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A higher-dimensional analogue of the fundamental groupoid

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§1. Introduction.

This is a brief report on joint work of the three authors. The senior author (Meyer) wishes to express his gratitude to Japanese colleagues for their hospitality during his Fall 1991 visit.

The problem which we address is an old one: how to compute $\pi_k(X)$ or, more generally, to determine the k-homotopy type of a simplicial set X directly from the simplicial structure. D. Kan, in his pioneering work, [6], [7] showed how to obtain $\pi_k(X)$ when X is a Kan complex. Since many interesting X do not satisfy the Kan extension condition, Kan defined two functors Ex^∞ and G which yield Kan complexes $Ex^\infty X, GX$, for any X and thus obtained answers to the first part of the above problem. These answers, however, are not as direct as might be wished.

Our approach is to reconsider the classical fundamental groupoid, looking at it from a categorical rather than geometric point of view, much as Gabriel-Zisman, [4], did and to develop higher-dimensional versions of the fundamental groupoid.

While our results have no intersection with those of R. Brown and his coworkers, [1], [2], we were clearly influenced by their use of multiple groupoids in homotopy theory. In [8], Kapranov and Voevodsky use techniques related to ours, although their weak n-groupoids are quite different from our n-tuple groupoids; they are "globular" whereas ours are "cubical".

§2. The Fundamental Groupoid, revisited.

Consider the following diagram

$$\begin{tikzcd} SS \arrow[r, shift right, "G"'] \arrow[rr, bend right=40, "G^1"'] & \mathrm{Cat} \arrow[l, shift right, "N"'] \arrow[r, shift right, "b"'] & \mathrm{Gpd} \arrow[l, shift right, "a"'] \arrow[ll, bend right=40, "N^1"'] \end{tikzcd}$$

where SS is the category of simplicial sets, Gpd that of groupoids, N is the nerve functor, G its left-adjoint (note that there is no connection between this G and Kan's functor mentioned in §1), a the inclusion functor and b its left-adjoint (which inverts all morphisms); $N^1 = N \circ a$, $G^1 = b \circ G$. It is easily seen from [4], Chapter II and [9], §3 that G^1 is a "geometric realization" functor, with $G^1X = \mathrm{coend}_n X_n \times \varphi_1(n)$ where the models $\varphi_1(n)$ are obtained from

$$\begin{array}{ccccccccc} \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \rightarrow \cdots & \cdot & \rightarrow & \cdot \\ 0 & & 1 & & 2 & & n-1 & & n \end{array}$$

by adjoining the inverses of all arrows. Since Gpd is cocomplete, both the boundary of $\varphi_1(n)$, $\dot{\varphi}_1(n)$ can be defined, and also the "horns" $\lambda_1^i(n)$, $0 \leq i \leq n$, which are the colimits of all but the i-th "face" of $\varphi_1(n)$. The following statements then hold:

(i) $\dot{\varphi}_1(n) \overset{\rightarrow}{\approx} \varphi_1(n)$, if $n \geq 3$.

(ii) $G^1(X) \approx G^1(sk^2X)$.

(iii) $G^1(X)$ can be described in terms of generators $\eta(x)$ corresponding to $x \in X_1$ and relations $\eta(d_2x_2) \cdot \eta(d_0x_2) = \eta(d_1x_2)$, for each $x_2 \in X_2$.

(iv) $\lambda_1^i(n) \overset{\rightarrow}{\approx} \varphi_1(n)$, $0 \leq i \leq n, n \geq 2$; hence, $N^1G^1(X)$ is a Kan complex, $\pi_i N^1G^1(X) = 0$, $i \geq 2$ and so $N^1G^1(X)$ is equivalent to a disjoint union of Eilenberg-MacLane spaces of type $(\pi, 1)$. N^1G^1X is the first Postnikov space of X and G^1X is an algebraic "model" for the 1-type of X.

(v) T obtain the classical description of the fundamental group in terms of a maximal tree, one can use the structure theorem, valid for any groupoid G, [5], p. 94:

$$G \simeq \pi * T$$

where $*$ denotes free product, i.e., coproduct in the category of groupoids, π is a totally disconnected subgroupoid consisting of one vertex group (fundamental group) for each component of G and T is a maximal unicursal subgroupoid (i.e., generated by the arrows of a maximal tree in each component of G).

§3. Fundamental k-groupoids.

