<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>On real James numbers (Recent development of algebraic topology)</td>
</tr>
<tr>
<td>著者</td>
<td>Oshima, Hideaki</td>
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On real James numbers

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1. Introduction

The purpose of this note is to determine the real James numbers. Throughout the note, $n$, $l$, $k$ denote integers with $n \geq l \geq k \geq 1$ and $n \geq 2$. Let $P_k$ denote the real projective space of dimension $k - 1$, $P_{l,k} = P_l/P_{l-k}$ the stunted projective space, and $V_{l,k} = O(l)/O(l-k)$ the Stiefel manifold of orthonormal $k$-frames in $\mathbb{R}^l$. Note that $P_{k,k}$ is the union of $P_k$ and a disjoint base point. Write $q : V_{l,k} \to V_{l,1} = S^{l-1}$ for the projection on to the last component, and $q : P_{l,k} \to P_{l,1} = S^{l-1}$ for the quotient map. There is a commutative square [6]:

$$
\begin{array}{ccc}
V_{n,k} & \xrightarrow{q} & V_{n,1} \\
\uparrow & & \uparrow \\
P_{n,k} & \xrightarrow{q} & P_{n,1}
\end{array}
$$

The unstable real James numbers $V\{n,k\}$ and $P\{n,k\}$ are non-negative integers which generate respectively the images of

$$
q_* : \pi_{n-1}(V_{n,k}) \to \pi_{n-1}(S^{n-1}) = \mathbb{Z},
$$

$$
q_* : \pi_{n-1}(P_{n,k}) \to \pi_{n-1}(S^{n-1}) = \mathbb{Z}.
$$

In the same way, replacing homotopy group $\pi_{n-1}(\cdot)$ by stable homotopy group $\pi_{n-1}(\cdot)$ we have the stable real James numbers $V^s\{n,k\}$ and $P^s\{n,k\}$. Let us denote the exponent of 2 in a positive integer $k$ by $\nu_2(k)$; define $\varphi(k)$ to be the number of integers $s$ such that $0 < s < k$ and $s \equiv 0, 1, 2, 4 \pmod{8}$. Our results are
THEOREM (1.1). We have $P^s\{n,k\} = V^s\{n,k\} = V\{n,k\}$ which is equal to 0, 1, or 2 according as $n \equiv 1 \pmod{2}$ and $k \geq 2$, $\nu_2(n) \geq \varphi(k)$, or $1 \leq \nu_2(n) < \varphi(k)$.

THEOREM (1.2). We have $V\{n,k\} = P\{n,k\}$ except for the following cases: (1) if $(n,k) = (4,3),(8,5),(8,6),(8,7),(16,9)$, then $V\{n,k\} = 1$ and $P\{n,k\} = 2$; (2) if $n = k = 2m$ with $m = 1,2,4$, then $V\{n,k\} = 1$ and $P\{n,k\} = 0$; (3) if $n = k = 2m$ with $m \neq 1,2,4$, then $V\{n,k\} = 2$ and $P\{n,k\} = 0$.

Let $p_n : S^{n-1} \rightarrow P_n$ be the canonical double covering map and $p_{n,k} : S^{n-1} \rightarrow P_{n,k}$ ($n > k$) the composition of $p_n$ with the quotient map.

COROLLARY (1.3). The rank of $\pi_{n-1}(P_{n,k})$, for $n > k$, is 0, 2, or 1 according as $n \equiv 1 \pmod{2}$ and $2 \leq k \leq n-2$, $n = 2k$ and $k \equiv 0 \pmod{2}$, or otherwise. The map $p_{n,k}$ generates a free direct summand of $\pi_{n-1}(P_{n,k})$ if and only if $n = k + 1 \geq 3$, $P\{n,k\} = 2$ or $(n,k) = (4,2),(8,4),(16,8)$.

Note that a part of (1.1) is not new. Indeed $V\{n,k\}$ was already known [1,4,5]. We shall calculate it again by using codegree [3,8,9]. We shall prove (1.1) in §2, and (1.2), (1.3) in §3.

2. $V\{n,k\}$

The symbol $a \mid b$ means that $b = ma$ for some integer $m$.

