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<th>On real James numbers (Recent development of algebraic topology)</th>
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<td>Author(s)</td>
<td>Oshima, Hideaki</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1992), 781: 18-24</td>
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<tr>
<td>Issue Date</td>
<td>1992-04</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/82510">http://hdl.handle.net/2433/82510</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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On real James numbers

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1. Introduction

The purpose of this note is to determine the real James numbers. Throughout the note $n$, $l$, $k$ denote integers with $n \geq l \geq k \geq 1$ and $n \geq 2$. Let $P_k$ denote the real projective space of dimension $k - 1$, $P_{l,k} = P_l/P_{l-k}$ the stunted projective space, and $V_{l,k} = O(l)/O(l-k)$ the Stiefel manifold of orthonormal $k$-frames in $\mathbb{R}^l$. Note that $P_{k,k}$ is the union of $P_k$ and a disjoint base point. Write $q : V_{l,k} \to V_{l,1} = S^{l-1}$ for the projection on to the last component, and $q : P_{l,k} \to P_{l,1} = S^{l-1}$ for the quotient map. There is a commutative square [6]:

$$
\begin{array}{ccc}
V_{n,k} & \overset{q}{\longrightarrow} & V_{n,1} \\
\uparrow & & \uparrow \\
P_{n,k} & \overset{q}{\longrightarrow} & P_{n,1}
\end{array}
$$

The unstable real James numbers $V\{n, k\}$ and $P\{n, k\}$ are non-negative integers which generate respectively the images of

$$
q_* : \pi_{n-1}(V_{n,k}) \to \pi_{n-1}(S^{n-1}) = \mathbb{Z},
$$

$$
q_* : \pi_{n-1}(P_{n,k}) \to \pi_{n-1}(S^{n-1}) = \mathbb{Z}.
$$

In the same way, replacing homotopy group $\pi_{n-1}(-)$ by stable homotopy group $^{s}\pi_{n-1}(-)$ we have the stable real James numbers $V^s\{n, k\}$ and $P^s\{n, k\}$. Let us denote the exponent of 2 in a positive integer $k$ by $\nu_2(k)$; define $\varphi(k)$ to be the number of integers $s$ such that $0 < s < k$ and $s \equiv 0, 1, 2, 4 \pmod{8}$. Our results are
Theorem (1.1). We have $P^s\{n, k\} = V^s\{n, k\} = V\{n, k\}$ which is equal to 0, 1, or 2 according as $n \equiv 1 \pmod{2}$ and $k \geq 2$, $\nu_2(n) \geq \varphi(k)$, or $1 \leq \nu_2(n) < \varphi(k)$.

Theorem (1.2). We have $V\{n, k\} = P\{n, k\}$ except for the following cases: (1) if $(n, k) = (4, 3), (8, 5), (8, 6), (8, 7), (16, 9)$, then $V\{n, k\} = 1$ and $P\{n, k\} = 2$; (2) if $n = k = 2m$ with $m = 1, 2, 4$, then $V\{n, k\} = 1$ and $P\{n, k\} = 0$; (3) if $n = k = 2m$ with $m \neq 1, 2, 4$, then $V\{n, k\} = 2$ and $P\{n, k\} = 0$.

Let $p_n : S^{n-1} \to P_n$ be the canonical double covering map and $p_{n,k} : S^{n-1} \to P_{n,k} (n > k)$ the composition of $p_n$ with the quotient map.

Corollary (1.3). The rank of $\pi_{n-1}(P_{n,k})$, for $n > k$, is 0, 2, or 1 according as $n \equiv 1 \pmod{2}$ and $2 \leq k \leq n-2$, $n = 2k$ and $k \equiv 0 \pmod{2}$, or otherwise. The map $p_{n,k}$ generates a free direct summand of $\pi_{n-1}(P_{n,k})$ if and only if $n = k + 1 \geq 3$, $P\{n, k\} = 2$ or $(n, k) = (4, 2), (8, 4), (16, 8)$.

Note that a part of (1.1) is not new. Indeed $V\{n, k\}$ was already known [1, 4, 5]. We shall calculate it again by using codegree [3, 8, 9]. We shall prove (1.1) in §2, and (1.2), (1.3) in §3.

