applies essentially without change to the case of holomorphic maps of degree d from S^2 to $\mathbb{C}P^n$, with $n > 1$. This space can be identified with the space of $(n+1)$-tuples of divisors $(\xi_0, \xi_1, \ldots, \xi_n)$, with empty intersection. Intuitive considerations suggest that in this case the stabilization dimension should increase with the number of polynomials. Indeed, in the filtration $P_{n,0} \supset P_{n,1} \supset \ldots \supset P_{n,n} = Sp^n(\mathbb{C})$ where $P_{n,k} = \{(\xi_0, \ldots, \xi_n) : \deg(\xi_0 \cap \cdots \cap \xi_n) \geq k\}$, each layer has complex codimension n in the preceding one and we get the following

Proposition 3. *The stabilization map $Q_d \longrightarrow Q_{d+1}$ given by adjoining distinct roots to the divisors, is a homology equivalence up to dimension $(2n-1)d$.*

So far we have not distinguished between monic polynomials and based rational maps on the one hand, and divisors or tuples of divisors on the other. These concepts are equivalent as long as we are concerned with based rational functions on S^2, or tuples of divisors in $\mathbb{C}$. For Riemann surfaces the relation between rational functions and divisors is described in the following

Proposition 4. *Let X be a Riemann surface of genus $g > 0$, and let $X' = X \setminus x_0$. There is a map $j : Sp^n(X) \longrightarrow J$, where J is a torus of complex dimension g associated to X (its Jacobian variety) such that*

(a) *A pair $(\xi, \eta) \in Q_n(X)$ arises from a rational function on X if and only if $j(\xi) = j(\eta)$*
(b) *$j : Sp^n(X) \longrightarrow J$ is a smooth fibre bundle with fibre $\mathbb{C}P^{n-g}$ if $n \geq 2g-1$.*
(c) *$j : Sp^n(X') \longrightarrow J$ is a smooth fibre bundle with fibre $\mathbb{C}^{n-g}$ if $n \geq 2g$.*

Here part (a) is Abel's theorem, part (b) is proved in [M] and part (c) follows from (b). As a consequence of Proposition 4 one can deduce results about rational functions from results about divisors provided $n \geq 2g$. Using Proposition 4 and Arnold's method Segal obtained the following generalization of Proposition 2 for Riemann surfaces of non zero genus:

Theorem 5 (Segal). *Let X be a Riemann surface of genus $g > 0$, and let $X' = X \backslash x_0$. Let Q_d^g denote the space of $(n+1)$-tuples of divisors of degree d on X', with empty intersectio. Then the stabilization map $Q_d^g \longrightarrow Q_{d+1}^g$ is a homology equivalence up to dimension $(d-2g+1)(2n-1)$ when $g \neq 0$ and $(2n-1)d$ when $g = 0$.*

Of course this theorem could equally well be stated for holomorphic functions $\mathrm{Hol}_d(M_g; \mathbb{C}P^n)$.

However, another natural way to generalize Proposition 2 is to consider n-tuples of divisors which are mutually disjoint, i. e. the set $E_d^n = \{(\xi_1, \xi_2, \dots \xi_n) : \deg \xi_i = d \text{ and } \xi_i \cap \xi_j = \emptyset \text{ for } i \neq j\}$. This space is in some ways more analogous to the case of pairs of divisors (and like the latter but unlike the case of $(n+1)$-tuples of divisors $(n > 1)$, it is not simply-connected). We refer to these spaces as "Epshtein spaces", after S. I. Epshtein, who in [E] first computed their fundamental groups (His results include that of Jones, which states that $\pi_1(Q_d) = \mathbb{Z}$. Jones' argument is reproduced in [S]). For Epshtein spaces we shall prove

Theorem 6. *Let M be an Riemann surface and let E_d denote the space of n-tuples of mutually disjoint divisors of degree d on M. Then the stabilization map $E_d \longrightarrow E_{d+1}$ is a homology equivalence up to dimension d.*

As explained in Yamaguchi's talk, when M is the Riemann sphere we can in fact determine the homotopy type of these spaces up to dimension d. Note that $M = \mathrm{S}^2$ and $n = 2$ this result coincides with Segal's. When M is a Riemann surface of genus $g > 0$ our result is an improvement on the Segal result for divisors (though not for holomorphic maps!)