LEMMA (2.1). (1) $V\{n,n\} = V\{n,n-1\}$; $P^s\{n,1\} = V^s\{n,1\} = V\{n,1\} = P\{n,1\} = 1$; $P\{n,n\} = 0$.

(2) $V^s\{n,k\} \mid V\{n,k\} \mid P\{n,k\}$; $V\{n,k\} \mid V\{n,l\}$ and $P\{n,k\} \mid P\{n,l\}$ if $n \geq l \geq k \geq 1$.

(3) $V\{2,2\} = V\{4,4\} = V\{8,8\} = V\{16,9\} = 1$.

(4) ([7; 4.2]) $P^s\{n,k\} = V^s\{n,k\}$. 
(5) If \( n \geq 2k \), then \( P^s\{n, k\} = V^s\{n, k\} = V\{n, k\} = P\{n, k\} \).

(6) ([10; 23.4, 25.6], [5; 2.3]) If \( n \) is even or \( k = 1 \), then \( V\{n, k\} = 1 \) or \( 2 \). If \( n \) is odd and \( k \geq 2 \), then \( V\{n, k\} = 0 \).

**Proof.** By definition, (1) and (2) are obvious. As is well-known, if \( n = 2, 4, 8 \), then \( V\{n, n\} = 1 \) (cf., [11; p. 200]). By [6; p. 4], we have \( V\{16, 9\} = 1 \). This proves (3). Since \( P_{n,k} \) is \((n-k-1)\)-connected, it follows from suspension theorem that \( P^s\{n, k\} = P\{n, k\} \) if \( n \geq 2k \). Hence (5) follows from (2) and (4).

**Proposition (2.2).** The number \( V^s\{n, k\} \) is 0, 1, or 2 according as \( n \equiv 1 \) (mod 2) and \( k \geq 2 \), \( \nu_2(n) \geq \varphi(k) \), or \( 1 \leq \nu_2(n) < \varphi(k) \).

**Proof.** Let \( L_k \to P_k \) be the canonical line bundle. Then \( L_k \) is of order \( 2^{\varphi(k)} \) in the \( J \)-group of \( P_k \) [2]. If a positive integer \( m \) satisfies \( m + n \equiv 0 \) (mod \( 2^{\varphi(k)} \)), then \( P^s\{n, k\} = \ast \text{cdg}(P_k^{mL}, m) \) by stable duality [6; (7.9)], where \( \ast \text{cdg}(\cdot) \) is the stable codegree [3, 8, 9] which was denoted by \( \text{cd}(mL_k) \) in [8], and \( P_k^{mL} \) is the Thom space of \( mL_k \). Then the assertion follows from (2.1)(4) and [8; 3.5] (cf., [3]).

**Proposition (2.3).** \( V^s\{n, k\} = V\{n, k\} \).

To prove (2.3), we need

**Lemma (2.4).** (1) If \( k \geq 10 \), then \( 2^{\varphi(k)} > 2k \). If \( 1 \leq k \leq 9 \), then \( 2^{\varphi(k)} < 2k \).

(2) Conditions \( 2k > n \geq k \geq 2 \) and \( n \equiv 0 \) (mod \( 2^{\varphi(k)} \)) are satisfied if and only if \( (n, k) \) is \((2, 2), (4, 3), (4, 4), (8, 5), (8, 6), (8, 7), (8, 8) \) or \((16, 9)\).

(3) \( V\{n, k\} = 1 \) for every \( (n, k) \) in (2).

**Proof.** Write \( k - 1 = 8x + y \) with \( 0 \leq y \leq 7 \). Then \( \varphi(k) = 4x + z \) such that \( z \) is 0 (if \( y = 0 \)), 1 (if \( y = 1 \)), 2 (if \( y = 2, 3 \)), and 3 (if \( 4 \leq y \leq 7 \)).
If \( x \geq 2 \), that is, if \( k \geq 17 \), then \( 2^{\varphi(k)} \geq 2^{4x} > 16(x + 1) \geq 2k \). Hence the following table completes the proof of (1).

