TRANSFER IMAGE FOR STUNTED PROJECTIVE SPACES

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Let $N_i \rightarrow M_i = L_i N_i \rightarrow N_{i+1} \rightarrow \Sigma N_i$ be the cofiber sequence such that $S^0 = N_0 \rightarrow M_0 \rightarrow M_1 \rightarrow \ldots$ is the geometric realization by Ravenel [Ra] of the chromatic resolution, where $L_i$ is the Bousfield localization [Bo] with respect to the $v_i^{-1} BP_*$-homology. Throughout this note, we assume that spectra are always localized at an odd prime $p$. The authors of [BC] revealed that the double $S^1$-transfer

$$t_2 : \Sigma^2 CP_0^\infty \wedge CP_0^\infty \rightarrow S^0$$

factors through $\delta_1 \delta_2 : \Sigma^{-2} N_2 \rightarrow S^0$. The purpose of the present note is to remark that their result is also valid for the double $S^1$-transfers

$$t_2 : \Sigma^{-2(m+n-1)} CP_m^\infty \wedge CP_n^\infty \rightarrow S^0$$

for all integers $m$ and $n$.

Here $CP_k^\infty$ denotes the suspension spectrum of the Thom space for the $k$-Whitney sum $k \xi$ of the canonical complex line bundle $\xi$ over $CP^\infty$.

**Theorem.** For the second stage $N_2$ of the chromatic filtration at an odd prime $p$, there is a map $u : \Sigma^{-2(m+n-1)} CP_m^\infty \wedge CP_n^\infty \rightarrow \Sigma^{-2} N_2$ satisfying $\delta_1 \delta_2 \circ u = t_2$.

We will prove the theorem using an analogous method as in [BC], that is, we will see that the strategy in [BC; Th.5.2] is applicable to our situation. The crucial point is to prove the following lemma, in which a map $\bar{U}_1$ is defined later in (2).
Lemma 1. There is a map $U_2$ satisfying the following homotopy commutative diagram:

$$
\begin{array}{ccc}
\Sigma^{2m}CP^\infty_{n+1} & \xrightarrow{i \wedge 1} & CP^\infty_m \wedge CP^\infty_{n+1} \\
\downarrow \varrho_1 & & \downarrow \varrho_2 \\
\Sigma^{2(m+n)}N_1 & \xrightarrow{r} & \Sigma^{2(m+n)}M_1
\end{array}
$$

where $i$ is the bottom inclusion.

$t_2$ is the composition $t(t \wedge 1)$ of the transfer maps $t : \Sigma^{-2k+1}CP^\infty_k \to S^0$ for $k = m$ and $n$, and $t$ is homotopic to the attaching map to the bottom cell of $\Sigma^{-2k+2}CP^\infty_{k-1}$ ([Kn]). Thus the lemma induces a required map $u : CP^\infty_m \wedge CP^\infty_{n+1} \to \Sigma^{2(m+n)}N_2$ in our theorem, and the rest of this note is devoted to the proof of this lemma.

Let $U_1 \in \pi^{2k}(CP^\infty_k; Q)$ be the Thom class of $k\xi$ for the rational theory $SQ = HQ$, $q : SQ \to SQ/Z(p)$ the mod $Z(p)$ quotient and $c : CP^\infty_k \to CP^\infty_{k+1}$ the collapsing map. Then, we have an element $\bar{U}_1 \in \pi^{2k}(CP^\infty_k; Q/Z(p))$ satisfying $q_*(U_1) = c^*(\bar{U}_1)$. Since $N_1 = SQ/Z(p)$, $\bar{U}_1$ represents a map

$$
(2) \quad \bar{U}_1 : CP^\infty_{n+1} \to \Sigma^{2n}N_1
$$

which is the map in Lemma 1.