2. $V\{n, k\}$

The symbol $a | b$ means that $b = ma$ for some integer $m$.

Lemma (2.1). (1) $V\{n, n\} = V\{n, n-1\}; P^s\{n, 1\} = V^s\{n, 1\} = V\{n, 1\} = P\{n, 1\} = 1; P\{n, n\} = 0$.

(2) $V^s\{n, k\} | V\{n, k\} | P\{n, k\}; V\{n, k\} | V\{n, l\}$ and $P\{n, k\} | P\{n, l\}$ if $n \geq l \geq k \geq 1$.

(3) $V\{2, 2\} = V\{4, 4\} = V\{8, 8\} = V\{16, 9\} = 1$.

(4) ([7; 4.2]) $P^s\{n, k\} = V^s\{n, k\}$. 


(5) If \( n \geq 2k \), then \( P^s\{n,k\} = V^s\{n,k\} = V\{n,k\} = P\{n,k\} \).

(6) ([10; 23.4, 25.6], [5; 2.3]) If \( n \) is even or \( k = 1 \), then \( V\{n,k\} = 1 \) or 2. If \( n \) is odd and \( k \geq 2 \), then \( V\{n,k\} = 0 \).

PROOF. By definition, (1) and (2) are obvious. As is well-known, if \( n = 2, 4, 8 \), then \( V\{n,n\} = 1 \) (cf., [11; p. 200]). By [6; p. 4], we have \( V\{16,9\} = 1 \). This proves (3). Since \( P_{n,k} \) is \((n-k-1)\)-connected, it follows from suspension theorem that \( P^s\{n,k\} = P\{n,k\} \) if \( n \geq 2k \). Hence (5) follows from (2) and (4).

PROPOSITION (2.2). The number \( V^s\{n,k\} \) is 0, 1, or 2 according as \( n \equiv 1 \pmod{2} \) and \( k \geq 2 \), \( \nu_2(n) \geq \varphi(k) \), or \( 1 \leq \nu_2(n) < \varphi(k) \).

PROOF. Let \( L_k \to P_k \) be the canonical line bundle. Then \( L_k \) is of order \( 2^{\varphi(k)} \) in the J-group of \( P_k \) [2]. If a positive integer \( m \) satisfies \( m + n \equiv 0 \pmod{2^{\varphi(k)}} \), then \( P^s\{n,k\} = *\text{cdg}(P^m_{kL}) \) by stable duality [6; (7.9)], where \( *\text{cdg}(-) \) is the stable codegree [3, 8, 9] which was denoted by \( \text{cd}(mL_k) \) in [8], and \( P^m_{kL} \) is the Thom space of \( mL_k \). Then the assertion follows from (2.1)(4) and [8; 3.5] (cf., [3]).

PROPOSITION (2.3). \( V^s\{n,k\} = V\{n,k\} \).

To prove (2.3), we need

LEMMA (2.4). (1) If \( k \geq 10 \), then \( 2^{\varphi(k)} > 2k \). If \( 1 \leq k \leq 9 \), then \( 2^{\varphi(k)} < 2k \).

(2) Conditions \( 2k > n \geq k \geq 2 \) and \( n \equiv 0 \pmod{2^{\varphi(k)}} \) are satisfied if and only if \((n,k)\) is \((2,2),(4,3),(4,4),(8,5),(8,6),(8,7),(8,8)\) or \((16,9)\).

(3) \( V\{n,k\} = 1 \) for every \((n,k)\) in (2).

PROOF. Write \( k - 1 = 8x + y \) with \( 0 \leq y \leq 7 \). Then \( \varphi(k) = 4x + z \) such that \( z \) is 0 (if \( y = 0 \)), 1 (if \( y = 1 \)), 2 (if \( y = 2,3 \)), and 3 (if \( 4 \leq y \leq 7 \)).
If $x \geq 2$, that is, if $k \geq 17$, then $2^{\varphi(k)} \geq 2^4x > 16(x + 1) \geq 2k$. Hence the following table completes the proof of (1).