<table>
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<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
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<tbody>
<tr>
<td>( 2k )</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td>22</td>
<td>24</td>
<td>26</td>
<td>28</td>
<td>30</td>
<td>32</td>
</tr>
<tr>
<td>( 2^{\varphi(k)} )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>64</td>
<td>128</td>
<td>128</td>
<td>128</td>
<td>128</td>
<td>128</td>
</tr>
</tbody>
</table>

If \( 2k > n \geq k \geq 2 \) and \( n \equiv 0 \pmod{2^{\varphi(k)}} \), then \( k \leq 9 \) by (1), hence (2) follows from the table. We have (3) by (2.1)(2)(3).

Proof of Proposition (2.3). By (2.1)(5)(6) and (2.2), it suffices to consider the case: \( 2k > n \equiv 0 \pmod{2} \). If \( 1 \leq \nu_2(n) < \varphi(k) \) and \( n < 2k \), then \( V^s\{n, k\} = V\{n, k\} = 2 \) by (2.1)(2)(6) and (2.2). If \( 2k > n \equiv 0 \pmod{2^{\varphi(k)}} \), then \( V^s\{n, k\} = V\{n, k\} = 1 \) by (2.2) and (2.4)(3).

Proof of Theorem (1.1). This follows from (2.1)(4), (2.2) and (2.3).

Let us write \( n = (2a + 1)2^b4c \), where \( a, b, c \) are integers and \( 0 \leq b \leq 3 \); let us define \( \rho(n) = 2^b + 8c \). As is easily shown, \( \nu_2(n) \geq \varphi(k) \) if and only if \( \rho(n) \geq k \). Hence we have the following by Theorem (1.1).

**Theorem** (2.5) (Eckmann, Adams). The fibration \( q : V_{n,k} \to V_{n,1} \) has a cross section if and only if \( \rho(n) \geq k \).

3. \( P\{n, k\} \)

Let \( \iota_k \in \pi_k(S^k) \) be the class of the identity map of \( S^k \). Then the following is well-known.

**Lemma** (3.1). The homotopy class of \( p_{n,1} \) is \( 2\iota_{n-1} \) or 0 according as \( n \) is even or odd; \( \pi_{n-1}(P_n) = \mathbb{Z}\{\epsilon \cdot p_n\} \), where \( \epsilon \) is 1 or 1/2 according as \( n \geq 3 \) or \( n = 2 \).

**Lemma** (3.2). If \( n \) is even, then \( P\{n, n-1\} = 2 \) for \( n \geq 4 \) and \( P\{n, k\} = 1 \) or 2 for \( n > k \). If \( n \) is odd and \( k \geq 2 \), then \( P\{n, k\} = 0 \).
PROOF. If \( n \) is even, then \( P\{n, n-1\} \) is 1 or 2 according as \( n = 2 \) or \( n \geq 4 \) by (2.1)(1) and (3.1), hence \( P\{n, k\} \mid 2 \) provided \( n > k \) by (2.1)(2). The second assertion follows from (2.1)(2)(6).

**Lemma (3.3).** If \( n = 2, 4, 8 \), then \( \pi_{2n-1}(P_{2n,n+1}) = \mathbb{Z}\{p_{2n,n+1}\} \oplus \text{Tor} \).

**Proof.** Let \( n = 2, 4, 8 \). The assertion is obvious by (3.1) when \( n = 2 \). Let \( \omega_n : S^{2n-1} \to S^n \) be the Hopf map. We denote by Tor the torsion subgroup of any group. Then \( \pi_{2n-1}(S^n) = \mathbb{Z}\{\omega_n\} \oplus \text{Tor} \). Let \( T\text{OR} \) be the class of torsion groups. By mod \( T\text{OR} \) Hurewicz theorem, \( \pi_*(P_{2n-1,n}) \) is a torsion group for \( n = 4, 8 \). It then follows from the homotopy exact sequence of the pair \( (P_{2n,n+1}, P_{2n-1,n}) \) that the rank of \( \pi_{2n-1}(P_{2n,n+1}) \) is 1 and \( p_{2n,n+1} \) is of infinite order for \( n = 4, 8 \). To complete the proof, it suffices to prove

\[
\pi_{2n-1}(P_{2n,n}) = \mathbb{Z}\{p_{2n,n}\} \oplus \mathbb{Z} \oplus \text{Tor} \quad \text{for} \quad n = 2, 4, 8.
\]

