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SOME RESULTS AND PROBLEMS ON ANR'S FOR STRATIFIABLE SPACES

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Stratifiable spaces are also called $M_3$-spaces, which were introduced by Ceder [Ce] and renamed by Borges [Bo]. The class $S$ of stratifiable spaces contains both metrizable spaces and CW-complexes and has many desirable properties (cf. [Bo]). And CW-complexes are ANR for the class $S$ [Ca1]. Hence it has been expected that ANR theory for the class $S$ is established so successfully as the class $M$ of metrizable spaces. An absolute (neighborhood) retract for a class $C$ is simply called an AR($C$) (resp. ANR($C$)). Although ANR($S$)'s have been studied by Borges, Cauty and Miwa, etc., many problems are still left. In this note, we present the result of [GS] and some related problems.

The join of spaces $X$ and $Y$ is defined as the space

$$X * Y = X \cup X \times Y \times (0,1) \cup Y$$

admitting the topology generated by all open sets in the product space $X \times Y \times (0,1)$ and the following sets:

$$U \cup U \times Y \times (0,t) \quad \text{and} \quad X \times V \times (t,1) \cup V,$$

where $U$ and $V$ are open in $X$ and $Y$, respectively, and $0 < t < 1$. In [Ca3], this join is denoted by $X \# Y$ in order to distinguish from the join as the quotient space of $X \times Y \times I$.

The mapping cylinder of a map $f : X \rightarrow Y$ is defined as the space

$$M(f) = X \times [0,1) \cup Y$$

admitting the topology generated by all open sets in the product space $X \times [0,1)$ and the following sets:

$$f^{-1}(V) \times (t,1) \cup V,$$

where $V$ is open in $Y$ and $0 < t < 1$. Notice that $M(f)$ is not a quotient space of $X \times I \oplus Y$. It is easily observed that $X \# Y$ is homeomorphic to

$$M(pr_X) \cup_{X \times Y \times \{0\}} M(pr_Y),$$

where $pr_X : X \times Y \rightarrow X$ and $pr_Y : X \times Y \rightarrow Y$ are the projections. By using the Bing Metrization Theorem, it is easy to see that $M(f)$ (hence $X \# Y$) is metrizable if so are $X$ and $Y$. Extending [Ca3, Lemma 6.3], we can show the following:
LEMMA. For any map $f: X \to Y$, the mapping cylinder $M(f)$ is stratifiable if so are $X$ and $Y$.

By [Hy] (cf. [KL]), $M(f)$ (hence $X \ast Y$) is an $ANR(M)$ if so are $X$ and $Y$. This is expected to be true for $ANR(S)$'s. However we cannot apply this method to stratifiable spaces (cf. [Ca1]). In fact, San-ou [Sa] constructed a stratifiable space $X$ with $A$ a closed set such that $(X, A)$ is not semi-canonical. (For the definition of semi-canonical pairs, refer to [Hy].) In his construction, by replacing $N$ and $Q$ by $R$, we have a stratifiable locally convex linear topological space $X$, hence $X$ is an $AR(S)$, such that $(X, A)$ is not semi-canonical, where $A = \{0\}$. Consider the mapping cylinder $M(i)$ of the inclusion $i: X \setminus A \subset X$. Then $(M(i), X)$ is not semi-canonical. And $((X \setminus A) \ast X, X)$ is not semi-canonical. Thus we need another approach.

To characterize $AR$'s, Borges [Bo] introduced the concept of hyperconnectedness. For a space $X$, let $F(X)$ be the full simplicial complex with $X$ the set of vertices, i.e., $X = F(X)^{(0)}$. Introducing a topology on $|F(X)|$, Cauty [Ca] constructed a test space $Z(X)$ such that a stratifiable space $X$ is an $ANR(S)$ if and only if $X$ is a neighborhood retract of $Z(X)$. Improving the construction of $Z(X)$, Miwa [Mi] constructed a hyperconnected space $E(X)$ containing $X$ as a closed set and proved that $E(X)$ is stratifiable if so is $X$. Then any stratifiable space $X$ can be embedded in an $AR(S) E(X)$ as a closed set. By his construction, any map $f: X \to Y$ extends to the map $\tilde{f}: E(X) \to E(Y)$ which is a simplicial map from $F(X)$ to $F(Y)$. For this extension $\tilde{f}$, we have the following:

**Theorem 1.** Let $\tilde{f}: E(X) \to E(Y)$ be the extension of a map $f: X \to Y$. Then $M(\tilde{f})$ is hyperconnected. Hence $M(\tilde{f})$ is an $AR(S)$ in case $X$ and $Y$ are stratifiable.

Since $M(f)$ is a closed subset of $M(\tilde{f})$, the following problem reduces to prove that $M(f)$ is a neighborhood retract of $M(\tilde{f})$.

**Problem 1.** Let $f: X \to Y$ be a map between $ANR(S)$'s. Is the mapping cylinder $M(f)$ an $ANR(S)$?

Although this has not yet been succeeded, the following holds:

**Theorem 2.** Let $X$ and $Y$ be $ANR(S)$'s and $f: X \to Y$ a Hurewicz fibration. Then the mapping cylinder $M(f)$ is an $ANR(S)$.

Since the projection $pr_X: X \times Y \to X$ is a Hurewicz fibration, we have the following generalization of [Ca3, Corollary 6.2]:

**Theorem 3.** If $X$ and $Y$ are $ANR(S)$'s then so is the join $X \ast Y$. 
Remark. We can also prove Theorem 3 by showing that $E(X) * E(Y)$ is hypercon- 
ected and that $X * Y$ is a neighborhood retract of $E(X) * E(Y)$. This approach is 
 easier than the above approach.

In [Ca$_2$], Cauty asserted that the adjunction space of ANR($S$)'s is also an ANR($S$), 
but his key lemma is false [Sa] (even if $(X, A)$ is a pair of ANR($S$)'s as shown in the 
above). Thus his assertion is still a conjecture and Theorem 3 is still open for the 
quotient topology:

**Problem 2.** Let $X$ and $Y$ be ANR($S$)'s. Is the join $X * Y$ with the quotient topology an ANR($S$)? For any map $f : X \rightarrow Y$, is the mapping cylinder $M(f)$ with the quotient topology an ANR($S$)?

In [Ca$_3$], Cauty proved that the direct limit of the tower of compact ANR($M$)'s is an 
ANR($S$). It is natural to ask the following:

**Problem 3.** Let $X_1 \subset X_2 \subset \cdots$ be a tower of ANR($S$)'s such that each $X_{n+1}$ is a closed subspace of $X_n$. Is the direct limit $\lim X_n$ an ANR($S$)?

**References**