We now intend to generalize, insofar as possible, the statements of §2 to higher dimensions. We do this by replacing Cat by the category of small k-tuple categories, k-Cat and Gpd by the category of small k-tuple groupoids, k-Gpd.

A *k-tuple category C* is a set C (of "k-cubes", or "squares" if $k = 2$) which admits k ordinary category structures $(S_i, T_i, I_i, \cdot_i)$ and such that the compositions $\cdot_i$ are compatible in the sense that

$$(a \cdot_i c) \cdot_j (b \cdot_i d) = (a \cdot_j b) \cdot_i (c \cdot_j d) \tag{3.1}$$

whenever both sides are defined. We often exhibit the fact that these compositions are defined by drawing:

$$\begin{array}{|c|c|}\hline \mathrm{c} & \mathrm{d} \\ \hline \mathrm{a} & \mathrm{b} \\ \hline \end{array} \qquad \begin{array}{l} i \\ \uparrow \\ \llcorner\!\!\longrightarrow j \end{array}, \tag{3.2}$$

[1], [2], [3], [10], and we say that this is a 2×2 array of composable k-cubes; similarly, for higher-dimensional arrays.

We will also write

$$\begin{array}{ccc} c & & d \\ & \overset{i}{\llcorner}\!\!\rightarrow j & \\ a & & b \end{array}, \quad \text{or simply} \quad \begin{array}{ccc} c & & d \\ & \cdot & \\ a & & b \end{array},$$

if i, j are clear, for the common value of (3.1) and call it the *total composite* of the array (3.2).

A *k-tuple groupoid* is a k-tuple category such that every k-cube σ has an inverse $\sigma^{(i)}$ with respect to each composition $\cdot_i$. Note, therefore, that in a k-tuple groupoid, to every k-cube σ are associated 2^k k-cubes, namely all possible combinations of inverses of σ. Thus, for $k = 2$, we have $\sigma, \sigma^{(1)}, \sigma^{(2)}$ and $\sigma^{(1,2)} = (\sigma^{(1)})^{(2)}$.

The nerve functor $N : k\text{-Cat} \to SS$ is defined in a fairly obvious way: $N_n(C)$ is the set of $n \times n \times \cdots \times n$ (k factors) arrays of composable k-cubes; the simplicial operators are defined by composing in the i-th positions or by inserting identity cubes in the i-th position. There is also a k-simplicial nerve of which $N(C)$ is the diagonal.

We then have the diagram

$$\begin{tikzcd} SS \arrow[r, shift right, "G"'] \arrow[rr, bend right, "G^k"'] & k\text{-Cat} \arrow[l, shift right, "N"'] \arrow[r, shift right, "b"'] & k\text{-Gpd} \arrow[l, shift right, "a"'] \arrow[ll, bend right, "N^k"'] \end{tikzcd}$$

just as in §2. We have models $\varphi_k(n)$ in k-Gpd; we simply illustrate the case $k = 2$ where $\varphi_2(n)$ consists of the squares below

$$\begin{array}{ccccccccccccc}
n-1 & \bullet & \rightarrow & \bullet & \cdots & \cdots & \cdots & \cdots & \cdots & \bullet & \rightarrow & \bullet \\
 & \uparrow & & \uparrow & \cdots & \cdots & \cdots & \cdots & \cdots & \uparrow & & \uparrow \\
n & \bullet & \rightarrow & \bullet & & & & & & \bullet & \rightarrow & \bullet \\
 & \vdots & & \cdots & & \cdots & & \cdots & & \cdots & & \vdots \\
 & \uparrow & & & & & & & & & & \vdots \\
1 & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \cdots & \bullet & \rightarrow & \bullet \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \cdots & \bullet & \rightarrow & \bullet \\
 & 0 & & 1 & & 2 & & & & n-1 & & n
\end{array}$$

together with all their inverses. Again, k-Gpd is cocomplete and we can define the boundary $\dot{\varphi}_k(n)$ of $\varphi_k(n)$ and its horns $\lambda_k^i(n)$, $0 \leq i \leq n$.

We then have:

(i) $\dot{\varphi}_k(n) \overrightarrow{\approx} \varphi_k(n)$, if $n \geq k+2$.

(ii) $G^k(X) \approx G^k(sk^{k+1}X)$

(iii) $G^k(X)$ can be described in terms of generators and relations.