We shall prove (3.4). Since the manifold \( P_n \) is parallelizable and the Whitney sum of the tangent bundle of \( P_n \) with a trivial line bundle is \( nL_n \), we have \( P_{2n,n} = P_{n}^{nL} = S^n \wedge P_{n,n} = S^n \vee (S^n \wedge P_n) = S^n \vee (S^n \wedge P_{n-1}) \vee S^{2n-1} \) up to homotopy. Hence \( \pi_{2n-1}(P_{2n,n}) \cong \pi_{2n-1}(S^n) \oplus \pi_{2n-1}(S^n \wedge P_{n-1}) \oplus \pi_{2n-1}(S^{2n-1}) \) by [11; (1.5) in p.492, (7.12) in p.368], where the isomorphism is induced by inclusion maps, and the rank of \( \pi_{2n-1}(P_{2n,n}) \) is 2, since \( \pi_*(S^n \wedge P_{n-1}) \) is a torsion group by mod \( T\text{OR} \) Hurewicz theorem. We can write \( p_{2n,n} \equiv i_1*(a_n \omega_n) + i_3*(2\iota_{2n-1}) \pmod{\text{Tor}} \) by (3.1), where \( a_n \in \mathbb{Z} \) and \( i_k \) is a respective inclusion map. As is well-known, \( \omega_n = f \circ p_{2n,n} \) for some map \( f : P_{2n,n} \to S^n \). Write \( f|_{S^n} = x \iota_n \) and \( f|_{S^{2n-1}} = z \omega_n \pmod{\text{Tor}} \) with \( x, z \in \mathbb{Z} \). Then \( \omega_n = f \circ p_{2n,n} \equiv (a_n x^2 + 2z) \omega_n \pmod{\text{Tor}} \), hence \( a_n \) is odd, therefore (3.4) follows. This completes the proof of (3.3).

**Proof of Theorem (1.2).** As is easily shown, \( \nu_2(n) \geq \varphi(n) \) if and only if \( n = 2, 4, 8 \). Then the assertion for \( n = k \) follows from (1.1) and (2.1)(1).
If $V\{n, k\}$ is 0 or 2, then $P\{n, k\} = V\{n, k\}$ by (1.1), (2.1)(2) and (3.2). Suppose that $V\{n, k\} = 1$ and $n > k$. Then $n \equiv 0 \pmod{2^\varphi(k)}$ by (1.1). If $k \geq 10$ or $k \leq 9$ and $n \geq 2k$, then $V\{n, k\} = P\{n, k\}$ by (2.1)(5) and (2.4)(1). If $k \leq 9$ and $n < 2k$, then $(n, k)$ is $(4, 3)$, $(8, 5)$, $(8, 6)$, $(8, 7)$, or $(16, 9)$ by (2.4)(2), and $P\{n, k\} = 2$ except for $(n, k) = (8, 6)$ by (3.1), (3.2) and (3.3). We then have $P\{8, 6\} = 2$ by (2.1)(2).

Proof of Corollary (1.3). The assertions are obvious when $k = 1$ or $k = n - 1$, by (3.1). Suppose $2 \leq k \leq n - 2$. If $n$ is odd, then $\pi_*(P_{n,k}) \in T\mathcal{O}R$ for $k$ even by mod $T\mathcal{O}R$ Hurewicz theorem, and $i_* : \pi_*(S^{n-k}) \to \pi_*(P_{n,k})$ is a $T\mathcal{O}R$-isomorphism for $k$ odd by mod $T\mathcal{O}R$ Whitehead theorem. Let $j : P_{n-1,k-1} \to P_{n,k}$ be the inclusion map and $f : S^{n-1} \to P_{n,k}$ a map with $g_*(f) = P\{n, k\}i_{n-1}$. If $n$ is even, then, by mod $T\mathcal{O}R$ Whitehead theorem, $g_* : \pi_*(P_{n,k}) \to \pi_*(S^{n-1})$ and $(f \vee j)_* : \pi_*(S^{n-1} \vee P_{n-1,k-1}) \to \pi_*(P_{n,k})$ are $T\mathcal{O}R$-isomorphisms when $k$ is odd and even respectively. Then the assertions can be proved easily by using (1.1), (1.2) and (3.4).

References