Lemma 1 will be established by using Lemma 8 below, and before it we need to prepare some generalization of a result due to Miller [Mi]. Let $E$ be a ring spectrum such that $E_*E$ is flat over $E_* = \pi_*E$ and $H^0(E; Q) \cong Q$. Furthermore, we assume that $E$ is oriented by $x \in E^2(CP^\infty)$. Then, $E^*(CP^\infty_k) \cong E^*[x]\{U\}$ for a Thom class $U$ of $k\xi$, and $E_*(CP^\infty_k) \cong E_*\{\beta_k, \beta_{k+1}, \ldots\}$, where $\beta_i$ is the dual element of $Ux^{i-k}$. We put $\hat{\beta}_k(T) = \sum_{i \geq k} \beta_i T^{i-k} \in E_*(CP^\infty_k)[[T]]$.

Let $\log^E T$ be the power series which gives a strict isomorphism from the formal group law defined by the orientation class $x$ of $E$ to the additive formal group law, over $E_* \otimes Q$. Then, by the method designed in [Mi], we have the following:
Lemma 3. \((\bar{U}_1)_*(T\hat{\beta}_{k+1}(T)) = \left(\frac{\log^{E} T}{T}\right)^k - 1 \text{ in } \pi_*(E; \mathbb{Q}/\mathbb{Z})[[T]].\)

Now, we consider the spectrum \(E(1)\) which represents a wedge summand of the complex K-theory \(K(p)\) localized at \(p\) and whose coefficient group is \(E(1)_* = \mathbb{Z}(p)[v_1, v_1^{-1}]\) for \(v_1 \in E(1)_2(p - 1)\). Then Lemma 3 holds for the case of \(E = E(1)\), and in this case the formula of \(\log^{E(1)} T\) is given by

**Theorem 4 (S. Araki [Ar]).**

\[
\log^{E(1)} T = \sum_{i \geq 0} \frac{v_1}{p^i - 1} T^{p^i}
\]

Let \(\exp^{E(1)} T\) be the formal power inverse of \(\log^{E(1)} T\), and put

\[
(5) \quad \left(\frac{T}{\exp^{E(1)} T}\right)^k = \sum_{i \geq 0} B(k, i)T^i \in (E(1)_* \otimes \mathbb{Q})[[T]]
\]

for \(B(k, i) \in E(1)_{2i} \otimes \mathbb{Q}\). Clearly \(B(m, 0) = 1\), and \(B(1, i)\) is the \(E(1)\)-theory Bernoulli number in the sense of [Mi].

Let \(\psi^\gamma : E(1) \rightarrow E(1)\) be the stable Adams operation for a positive integer \(\gamma\) which generates the unit group of \(\mathbb{Z}/p^2\). Then \(\psi^\gamma\) is a ring homomorphism on \(E(1)^*()\), and it holds that \(\psi^\gamma(v_1) = \gamma^{p-1}v_1\) and \(\psi^\gamma(x) = (1/\gamma)[\gamma](x)\), where \(x \in E(1)^2(\mathbb{C}P^\infty)\) is the orientation class and \([\gamma](x)\) means the formal group sum of \(\gamma\) numbers of \(x\). The following is easy to see by these properties, (5) and Theorem 4.

**Lemma 6.** Let \(U_k^{E(1)} \in E(1)^2(\mathbb{C}P_k^\infty)\) be a Thom class of \(k\xi\). Then we have the following:

1. \(\psi^\gamma(U_k^{E(1)}) = U_k^{E(1)}(1 + \sum_{i > 0} (\gamma^i - 1)B(-k, i)x^{-k}(\log^{E(1)} x)^{k+i});\)
2. \(\psi^\gamma(\log^{E(1)} x) = \log^{E(1)} x.\)
Let $Ad$ be the fiber spectrum of $\psi^{\gamma} - 1 : E(1) \to E(1)$, and $j : Ad \to E(1)$ the inclusion. A unit $\iota \in E(1)_0$ induces a map $\iota : N_1 = SQ/Z(p) \to E(1) \wedge SQ/Z(p)$, and we have $\iota = j_*(\iota')$ for a unique $\iota' \in \pi_0(Ad; Q/Z)$, since $(\psi^{\gamma} - 1)_*(\iota) = 0$ and $j_*$ is monomorphic. By [Ra], there is an equivalence

(7) $\zeta : M_1 \simeq Ad Q/Z(p)$ with $\iota' \simeq \zeta \circ \tau : SQ/Z(p) = N_1 \to Ad \wedge SQ/Z(p)$.