For example, if $k=2$, we have vertices $\overline{x_0}$ for each $x_0 \in X_0$, generators $\eta(x_1)$ for each $x_1 \in X_1$:

$$\begin{tikzcd} & & \overline{d_0x_1} \\ \bullet \arrow[r] & \bullet & \\ \bullet \arrow[u] \arrow[r] \arrow[ur, phantom, "\eta(x_1)"] & \bullet \arrow[u] & \\ \overline{d_1x_1} & & \end{tikzcd}$$

and 2×2-arrays

$\sigma(x_2)$	$\eta(d_0x_2)$
$\eta(d_2x_2)$	$\sigma'(x_2)$

whose total composite is $\eta(d_1x_2)$.

We can then solve for $\sigma'(x_2)$ in terms of the η's, $\sigma(x_2)$ and inverses, so that $\sigma(x_2)$ is the only new generator. It turns out that there are no further generators but each $x_3 \in X_3$ yields the relation: $\sigma(d_1x_3) \cdot_1 \sigma(d_0x_3)^{(1)} = \sigma(d_3x_3)^{(2)} \cdot_2 \sigma(d_2x_3)$.

(iv) $\lambda_k^i(n) \to \varphi_k(n)$, $0 \leq i \leq n$, is an epimorphism if $n \geq k+1$. We conjecture that this is actually an isomorphism. *However*, for $k > 1$, $N^kG^k(X)$ is *not* Kan in general so that the above does not imply $\pi_i N^kG^kX = 0$, $i \geq k+1$ although this seems quite likely.

(v) For groupoids, the structure theorem $G \simeq \pi * T$ arises from the formula $xw = xwx^{-1} \cdot x$, where w is a loop, i.e., a loop can be moved to the left in a word, upon suitable conjugation. Similar results hold in k-groupoids and yield similar structure theorems.

But the universal sub-k-groupoid T is much more difficult to construct than in dimension 1.

§4. Study of G^kX and N^kG^kX.

As mentioned in §3, N^kG^kX does not satisfy the Kan condition in general; nevertheless, let's proceed as if it did. We then find:

(a) The set of k-simplices s of N^kG^kX with $d_is = *$, all i, is in one-one correspondence with the set of k-cubes of G^kX with trivial faces (at $*$).

(b) Two such k-simplices are *homotopic* if and only if the corresponding k-cubes are *equal.*

Therefore it makes sense to define $\bar{\pi}_kX = \bar{\pi}_k(X,*)$ to be the set of all k-cubes of G^kX with trivial faces (at $*$).

(c) $\bar{\pi}_k(X,*)$ is a group, abelian if $k \geq 2$; it admits an action of $\pi_1(X,*)$. More generally, the union of $\bar{\pi}_k(X,*)$ over all basepoints $*$ of X (=vertices of the form $\bar{*}$, $* \in X_0$) admits an action of G^1X, the fundamental groupoid of X. We illustrate this for $k = 2$:

If $x \in X_1$, $\eta(x)$ is the corresponding generator of G^1X and σ is a square of G^2X with trivial faces (at d_1x), so that $\sigma \in \bar{\pi}_2(X, d_1x)$, then $\eta(x) \cdot \sigma \in \bar{\pi}_2(X, d_0x)$ is given by the total composite of

$$\begin{array}{|c|c|c|}
\hline
\eta(x)^{(1)} & & \eta(x) \\
\hline
 & \sigma & \\
\hline
\eta(x)^{(1,2)} & & \eta(x)^{(2)} \\
\hline
\end{array}$$

Here, the four blank squares are the obvious identity squares (in directions 1 or 2); this is to be thought of as a kind of 2-dimensional conjugate of σ by $\eta(x)$.

(d) If $sk^ix = *$, $i \leq k-1$, then $N^kG^kX \simeq K(H_kX, k)$ and therefore $\bar{\pi}_k(X) \simeq H_k(X)$. This is a form of the Hurewicz theorem.

We are thus led to the:

MAIN CONJECTURE.

(1) This is a natural isomorphism $\pi_k(X) \simeq \bar{\pi}_k(X)$, for any simplicial set X.

(2) $G^k X$ is an "algebraic model" for the k-th Postnikov space of X.

§5. Program for proving the Main Conjecture.

The Main Conjecture would follow immediately from the two statements below:

(a) $\bar{\pi}_k(X) \simeq \bar{\pi}_k(ExX)$

(b) If X is a Kan complex, then $\bar{\pi}_k(X) \simeq \pi_k(X)$.

We are attempting to prove (a) and (b).

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