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ON THE CLOSED IMAGES OF A DEVELOPABLE SPACE

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ABSTRACT

We study the properties of the image of a developable space and an orthocompact developable space under a closed mapping, comparing with Lašnev spaces. Two classes $C$ and $C'$ are defined and their properties are given.

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1. Introduction.

Throughout this paper, all spaces are assumed to be $T_1$ topological ones and mappings to be continuous and onto. The letter $N$ always denotes natural numbers. The letter $\mathbb{Z}$ always denotes a convergent sequence of points of a space such that $\mathbb{Z} = \{z_n : n \leq N\}$ and $z \rightarrow p$ implies that $\mathbb{Z}$ converges to $p$ as $n \rightarrow \infty$. We denote the topology of $X$ by $\tau_X$. We use the brief expressions HCP and IP in place of "hereditarily closure-preserving" and "interior-preserving", respectively.

As a nice generalization of metric spaces, we have a class of developable spaces, which are defined to be ones $X$ having a sequence $\{ U_n : n \in N\}$ of open covers of $X$ such that for each point $p \in X$, $\{S(p, U_n) : n \in N\}$ is a local base at $p$ in $X$. Until now, the image of a metric space under a closed mapping, called a Lašnev space, is widely studied. But the study of the image of a developable space, briefly called the closed image of a developable space, has not been published yet. In this paper, we begin on its study, especially using the notion of pair-networks. This is our aim of this paper.

To start with, we give the meanings to the special spaces used later. A space $X$ is called semi-stratifiable if there exists a function $\mathcal{O} : \{\text{closed subsets of } X\} \times N \rightarrow \tau_X$, called the semi-stratification of $X$, satisfying the following conditions:

(1) For each closed subset $F$ of $X$,
$$F = \bigcap \{\mathcal{O}(F, n) : n \in N\}$$
and \( O(F, n + 1) \subseteq O(F, n) \) for each \( n \).

(2) If \( F, G \) are closed subsets of \( X \) such that \( F \subseteq G \),
then \( O(F, n) \subseteq O(G, n) \) for each \( n \).

2. The closed image of a developable space.

DEFINITION [2]. Let \( P = \{ (F_\alpha, V_\alpha) : \alpha \in A \} \) be a
collection of ordered pairs of subsets of a space \( X \).

\( P \) is called a \textit{pair-network} for \( X \) if whenever \( p \in U \in \tau_X \),
there exists \( \alpha \in A \) such that \( p \in F_\alpha \subseteq V_\alpha \subseteq U \). \( P \) is called
discrete (resp. HCP) if the family \( \{ F_\alpha : \alpha \in A \} \) is discrete
(resp. HCP) in \( X \). \( P \) is called \textit{\( \sigma \)-discrete} (resp. \( \sigma \)-HCP)
in \( X \) if \( P = \bigcup \{ P_n : n \in \mathbb{N} \} \) with each \( P_n \) discrete
(resp. HCP) in \( X \). The other terms for \( P \) are similar.

In this paper, we assume that every \( F_\alpha \) is \textit{closed} in \( X \), but
every \( V_\alpha \) is \textit{not necessarily open} in \( X \). Unless otherwise is
stated explicitly, we assume that \( P \) has the members
\( \{ (F_\alpha, V_\alpha) : \alpha \in A \} \) or \( \{ (F_\alpha, V_\alpha) : \alpha \in A, n \in \mathbb{N} \} \).

THEOREM 1. For a Fréchet space \( X \), the following are equivalent:

(1) \( X \) has a \( \sigma \)-HCP pair-network \( P \) such that if \( Z \rightarrow p \)
\( \in U \in \tau_X \), then there exists \( \alpha \in A \) such that \( p \in F_\alpha \subseteq V_\alpha \subseteq U \)
and \( Z \) is cofinal in \( V_\alpha \), where \( Z \) is cofinal in \( V_\alpha \) means
\( z_n \in V_\alpha \) for infinitely many \( n \).