Lemma 8. There is an element $u_2 \in E(1)^{2(m+n)}(CP_{m}^\infty \wedge CP_{n+1}^\infty) \otimes Q$ satisfying

1. $(i \wedge 1)^* q_*(u_2) = i_*(\bar{U}_1)$ in $E(1)^{2(m+n)}(\Sigma^{2m} CP_{n+1}^\infty; Q/Z(p))$ and
2. $(\psi^{\gamma} - 1)_*(u_2) \in E(1)^{2(m+n)}(CP_{m}^\infty \wedge CP_{n+1}^\infty)$,

where $i : S^{2m} \to CP_{m}^\infty$ and $q : E(1) \wedge SQ \to E(1) \wedge SQ/Z(p)$ are the bottom inclusion and the mod $Z(p)$ quotient respectively.

By (2) in this lemma, we have $q_*(u_2) = j_*(u_2')$ for some $u_2' \in Ad^{2(m+n)}(CP_{m}^\infty \wedge CP_{n+1}^\infty; Q/Z)$. Since $j_* : Ad^{2(m+n)}(CP_{m}^\infty \wedge CP_{n+1}^\infty; Q/Z) \to E(1)^{2(m+n)}(CP_{m}^\infty \wedge CP_{n+1}^\infty; Q/Z)$ is monomorphic, we have $(u_2') \circ (i \wedge 1) = (\iota' \circ \bar{U}_1)$ in $Ad^{2(m+n)}(\Sigma^{2m} CP_{n+1}; Q/Z)$ by (1) in Lemma 8. Thus, by (7), we can take $U_2$ in Lemma 1 to be $(\zeta)^{-1} \circ u_2'$, and thus Lemma 8 yields Lemma 1.

We put

$$g_{n}(T) = T^{-1} \left( \left( \frac{\log E(1) T}{T} \right)^n - 1 \right) \in (E(1)_* \otimes Q)[[T]],$$

and consider the following element of $(E(1)_* \otimes Q)[[S, T]]$ by using $B(k, i)$ in (5):

$$h_{m,n}(S, T) = \sum_{k, l > 0} a_{k, l} B(-m, k) B(-n, l) S^{-m} (\log E(1) S)^{m+k} T^{-n-1} (\log E(1) T)^{n+l},$$

where $a_{k, l} = (\gamma^k - 1)/(\gamma^{k+l} - 1)$. Then $g_{n}(T) = (\bar{U}_1)_* (\hat{\beta}_{n+1}(T))$ in $\pi_*(E(1); Q/Z)[[T]]$ by Lemma 3. By putting $E(1)^*(CP^\infty_0 \wedge CP^\infty_0) = E(1)^*[[x, y]]$, we regard $E(1)^*(CP_{m+1}^\infty \wedge CP_{n+1}^\infty)$ as a free $E(1)^*[[x, y]]$-module with $U_{m+1} E(1) U_{n+1}$ as a base. Then Lemma 8 follows from the next Lemma.
Lemma 9. The element $u_2 = U_m^{E(1)}U_{n+1}^{E(1)}(g_n(y) + h_{m,n}(x,y))$ of $E(1)^{2(m+n)}(CP_m^\infty \land CP_{n+1}^\infty) \otimes Q$ satisfies (1) and (2) of Lemma 8.

The proof of Lemma 9 is straightforward using Lemmas 3 and 6, and we can complete the proof.

References