Before proving Theorem 6, we shall use the same technique to prove a natural generalization of Proposition 1 of Arnold. Let $\hat{E}_d^n$ denote the Epshtein space of $(n+1)$-tuples of monic polynomials, with the additional condition that they have no repeated roots, or more generally the set of $(n+1)$-tuples of distinct subsets of cardinality d of points of any connected open manifold M. We can prove

Theorem 7. *For any n the space $\hat{E}_d^n$ stabilizes in homology up to dimension $[d/2]$.*

Theorem 7 will follow from a more general result. First, for positive integers $d_1, d_2, \dots, d_m$ and an open manifold M, let $\hat{E}(d_1, d_2, \dots d_m, M)$ be the space of mutually disjoint sets of points in M of cardinality d_1, d_2, … d_m. (These spaces have recently played an important role in the study of representations of braid groups.) By adding a particle "at infinity" in a standard way, for each $1 \leq s \leq m$, we have the s-th stabilization map

$$j_s : \hat{E}(d_1, d_2, \dots, d_m : M) \longrightarrow \hat{E}(d_1, d_2, \dots, d_s + 1, \dots d_m : M).$$

Theorem 8. *If M is a connected open manifold of dimension ≥ 2, the stabilization map*

$$j_s : \hat{E}(d_1, d_2, \dots, d_m : M) \longrightarrow \hat{E}(d_1, d_2 \dots, d_s + 1, d_m : M).$$

is a homology equivalence up to dimension $[d_s/2]$.

Corollary. *If M is a connected open manifold of dimension ≥ 2, the stabilization map*

$$j : \hat{E}(d_1, d_2, \ldots, d_m : M) \to \hat{E}(d_1 + 1, d_2 + 1 \ldots, d_m + 1 : M).$$

is a homology equivalence up to dimension $[d/2]$, where $d = \min\{d_s : 1 \leq s \leq m\}$.

Proof. Consider the stabilization map $\hat{E}(d_1, d_2, \ldots, d_n) \to \hat{E}(d_1 + 1, d_2, \ldots, d_n)$ with respect to the first set of points, and the projection map onto the remaining set of points. The projection $\hat{E}(d_1, d_2, \ldots, d_n) \to \hat{E}(d_2, d_3, \ldots, d_n)$ is a bundle map, with the fibre $C_{d_1}(M \setminus \{(d_2 + d_3 + \ldots d_n) \text{ points}\})$ (where $C_k(M)$ denotes the space of configurations of n-points on M), which is a connected open manifold. The stabilization map induces a map of bundles

$$\begin{array}{ccc} \hat{E}(d_1, d_2, \ldots, d_n) & \longrightarrow & \hat{E}(d_1 + 1, d_2, \ldots, d_n) \\ \downarrow & & \downarrow \\ \hat{E}(d_2, \ldots, d_n) & = & \hat{E}(d_2, \ldots, d_n) \end{array}$$

with the induced map on the fibres the stabilization map $C_{d_1}(M \setminus \{(d_2 + d_3 + \cdots + d_n) \text{ points}\}) \to C_{d_1+1}(M \setminus \{(d_2 + d_3 + \cdots + d_n) \text{ points}\})$. By Segal's generalization of Arnold's theorem this map is a homology equivalence up to dimension $[d_1/2]$, hence so is the map on $\hat{E}(d_1, d_2, d_3, \ldots, d_n)$. Theorem 8 follows. $\square$

We prove Theorem 6 by using essentially the same method.

First, for positive integers $d_1, d_2, \ldots, d_m$ and a Riemann surface M, let $E(d_1, d_2, \ldots d_m : M)$ be the space

$$\{(\xi_1, \xi_2, \ldots, \xi_m) : \xi_k \in Sp^{d_k}(M) \text{ and } \xi_i \cap \xi_j = \emptyset \text{ if } i \neq j\}.$$

By adding a particle at infinity in a standard way, for each $1 \leq s \leq m$, we have the s-th stabilization map

$$j_s : E(d_1, d_2, \ldots, d_m : M) \to E(d_1, d_2, \ldots, d_s + 1, \ldots d_m : M).$$

Again, Theorem 6 will follow from a more general result:

Theorem 9. *If M is a connected Riemann surface, the stabilization map*

$$j_s : E(d_1, d_2, \ldots, d_s \ldots d_m : M) \to E(d_1, \ldots, d_s + 1, \ldots, d_m : M),$$

is a homology equivalence up to dimension d_s.