(2) \( X \) has a \( \sigma \)-HCP pair-network \( P \) such that if \( Z \rightarrow p \)
\( \in U \in \tau_X \), then there exists \( \alpha \in A \) such that \( p \in F_\alpha \subseteq V_\alpha \subseteq U \)
and \( Z \) is residual in \( V_\alpha \), where \( Z \) is residual in \( V_\alpha \) means
\( \{ z_n : n \geq m \} \subseteq V_\alpha \) for some \( m \in \mathbb{N} \).

(3) \( X \) has a \( \sigma \)-HCP pair-network \( P \) such that if \( Z \rightarrow p \)
\( \in U \in \tau_X \) and \( Z \subset X - \{ p \} \), then there exists \( \alpha \in A \) such that
\( p \in F_\alpha \subseteq V_\alpha \subseteq U \), \( F_\alpha - \{ p \} \subseteq \text{Int} V_\alpha \) and \( Z \) is residual in \( \text{Int} V_\alpha \).
PROOF. (3) + (2) + (1) is trivial. (1) + (3): Let \( P \) be a \( \sigma \)-HCP pair-network satisfying the condition of (1). Without loss of generality we can assume \( A_n \subseteq A_{n+1} \) for each \( n \in N \). For each \( \delta \subseteq A_n, n \in N \), let

\[
F(\delta) = \bigcap \{ F_\alpha : \alpha \in \delta \}, \quad V(\delta) = \bigcup \{ V_\alpha : \alpha \in \delta \}.
\]

Since the family of all intersections of members of a HCP family is also HCP in a Fréchet space [5, Remark 3.7], the pair-collection

\[
P' = \{ (F(\delta), V(\delta)) : \delta \subseteq A_n, n \in N \}
\]

is a \( \sigma \)-HCP pair-network for \( X \). We show that \( P' \) has the required properties in (3). Let \( Z \rightarrow p \in U \subseteq \tau_X \) and \( Z \subset X - \{ p \} \).

Set for each \( n \)

\[
\delta_n = \{ \alpha \in A_n : p \in F_\alpha \subseteq V_\alpha \subseteq U \}.
\]

Then \( p \in F(\delta_n) \subseteq V(\delta_n) \subseteq U \) for each \( n \).

Claim 1: \( F(\delta_n) - \{ p \} \subseteq \text{Int} V(\delta_n) \) for some \( n \).

Assume not. Take a sequence \( \{ p_n : n \in N \} \) of points such that \( p_n \in F(\delta_n) - \{ p \} - \text{Int} V(\delta_n) \) for each \( n \). Since \( X \) is Fréchet, for each \( n \) there exists a convergent sequence \( Z(n) \) of points of \( X - V(\delta_n) \) such that \( Z(n) \rightarrow p_n \). Note that \( \{ F(\delta_n) : n \in N \} \) forms a decreasing local network at \( p \) in \( X \). Then \( p_n \rightarrow p \) as \( n \rightarrow \infty \), implying

\[
p \in \bigcup \{ Z(n) : n \in N \}.
\]

Using Fréchet-ness of \( X \), we can take a convergent sequence \( Z \) of points of \( \bigcup \{ Z(n) : n \in N \} \) such that \( Z \rightarrow p \). Because \( p_n \nrightarrow p, n \in N, Z \cap Z(n) \nrightarrow \phi \) for infinitely many \( n \). We can take a convergent subsequence \( Z' = \{ Z(n_k) : k \in N \} \) of \( Z \) such that \( Z(n_k) \in Z(n_k) \) and \( k \leq n(k) < n(k+1), k \in N \).

By the property of \( P \) stated in (1), there exists \( \alpha \in A_n \),
n ∈ N, such that p ∈ F_a ⊂ V_a ⊂ U and Z' is cofinal in V_a. But this is a contradiction because V_a ⊂ V(δ_k) for every k ≥ n. Hence Claim 1 is established.

Claim 2: Z is residual in Int V(δ_m) for some m.