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
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<tbody>
<tr>
<td>$2k$</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td>22</td>
<td>24</td>
<td>26</td>
<td>28</td>
<td>30</td>
<td>32</td>
</tr>
<tr>
<td>$2^{\varphi(k)}$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>64</td>
<td>128</td>
<td>128</td>
<td>128</td>
<td>128</td>
<td></td>
</tr>
</tbody>
</table>

If $2k > n \geq k \geq 2$ and $n \equiv 0 \pmod{2^{\varphi(k)}}$, then $k \leq 9$ by (1), hence (2) follows from the table. We have (3) by (2.1)(2)(3).

**Proof of Proposition (2.3).** By (2.1)(5)(6) and (2.2), it suffices to consider the case: $2k > n \equiv 0 \pmod{2}$. If $1 \leq \nu_2(n) < \varphi(k)$ and $n < 2k$, then $V^s\{n, k\} = V\{n, k\} = 2$ by (2.1)(2)(6) and (2.2). If $2k > n \equiv 0 \pmod{2^{\varphi(k)}}$, then $V^s\{n, k\} = V\{n, k\} = 1$ by (2.2) and (2.4)(3).

**Proof of Theorem (1.1).** This follows from (2.1)(4), (2.2) and (2.3).

Let us write $n = (2a + 1)2^b3^c$, where $a, b, c$ are integers and $0 \leq b \leq 3$; let us define $\rho(n) = 2^b + 8c$. As is easily shown, $\nu_2(n) \geq \varphi(k)$ if and only if $\rho(n) \geq k$. Hence we have the following by Theorem (1.1).

**THEOREM (2.5) (Eckmann, Adams).** The fibration $q : V_{n,k} \rightarrow V_{n,1}$ has a cross section if and only if $\rho(n) \geq k$.

3. $P\{n, k\}$

Let $\iota_k \in \pi_k(S^k)$ be the class of the identity map of $S^k$. Then the following is well-known.

**LEMMA (3.1).** The homotopy class of $p_{n,1}$ is $2\iota_{n-1}$ or $0$ according as $n$ is even or odd; $\pi_{n-1}(P_n) = \mathbb{Z}\{\varepsilon\cdot p_n\}$, where $\varepsilon$ is $1$ or $1/2$ according as $n \geq 3$ or $n = 2$.

**LEMMA (3.2).** If $n$ is even, then $P\{n, n-1\} = 2$ for $n \geq 4$ and $P\{n, k\} = 1$ or $2$ for $n > k$. If $n$ is odd and $k \geq 2$, then $P\{n, k\} = 0$. 


PROOF. If $n$ is even, then $P\{n,n-1\}$ is 1 or 2 according as $n = 2$ or $n \geq 4$ by (2.1)(1) and (3.1), hence $P\{n,k\} | 2$ provided $n > k$ by (2.1)(2). The second assertion follows from (2.1)(2)(6).

LEMMA (3.3). If $n = 2, 4, 8$, then $\pi_{2n-1}(P_{2n,n+1}) = \mathbb{Z}\{p_{2n,n+1}\} \oplus \text{Tor}.$

PROOF. Let $n = 2, 4, 8$. The assertion is obvious by (3.1) when $n = 2$. Let $\omega_n : S^{2n-1} \to S^n$ be the Hopf map. We denote by Tor the torsion subgroup of any group. Then $\pi_{2n-1}(S^n) = \mathbb{Z}\{\omega_n\} \oplus \text{Tor}$. Let $TOR$ be the class of torsion groups. By mod $TOR$ Hurewicz theorem, $\pi_*(P_{2n-1,n})$ is a torsion group for $n = 4, 8$. It then follows from the homotopy exact sequence of the pair $(P_{2n,n+1}, P_{2n-1,n})$ that the rank of $\pi_{2n-1}(P_{2n,n+1})$ is 1 and $p_{2n,n+1}$ is of infinite order for $n = 4, 8$. To complete the proof, it suffices to prove

\begin{equation}
\pi_{2n-1}(P_{2n,n}) = \mathbb{Z}\{p_{2n,n}\} \oplus \mathbb{Z} \oplus \text{Tor} \quad \text{for } n = 2, 4, 8.
\end{equation}