Corollary. *If M is a connected Riemann surface, the stabilization map*

$$j : E(d_1, d_2, \ldots, d_m : M) \to E(d_1 + 1, d_2 + 1 \ldots, d_m + 1 : M),$$

is a homology equivalence up to dimension d, where $d = \min\{d_s : 1 \leq s \leq m\}$.

We shall only prove Theorem 9 in the case $m = 3$, as the general case is quite analogous.

We shall need a lemma.

Lemma 10. *If X is a connected based CW-complex, the natural inclusion map*

$$j : Sp^d(X) \to Sp^{d+1}(X)$$

is a homotopy equivalence up to dimension d.

Proof. We can suppose that X has only one zero cell. Then $Sp^{d+1}(X)$ is a cell complex, with a typical cell of dimension k of the form $[\sigma_1 \times \sigma_2 \times \dots \sigma_{d+1}]$ with $k_1 + k_2 + \dots + k_{d+1} = k$, where k_i is the dimension of the cell σ_i. Thus in a cell of dimension $\leq d$ at least one of the cells σ_i must be the zero dimensional cell. This means that the d skeleton of Sp^{d+1} lies in Sp^d, whence the result follows. □

Proof of Theorem 9. We shall consider only the case of triples of mutually prime monic polynomials (p, q, r), where $\deg p = d_1$, $\deg q = d_2$ and $\deg r = d_3$. We shall stabilize with respect to the first polynomial (by adjoining a root in the way described earlier) and consider the projection on to the remaining two polynomials. In other words, consider the map

$$\begin{aligned} E(d_1, d_2, d_3) &\xrightarrow{\Pi} E(d_2, d_3) \\ (p, q, r) &\mapsto (q, r) \end{aligned}$$

Let us now introduce a bi-filtration on $E(d_2, d_3)$ by defining $B_{i,j} = \{(q, r) : q$ has at least i distinct roots and r has at least j distinct roots$\}$. Let $E^d_{i,j} = \Pi^{-1}(B_{i,j})$ and let $X^d_{i,j} = E^d_{i,j} \setminus (E^d_{i+1,j} \cup E^d_{i,j+1})$. In other words, $X^d_{i,j}$ consists of triples (p, q, r) of mutually coprime monic polynomials of degrees d, d_2 and d_3 respectively, such that q has exactly i distinct roots and r has exactly j distinct roots. Note that $E^d_{d_2,d_3} = X^d_{d_2,d_3}$ and $E^d_{0,0} = E(d, d_2, d_3)$. Next, note that the map $\Pi_{|X_{i,j}} : X_{i,j} \to Y_{i,j}$ (where $Y_{i,j} = \Pi(X_{i,j})$) is a fibre bundle, with fibre $Sp^d(M \setminus \{(i+j) \text{ points}\})$.

Now, consider the stabilization map $E(d_1, d_2, d_3) \to E(d_1 + 1, d_2, d_3)$. It gives rise the following diagram of fibre bundles

$$\begin{array}{ccc} X^{d_1}_{i,j} & \longrightarrow & X^{d_1+1}_{i,j} \\ \downarrow & & \downarrow \\ Y_{i,j} & = & Y_{i,j} \end{array}$$

Since, by Lemma 10, the map induced on the fibres is a homology equivalence up to dimension d_1, this holds also for the total spaces (the base spaces being the same). So we can assume that each of the maps $X^{d_1}_{i,j} \to X^{d_1+1}_{i,j}$ is a homology equivalence up to dimension d. Note that $E^{d_1}_{d_2,d_3} \to E^{d_1+1}_{d_2,d_3}$ is a homology equivalence up to dimension d_1, since it coincides with the map $X^{d_1}_{d_2,d_3} \to X^{d_1+1}_{d_2,d_3}$. Consider the map $E^{d_1}_{i,j} \to E^{d_1+1}_{i,j}$, with $i + j = k$ and suppose by induction that $E^{d_1}_{i,j} \to E^{d_1+1}_{i,j}$ is a homology equivalence up to dimension d_1, for $i + j > k$.