Assume the contrary, i.e., Z is cofinal in X - Int V(δ_n) for every n. Then there exists a subset \{k(n) : n ∈ N\} of N such that \( z_{k(n)} \in X - \text{Int} V(\delta_n) \) and \( k \not\leq k(n) < k(n+1) \), n ∈ N. Using Fréchet-ness of X, for each n we can take a sequence Z(n) of points of X - V(δ_n) such that Z(n) → z_{k(n)}. Since \( z_{k(n)} \notin p \) for each n, we can use the same argument as above to get a contradiction, which implies the validity of Claim 2.

Now, let k be the maximum of n and m in Claims 1 and 2, respectively. Since F(δ_s) ⊂ F(δ_t) and V(δ_t) ⊂ V(δ_s) for every s, t with t ≥ s, and this k satisfies both claims. This completes the proof (1) → (3).

In the sequel, we denote by C the class of all Fréchet spaces satisfying one and hence all of (1) to (3) in Theorem 1. With respect to the properties of C, the following hold:

THEOREM 2. C has the following properties:

(1) C is closed under closed mappings.

(2) C is closed under subspaces.

(3) \{closed images of a developable space\} ⊂ C.

(4) C is not finitely productive.

All except (4) are easily seen from Theorem 1. (4) is a direct consequence of Theorem 9, stated later.

We give a characterization of developable spaces in terms of pair-networks some what different from the results of Burke [2, Theorem 2.1].
THEOREM 3. For a space X, the following are equivalent:

(1) X is a developable space.

(2) X is first countable and $X \subseteq \mathcal{C}$.

(3) X is a strongly Fréchet space having a $\sigma$-locally finite pair-network $P$ satisfying the same condition as in Theorem 1, (1).

(4) X has a $\sigma$-locally finite pair-network $P$ such that each $V_\alpha$ is open in $X$.

PROOF. As well-known, a space X is developable if and only if X has a $\sigma$-discrete pair-network $P$ such that each $V_\alpha$ is open in X, [4]. So, (1) $\Rightarrow$ (4) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are obvious. (2) $\Rightarrow$ (1): We shall show that X has a $\sigma$-discrete pair-network $P$ such that all $V_\alpha$ are open in X. Let $P$ be a $\sigma$-HCP pair-network for $X$ satisfying the same condition as in Theorem 1, (1). For each $n \in \mathbb{N}$, let

$$X_n = \{ p \in X : \text{ord}(p, F_n) \geq n \},$$

where $F_n = \{ F_\alpha : \alpha \in A_n \}$. Since $X$ is Fréchet and each $F_n$ is HCP in $X$, each $X_n$ is a discrete closed subset of $X$. Let

$$X_{0n} = \{ p \in X : F_\alpha - \text{Int} V_\alpha = \{ p \} \text{ for some } \alpha \in A_n, \ n \in \mathbb{N} \}.$$

Then obviously $\bigcup \{ X_{0n} : n \in \mathbb{N} \}$ is a $\sigma$-discrete closed subset of $X$. For each $n$, by the method of [10] we can construct a $\sigma$-discrete family $H_n$ of closed subsets of $X$ from

$$B_n = F_n \cup \{ (x) : x \in X_{0n} \cup X_n \}$$

such that $H_n$ satisfying the following: For each subfamily $B_0 \subseteq B_n$, if $p \in \bigcap B_0 - \bigcup (B_n - B_0)$, then $p \in H \subseteq \bigcap B_0 - \bigcup (B_n - B_0)$ for some $H \in H_n$. For each $H \in H_n$, $n \in \mathbb{N}$, with $H \cap (X_{0n} \cup X_n) = \emptyset$, choose an open subset $V(H)$ of $X$ such that

$$H \subseteq V(H) \subseteq \bigcap \{ \text{Int} V_\alpha : \alpha \in \delta \},$$
where $\delta$ is a finite subset of $A_n$ such that

$$H \subseteq \bigcap \{P_a : a \in \delta\} - \bigcup \{P_a : a \notin A_n - \delta\}.$$ 