We shall prove (3.4). Since the manifold $P_n$ is parallelizable and the Whitney sum of the tangent bundle of $P_n$ with a trivial line bundle is $nL_n$, we have $P_{2n,n} = P_n^n = S^n \wedge P_{n,n} = S^n \vee (S^n \wedge P_n) = S^n \vee (S^n \wedge P_{n-1}) \vee S^{2n-1}$ up to homotopy. Hence $\pi_{2n-1}(P_{2n,n}) \cong \pi_{2n-1}(S^n) \oplus \pi_{2n-1}(S^n \wedge P_{n-1}) \oplus \pi_{2n-1}(S^{2n-1})$ by [11; (1.5) in p.492, (7.12) in p.368], where the isomorphism is induced by inclusion maps, and the rank of $\pi_{2n-1}(P_{2n,n})$ is 2, since $\pi_*(S^n \wedge P_{n-1})$ is a torsion group by mod $TOR$ Hurewicz theorem. We can write $p_{2n,n} \equiv i_1^*(a_n \omega_n) + i_3^*(2\iota_{2n-1})$ (mod Tor) by (3.1), where $a_n \in \mathbb{Z}$ and $i_k$ is a respective inclusion map. As is well-known, $\omega_n = f \circ p_{2n,n}$ for some map $f : P_{2n,n} \to S^n$. Write $f|_{S^n} = \iota_n$ and $f|_{S^{2n-1}} \equiv z \omega_n$ (mod Tor) with $x, z \in \mathbb{Z}$. Then $\omega_n = f \circ p_{2n,n} \equiv (a_n x^2 + 2z) \omega_n$ (mod Tor), hence $a_n$ is odd, therefore (3.4) follows. This completes the proof of (3.3).

Proof of Theorem (1.2). As is easily shown, $\nu_2(n) \geq \varphi(n)$ if and only if $n = 2, 4, 8$. Then the assertion for $n = k$ follows from (1.1) and (2.1)(1).
If $V\{n,k\}$ is 0 or 2, then $P\{n,k\} = V\{n,k\}$ by (1.1), (2.1)(2) and (3.2). Suppose that $V\{n,k\} = 1$ and $n > k$. Then $n \equiv 0 \pmod{2^{\varphi(k)}}$ by (1.1). If $k \geq 10$ or $k \leq 9$ and $n \geq 2k$, then $V\{n,k\} = P\{n,k\}$ by (2.1)(5) and (2.4)(1). If $k \leq 9$ and $n < 2k$, then $(n,k)$ is $(4,3)$, $(8,5)$, $(8,6)$, $(8,7)$, or $(16,9)$ by (2.4)(2), and $P\{n,k\} = 2$ except for $(n,k) = (8,6)$ by (3.1), (3.2) and (3.3). We then have $P\{8,6\} = 2$ by (2.1)(2).

Proof of Corollary (1.3). The assertions are obvious when $k = 1$ or $k = n - 1$, by (3.1). Suppose $2 \leq k \leq n - 2$. If $n$ is odd, then $\pi_* (P_{n,k}) \in \text{TOR}$ for $k$ even by mod $\text{TOR}$ Hurewicz theorem, and $i_* : \pi_* (S^{n-k}) \rightarrow \pi_* (P_{n,k})$ is a $\text{TOR}$-isomorphism for $k$ odd by mod $\text{TOR}$ Whitehead theorem. Let $j : P_{n-1,k-1} \rightarrow P_{n,k}$ be the inclusion map and $f : S^{n-1} \rightarrow P_{n,k}$ a map with $q_*(f) = P\{n,k\}_{i_{n-1}}$. If $n$ is even, then, by mod $\text{TOR}$ Whitehead theorem, $q_* : \pi_* (P_{n,k}) \rightarrow \pi_* (S^{n-1})$ and $(f \vee j)_* : \pi_* (S^{n-1} \vee P_{n-1,k-1}) \rightarrow \pi_* (P_{n,k})$ are $\text{TOR}$-isomorphisms when $k$ is odd and even respectively. Then the assertions can be proved easily by using (1.1), (1.2) and (3.4).

References