Note that the stabilization map $E(d_1,d_2,d_3) \longrightarrow E(d_1+1,d_2,d_3)$ (and each of the other stabilization maps induced by it) extends to an open embedding $\mathbb{R}^2 \times E(d_1,d_2,d_3) \longrightarrow E(d_1+1,d_2,d_3)$, of orientable open manifolds of dimension $2(d_1+d_2+d_3+1)$. Hence, by Poincaré Duality, the statement we are trying to prove is equivalent to the assertion that $H^i_c(\mathbb{R}^2 \times E(d_1,d_2,d_3)) \longrightarrow H^i_c(E(d_1+1,d_2,d_3))$ is an isomorphism for $i > 2(d_1+d_2+d_3+1)-d_1$ and surjective for $i = 2(d_1+d_2+d_3+1)-d_1 = d_1+2d_2+2d_3+2$.

We shall next prove inductively that the map $H^i_c(\mathbb{R}^2 \times E^{d_1}_{i,j}) \longrightarrow H^i_c(E^{d_1+1}_{i,j})$ is an isomorphism for $i > 2(d_1+d_2+d_3+1)-d_1$ and is surjective for $i = 2(d_1+d_2+d_3+1)-d_1 = d_1+2d_2+2d_3+2$.

First note that this is true for $i+j = d_2+d_3$, since the space $E^{d_1}_{d_2,d_3} = X^{d_1}_{d_2,d_3}$ is an open orientable manifold of dimension $2(d_1+d_2+d_3+1)$. Now, suppose by induction it holds for all $i+j<k$. Then, since $E^{d_1}_{i+1,j} \cap E^{d_1}_{i,j+1} = E^{d_1}_{i+1,j+1}$, it follows from the Mayer-Vietoris sequence in cohomology with compact supports (see [B, p. 65]) that it holds for $\mathbb{R}^2 \times (E^{d_1}_{i+1,j} \cup E^{d_1}_{i,j+1}) \longrightarrow E^{d_1+1}_{i+1,j} \cup E^{d_1+1}_{i,j+1}$. Now consider the long exact sequence of cohomology with compact supports for the pair $(E^{d_1}_{i,j}, E^{d_1}_{i+1,j} \cup E^{d_1}_{i,j+1})$:

$$\begin{tikzcd}
\ldots \arrow[r] & H^k_c(\mathbb{R}^2 \times X^{d_1}_{i,j}) \arrow[r] \arrow[d] & H^k_c(\mathbb{R}^2 \times E^{d_1}_{i,j}) \arrow[r] \arrow[d] & H^k_c(\mathbb{R}^2 \times (E^{d_1}_{i+1,j} \cup E^{d_1}_{i,j+1})) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & H^k_c(\mathbb{R}^2 \times X^{d_1+1}_{i,j}) \arrow[r] & H^k_c(\mathbb{R}^2 \times E^{d_1+1}_{i,j}) \arrow[r] & H^k_c(\mathbb{R}^2 \times (E^{d_1+1}_{i+1,j} \cup E^{d_1+1}_{i,j+1})) \arrow[r] & \ldots
\end{tikzcd}$$

From the five lemma and the inductive hypothesis it now follows that the statement is valid for the map $E^{d_1}_{i,j} \longrightarrow E^{d_1+1}_{i,j}$. Hence, going down by induction it is valid for

$$\begin{tikzcd}
E^{d_1}_{0,0} \arrow[r] \arrow[d, equal] & E^{d_1+1}_{0,0} \arrow[d, equal] \\
E(d_1,d_2,d_3) \arrow[r] & E(d_1+1,d_2,d_3)
\end{tikzcd}$$

This proves our theorem. □

Finally, let me point out one possible generalization of the above results. Consider the set of all n-tuples of divisors of some fixed degree d on a Riemann surface M, in other words $\overbrace{Sp^d(M) \times \ldots \times Sp^d(M)}^{n \text{ times}}$. Of course, we have a stabilization map $\overbrace{Sp^d(M) \times \ldots \times Sp^d(M)}^{n \text{ times}} \longrightarrow \overbrace{Sp^d(M) \times \ldots \times Sp^d(M)}^{n+1 \text{ times}}$, which induces a homology isomorphism up to degree d. Inside the space $\overbrace{Sp^d(M) \times \ldots \times Sp^d(M)}^{n \text{ times}}$ one can consider various subspaces defined by "incidence conditions " among divisors, of which the Segal and the Epshtein conditions are the two extreme special cases, for example, one may require that for some fixed k, any collection of k divisors have an empty intersection, where $2 \le k \le n$. There should be a stabilization theorem for each of these cases.

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