For each point $p \in X$, let $\{O_n(p) : n \in \mathbb{N}\}$ be a local base at $p$ in $X$. Construct the pair-collection

$$P' = \{(p, O_n(p)) : p \in X_k, k, n \in \mathbb{N}\} \cup \{(p, O_n(p)) : p \in X_{0n}, k, n \in \mathbb{N}\} \cup \{(H, V(H)) : H \in H_n', n \in \mathbb{N}\},$$

where

$$H_n' = \{H \in H_n : H \cap (X_{0n} \cup X_n) = \emptyset\}, n \in \mathbb{N}.$$ 

Then it is easy to see that $P'$ is a $\sigma$-discrete pair-network for $X$ such that the second subset of each pair of $P'$ is open in $X$, proving that $X$ is developable.

(3) $\rightarrow$ (2): It suffices to show that $X$ is first countable.

Let $P = \bigcup \{P_n : n \in \mathbb{N}\}$ be a pair-network for $X$ satisfying the same condition as in Theorem 1, (1), where each $P_n = \{(F_a, V_a) : a \in A_n\}$ is locally finite in $X$. Without loss of generality we can assume $A_n \subseteq A_{n+1}$, $n \in \mathbb{N}$. For each point $p$, $A_n(p) = \{a \in A_n : p \in F_a\}, n \in \mathbb{N}$, is finite. For each $n$, set

$$\Delta_n = \{\delta \subseteq A_n(p) : p \in \text{Int } V(\delta)\},$$

where

$$V(\delta) = \bigcup \{V_a : a \in \delta\}, \delta \in \Delta_n.$$ 

We show that

$$\{\text{Int } V(\delta) : \delta \in \bigcup \{\Delta_n : n \in \mathbb{N}\}\}$$

is a local base at $p$ in $X$. Let $p \in U \subseteq \tau_X$. For each $n$, we take

$$\delta_n \subseteq A_n(p)$$

such that

$$\delta_n = \{a \in A_n(p) : V_a \subseteq U\}.$$ 

Assume $p \notin \text{Int } V(\delta_n)$ for each $n$. Since $X$ is strongly Fréchet, there exists a sequence $\{p_n : n \in \mathbb{N}\}$ of points of $X$ such that $p_n \rightarrow p$ and $p_n \notin V(\delta_n), n \in \mathbb{N}$. By the property of $P$, there
exists \( a \in A_n, \ n \in \mathbb{N} \), such that \( p \in F_a \subseteq V_a \subseteq U \) and \( \{ p_n \} \)

is cofinal in \( V_a \). But this is a contradiction. Hence we have \( p \in \text{Int} \ V(\delta_n) \subseteq U \) for some \( m \).

As the corollaries, we have two: The former is already known [9, Cor. to Proposition 4] and the latter is known for the case when \( X \) is an Moore space [3, Corollary 1.1]. The proof of the latter is the same as that of \((2) \Rightarrow (1)\).

**COROLLARY 1.** If a closed image of a developable space is first countable, then it is developable.

**COROLLARY 2.** If \( X \) is a closed image of a developable space, then \( X = X_0 \cup X_1 \), where \( X_0 \) is a \( \sigma \)-discrete closed subset and \( X_1 \) is a developable space.

The proof of \((3) \Rightarrow (2)\) above assures the following theorem:

**THEOREM 4.** If \( X \) is a strongly Fréchet space and \( X \in \mathcal{C} \), then \( X \) has a \( \sigma \)-HCP pair-network such that all \( V_a \) are open in \( X \).

But we do not know whether such a space is developable.

**QUESTION 1.** If \( X \) is a strongly Fréchet space and \( X \in \mathcal{C} \), then is \( X \) developable?

The following gives another characterization of the class \( \mathcal{C} \), which is similar to that of Lašnev spaces in terms of \( \sigma \)-HCP \( k \)-networks by Foged.
THEOREM 5. A space \( X \) belongs to \( C \) if and only if \( X \) is a Fréchet space which has a \( \sigma \)-HCP pair-network \( P \) such that if \( K \subset U \subset \tau_X \) with \( K \) compact in \( X \), then there exists a finite subcollection \( \{(F_a, V_a) : a \in \delta\} \) of \( P \) such that
\[
K \subset \bigcup \{V_a : a \in \delta\} \subset U
\]
and \( K \cap F_a \neq \emptyset \) for each \( a \in \delta \).

PROOF. If part is trivial. Only if part: Let \( P \) be a \( \sigma \)-HCP pair-network for \( X \) satisfying the condition of Theorem 1, (1). Assume \( A_n \subset A_{n+1} \) for each \( n \). For each \( \delta \subset A_n \), \( n \in \mathbb{N} \), set
\[
F(\delta) = \bigcap \{F_a : a \in \delta\}, \quad V(\delta) = \bigcup \{V_a : a \in \delta\}
\]
and
\[
Q = \{(F(\delta), V(\delta)) : \delta \subset A_n, \ n \in \mathbb{N}\}.
\]
Then \( Q \) is a \( \sigma \)-HCP pair-network for \( X \). We shall show that \( Q \) has the required property. Let \( K \subset U \subset \tau_X \) with \( K \) compact in \( X \). For each \( n \in \mathbb{N} \), let
\[
A_{0n} = \{a \in A_n : F_a \cap K \neq \emptyset \text{ and } V_a \subset U\}.
\]
Then \( \text{HCP-ness of } \{F_a : a \in A_n\} \) implies
\[
\{F_a : a \in A_{0n}\}|K = \{F_1, F_2, \ldots, F_{k(n)}\}
\]
with some \( k(n) \in \mathbb{N} \), [7, Proposition 3.7]. For each \( i \) with \( 1 \leq i \leq k(n) \), choose \( (F(\delta_{n_1}), V(\delta_{n_1})) \in Q \) such that
\[
\delta_{n_1} = \{a \in A_{0n} : F_a \cap K = F_i\}.
\]
Obviously \( \bigcup \{V(\delta_{n_1}) : 1 \leq i \leq k(n)\} \subset U \). Assume
\[
K \not\subset \bigcup \{V(\delta_{n_1}) : 1 \leq i \leq k(n)\}
\]
for each \( n \). Choose a sequence \( \{p_n : n \in \mathbb{N}\} \) of points of \( X \) such that
\[
p_n \in K \setminus \bigcup \{V(\delta_{n_1}) : 1 \leq i \leq k(n)\}, \ n \in \mathbb{N}.
\]
Since \( K \) is metrizable, \( \{p_n\} \) has a convergent subsequence \( Z \) to some point \( p \in K \) in \( X \). By the property of \( P \), there exists \( a_0 \in A_m, \ m \in \mathbb{N} \), such that \( p \in F_{a_0} \subset V_{a_0} \subset U \) and
Z is cofinal in $V_{\alpha_0}$. But this is a contradiction because 

$$V_{\alpha_0} \subset \bigcup \{V(\delta_{m_l}) : 1 \leq l \leq k(m)\}.$$

This completes the proof.

Viewing Theorem 1, (1), we can easily observe that 
a space $X$ belongs to $C$ if and only if $X$ is a Fréchet space 

having a $\sigma$-HCP pair-network $P$ such that the following 

conditions:

(C1) For each $\alpha \in A$, there exists an open subset $W_\alpha$ of $X$ 
such that $V_\alpha = F_\alpha \cup W_\alpha$.

(C2) If $Z \rightarrow p \in U \in \tau_X$ and $Z \subset X-\{p\}$, then there exists 

$\alpha \in A$ such that $p \in F_\alpha \subset V_\alpha \subset U$, $F_\alpha - \{p\} \subset W_\alpha$ and $Z$ is 

residual in $W_\alpha$.

By setting one more additional condition to $P$, we define 
a class $C'$ of spaces as follows: A space $X$ belongs to $C'$ 

if and only if $X$ is a Fréchet space having a $\sigma$-HCP pair-network 

$P$ satisfying the following additional condition (IP) besides 

(C1) and (C2):

(IP) For each $n$, $W_n = \{W_\alpha : \alpha \in A_n\}$ is an IP family 
of open subsets of $X$.

With respect to the properties of $C'$, the following 
holds and that corresponds to Theorem 2 for $C$.

THEOREM 6. $C'$ has the following properties:

(1) $C'$ is closed under closed mappings.

(2) $C'$ is closed under subspaces.

(3) A closed image of an orthocompact developable space 

belongs to $C'$.

(4) $C'$ is not finitely productive.
PROOF. (2) is obvious and (4) is a direct consequence of Theorem 9. So, we state the proofs of (1) and (3) only. First, we show (1). Let \( f : X \to Y \) be a closed mapping of \( X \) onto a space \( Y \) and let \( X \subseteq C' \). Let \( P \) be a \( \sigma \)-HCP pair-network for \( X \) assured by the definition of \( X \subseteq C' \). Assume \( A_n \subseteq A_{n+1} \), \( n \in \mathbb{N} \). For each \( \delta \subseteq A_n \), \( n \in \mathbb{N} \), set

\[
F(\delta) = \bigcap \{ f(P_\alpha) : \alpha \in \delta \},
\]

\[
W(\delta) = Y - f(X - \bigcup \{ W_\alpha : \alpha \in \delta \}),
\]

\[
V(\delta) = F(\delta) \cup W(\delta).
\]

Obviously \( \{ F(\delta) : \delta \subseteq A_n \} \) is a HCP family of closed subsets and \( \{ W(\delta) : \delta \subseteq A_n \} \) is an IP family of open subsets of \( Y \). Thus, the pair-collection

\[
P' = \{ (F(\delta), V(\delta)) : \delta \subseteq A_n, n \in \mathbb{N} \}
\]

is a \( \sigma \)-HCP pair-network for \( Y \) satisfying (C1) and (IP).

We show that \( P' \) satisfies the condition (C2) in \( Y \). Let \( Z \to y \subseteq U \subseteq \tau_X \) and \( Z \subseteq Y - \{ y \} \). For each \( n \), let

\[
\delta_n = \{ \alpha \subseteq A_n : P_\alpha \cap f^{-1}(y) \neq \phi \}
\]

and

\[
V_\alpha \subseteq f^{-1}(U).
\]

Then obviously, without loss of generality we can assume \( y \subseteq F(\delta_n) \subseteq V(\delta_n) \subseteq U \) for each \( n \in \mathbb{N} \).

Claim 1: \( F(\delta_n) - \{ y \} \subseteq W(\delta_n) \) for some \( m \).

To see it, assume the contrary. Then we can choose a point \( p_n \in F(\delta_n) - \{ y \} - W(\delta_n) \) for each \( n \). Since \( \{ F(\delta_n) : n \in \mathbb{N} \} \) forms a decreasing local network at \( y \) in \( Y \), \( p_n \to y \) as \( n \to \infty \) in \( Y \). Using the closedness of \( f \) and Fréchet-ness of \( X \), we can choose a sequence \( \{ q_n(k) : k \in \mathbb{N} \} \) of points of \( X - f^{-1}(y) \) such that \( \{ q_n(k) \} \) converges to some point of \( f^{-1}(y) \), \( f(q_n(k)) = p_n(k) \).
and
\[ q_n(k) \subseteq \bigcup \{ W_\alpha : \alpha \in \delta_n(k) \} \quad \text{for each } k \]
where \( k \leq n(k) < n(k+1) \), \( k \in \mathbb{N} \). By (C2) of \( C' \), there exists \( \alpha \in A_n \), \( n \in \mathbb{N} \), such that \( \{ q_n(k) \} \) is residual in \( W_\alpha \) and \( \alpha \in \delta_n \). But this is a contradiction. Hence Claim 1 is established.

By the same argument as above, we can show that \( Z \) is residual in \( W(\delta_m) \) for some \( m \). This completes the proof of (1).

Since an orthocompact developable space \( X \) has a \( \sigma \)-discrete pair-network \( P \) such that for each \( n \) \( \{ V_\alpha : \alpha \in A_n \} \) is an IP family of open subsets of \( X \), obviously \( X \in C' \), which combined with (1) implies (3).

We give two lemmas used in the proof of Theorem 7.

**Lemma 1.** Let \( X \in C' \). Then for each discrete family \( \{ F_\lambda : \lambda \in \Lambda \} \) of closed subsets of \( X \) there exist families \( \{ W_\lambda : \lambda \in \Lambda \} \) of open subsets of \( X \) satisfying the following:

1. For each \( \lambda \), \( W_\lambda \) is an outer base of \( F_\lambda \) in \( X \).
2. \( \bigcup \{ W_\lambda \mid (X - F_\lambda) : \lambda \in \Lambda \} \) is IP in \( X \).

**Proof.** For each \( \lambda \in \Lambda \), there exists a sequence \( \{ O(\lambda, n) : n \in \mathbb{N} \} \) of open subsets of \( X \) such that \( F_\lambda = \bigcap \{ O(\lambda, n) : n \in \mathbb{N} \} \), \( O(\lambda, n+1) \subseteq O(\lambda, n) \subseteq O(F_\lambda, n) \cap (X - \bigcup \{ F_\mu : \mu \uparrow \lambda \}) \).

Let \( P \) be a \( \sigma \)-HCP pair-network for \( X \) assured by \( X \in C' \). Let \( \lambda \in \Lambda \) be fixed for a while. Set
\[
W_n = \{ W_\alpha \cap O(\lambda, n) : \alpha \in A_n \}, \quad n \in \mathbb{N}.
\]
Let \( \{ W(\delta) : \delta \in \Delta(\lambda) \} \) be the totality of subfamilies of
\[ \bigcup \{ w_n : n \in \mathbb{N} \} \text{ such that} \]

\[ W(\delta) = F_\lambda \cup \left( \bigcup \{ w(\delta) \} \right) \]

is an open neighborhood of \( F_\lambda \) in \( X \). We show that \( \{ W(\delta) : \delta \in \Delta(\lambda) \} \) is an outer base of \( F_\lambda \) in \( X \). Let \( F_\lambda \subseteq O \subseteq \tau_X \). Let

\[ w_n' = \{ W \subseteq w_n : W \subseteq O \}, \quad n \in \mathbb{N}, \]

\[ w(\delta) = \bigcup \{ w_n' : n \in \mathbb{N} \}. \]

Then \( F_\lambda \subseteq W(\delta) \subseteq O \). To see that \( W(\delta) \) is open in \( X \), assume the contrary. Take a point \( p \in F_\lambda \setminus \text{Int } W(\delta) \). Since \( X \) is Préchet, there exists a sequence \( Z \) of points of \( X \setminus W(\delta) \) such that \( Z \to p \) in \( X \). By the property of \( P \), we can choose \( a \in A_n \), \( n \in \mathbb{N} \), such that

\[ p \in F_a \subset V_a \subset O, \quad F_a \setminus \{ p \} \subset W_a \]

and \( Z \) is residual in \( W_a \). This implies also that \( Z \) is residual in \( W_a \cap O(\lambda, n) \). But this is a contradiction. From the property \( (IP) \) of \( P \), we can easily see that \( (2) \) is satisfied for thus constructed

\[ w_\lambda = \{ W(\delta) : \delta \in \Delta(\lambda) \}, \quad \lambda \in \Lambda. \]

This completes the proof.

A space \( X \) is called d-IP-expandable [6] if for each discrete family \( \{ F_\lambda : \lambda \in \Lambda \} \) of closed subsets and each family \( \{ U_\lambda : \lambda \in \Lambda \} \) of open subsets of \( X \) such that \( F_\lambda \subseteq U_\lambda \), \( \lambda \in \Lambda \), there exists an IP family \( \{ V_\lambda : \lambda \in \Lambda \} \) of open subsets of \( X \) such that \( F_\lambda \subseteq V_\lambda \subseteq U_\lambda \), \( \lambda \in \Lambda \).

**Lemma 2.** If \( X \in C' \), then \( X \) is orthocompact.

**Proof.** By the lemma above, \( X \) is d-IP-expandable.

Since a submetacompact, d-IP-expandable space is orthocompact, [6, Theorem 2.5], \( X \) is orthocompact.
From Lemmas 2 and 3, we have a characterization of orthocompact developable spaces in terms of pair-networks as follows:

**THEOREM 7.** For a space $X$, the following are equivalent:

1. $X$ is an orthocompact developable space.
2. $X$ is a first countable space and $X \in C'$.

A space $X$ is called **d-paracompact** \([1]\) if for each open cover $\mathcal{U}$ of $X$, there exists a $\mathcal{U}$-mapping of $X$ onto a developable space. A space $X$ is called **subdevelopable** if $\tau_X$ contains a developable subtopology. With respect to the notions, we have the following:

**THEOREM 8.** If $X \in C'$, then $X$ is both d-paracompact and subdevelopable.

**PROOF.** If $X \in C'$, then by Lemma 1 $X$ is D-expandable and hence id d-paracompact \([1, \text{Theorem 1}]\). Since a d-paracompact space with a $G_δ$-diagonal is subdevelopable \([8, \text{Theorem 4}]\), $X$ is subdevelopable.

But we do not know whether the above holds for the class $C'$.

**QUESTION 2.** If $X \in C$, then is $X$ d-paracompact or subdevelopable?

It is well-known as Heyman's result that for any non-discrete spaces $X, Y$, the product space $X \times Y$ being Lašnev means both $X, Y$ are metrizable. This is true for the class $C'$. 
we state it more generally.

**Theorem 9.** Let $X$, $Y$ be non-discrete spaces. If 
$X \times Y \in C'$, then $X \times Y$ is an orthocompact developable space.

**Proof.** By the virtue of Theorem 7, it suffices to show that both $X$, $Y$ are first countable. Let $P$ be a $\sigma$-HCP pair-network for $X \times Y$ defining $X \times Y \in C'$. Let $Z$ be a sequence of points of $X$ such that $Z \rightarrow x$ and $Z \subseteq X - \{x\}$. Let $y$ be an arbitrary point of $Y$. We show that $y$ has a countable local base in $Y$. Obviously

$$Z' = \{(z_k, y) : k \in \mathbb{N}\} \rightarrow (x, y)$$

in $X \times Y$. Since $\{W_\alpha : \alpha \in A_n\}$, $n \in \mathbb{N}$, is IP in $X \times Y$ by (IP), for each pair $(m, n) \in \mathbb{N}^2$ with

$$(z_m, y) \in W_\alpha \quad \text{for some } \alpha \in A_n,$$

there exists an open subset $O(m, n)$ of $X$ such that

$$(z_m, y) \in O(m, n) \subseteq \bigcap \{W_\alpha : \alpha \in A_n\},$$

$$(z_m, y) \in W_\alpha.$$  

Let $N_0$ be the totality of such pairs $(m, n)$. Let $p : X \times Y \rightarrow Y$ be the projection. By the property of $P$, it is easily seen that $\{p(O(m, n)) : (m, n) \in N_0\}$ is a local base at $y$ in $Y$. This completes the proof.

**Corollary.** Let $X$, $Y$ be non-discrete spaces. If $X \times Y$ is the closed image of an orthocompact developable space, then $X \times Y$ is an orthocompact developable space.

But, we do not know whether Theorem 9 holds for the class $C$:
QUESTION 3. For non-discrete spaces $X, Y$, does $X \times Y$ imply that $X \times Y$ is developable?

Finally, we pose the following question about the characterization of a closed image of a developable space:

QUESTION 4. If a space $X$ belongs to $C$, then is $X$ a closed image of a developable space?

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