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ON THE CLOSED IMAGES OF A DEVELOPABLE SPACE

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ABSTRACT

We study the properties of the image of a developable space and an orthocompact developable space under a closed mapping, comparing with Lašnev spaces. Two classes $C$ and $C'$ are defined and their properties are given.

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1. Introduction.

Throughout this paper, all spaces are assumed to be \( T_1 \) topological ones and mappings to be continuous and onto. The letter \( N \) always denotes natural numbers. The letter \( Z \) always denotes a convergent sequence of points of a space such that \( Z = \{z_n : n \in N\} \) and \( Z \rightarrow p \) implies that \( Z \) converges to \( p \) as \( n \rightarrow \infty \). We denote the topology of \( X \) by \( \tau_X \). We use the brief expressions HCP and IP in place of "hereditarily closure-preserving" and "interior-preserving", respectively.

As a nice generalization of metric spaces, we have a class of developable spaces, which are defined to be ones \( X \) having a sequence \( \{U_n : n \in N\} \) of open covers of \( X \) such that for each point \( p \in X \), \( \{S(p, U_n) : n \in N\} \) is a local base at \( p \) in \( X \). Until now, the image of a metric space under a closed mapping, called a Lašnev space, is widely studied. But the study of the image of a developable space, briefly called the closed image of a developable space, has not been published yet. In this paper, we begin on its study, especially using the notion of pair-networks. This is our aim of this paper.

To start with, we give the meanings to the special spaces used later. A space \( X \) is called semi-stratifiable if there exists a function \( O : \{\text{closed subsets of } X\} \times N \rightarrow \tau_X \), called the semi-stratification of \( X \), satisfying the following conditions:

(1) For each closed subset \( F \) of \( X \),

\[ F = \bigcap \{O(F, n) : n \in N\} \]
and \( O(F, \ n+1) \subseteq O(F, \ n) \) for each \( n \).

(2) If \( F, \ G \) are closed subsets of \( X \) such that \( F \subseteq G \), then \( O(F, \ n) \subseteq O(G, \ n) \) for each \( n \).

2. The closed image of a developable space.

**DEFINITION [2].** Let \( P = \{(F_\alpha, V_\alpha) : \alpha \in A\} \) be a collection of ordered pairs of subsets of a space \( X \).

\( P \) is called a **pair-network for** \( X \) if whenever \( p \in U \in \tau_X \), there exists \( \alpha \in A \) such that \( p \in F_\alpha \subseteq V_\alpha \subseteq U \). \( P \) is called **discrete** (resp. **HCP**) if the family \( \{F_\alpha : \alpha \in A\} \) is discrete (resp. HCP) in \( X \). \( P \) is called **\( \sigma \)-discrete** (resp. **\( \sigma \)-HCP**) in \( X \) if \( P = \bigcup \{P_n : n \in \mathbb{N}\} \) with each \( P_n \) discrete (resp. HCP) in \( X \). The other terms for \( P \) are similar.

In this paper, we assume that every \( F_\alpha \) is closed in \( X \), but every \( V_\alpha \) is not necessarily open in \( X \). Unless otherwise is stated explicitly, we assume that \( P \) has the members \( \{(F_\alpha, V_\alpha) : \alpha \in A\} \) or \( \{(F_\alpha, V_\alpha) : \alpha \in A_n, \ n \in \mathbb{N}\} \).

**THEOREM 1.** For a Fréchet space \( X \), the following are equivalent:

1. \( X \) has a \( \sigma \)-HCP pair-network \( P \) such that if \( Z \to p \in U \in \tau_X \), then there exists \( \alpha \in A \) such that \( p \in F_\alpha \subseteq V_\alpha \subseteq U \) and \( Z \) is cofinal in \( V_\alpha \), where \( Z \) is cofinal in \( V_\alpha \) means \( z_n \in V_\alpha \) for infinitely many \( n \).

2. \( X \) has a \( \sigma \)-HCP pair-network \( P \) such that if \( Z \to p \in U \in \tau_X \), then there exists \( \alpha \in A \) such that \( p \in F_\alpha \subseteq V_\alpha \subseteq U \) and \( Z \) is residual in \( V_\alpha \), where \( Z \) is residual in \( V_\alpha \) means \( \{z_n : n \geq m\} \subseteq V_\alpha \) for some \( m \in \mathbb{N} \).

3. \( X \) has a \( \sigma \)-HCP pair-network \( P \) such that if \( Z \to p \in U \in \tau_X \) and \( Z \subseteq X - \{p\} \), then there exists \( \alpha \in A \) such that \( p \in F_\alpha \subseteq V_\alpha \subseteq U \), \( P_\alpha - \{p\} \subseteq \text{Int} \ V_\alpha \) and \( Z \) is residual in \( \text{Int} \ V_\alpha \).
PROOF. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is trivial. (1) $\Rightarrow$ (3): Let $P$ be a $\omega$-HCP pair-network satisfying the condition of (1). Without loss of generality we can assume $A_n \subset A_{n+1}$ for each $n \in N$. For each $\delta \subset A_n$, $n \in N$, let

$$F(\delta) = \bigcap \{F_\alpha : \alpha \in \delta\}, \quad V(\delta) = \bigcup \{V_\alpha : \alpha \in \delta\}.$$ 

Since the family of all intersections of members of a HCP family is also HCP in a Prékét space [5, Remark 3.7], the pair-collection

$$P' = \{ (F(\delta), V(\delta)) : \delta \subset A_n, \ n \in N \}$$

is a $\omega$-HCP pair-network for $X$. We show that $P'$ has the required properties in (3). Let $Z \rightarrow p \in U \in \tau_X$ and $Z \subset X - \{p\}.$

Set for each $n$

$$\delta_n = \{ \alpha \in A_n : p \in F_\alpha \subset V_\alpha \subset U \}.$$ 

Then $p \in P(\delta_n) \subset V(\delta_n) \subset U$ for each $n$.

Claim 1: $F(\delta_n) \setminus \{p\} \subset \text{Int} V(\delta_n)$ for some $n$.

Assume not. Take a sequence $\{ p_n : n \in N \}$ of points such that $p_n \in F(\delta_n) \setminus \{p\} \subset \text{Int} V(\delta_n)$ for each $n$. Since $X$ is Fréchet, for each $n$ there exists a convergent sequence $Z(n)$ of points of $X - V(\delta_n)$ such that $Z(n) \rightarrow p_n$. Note that $\{ F(\delta_n) : n \in N \}$ forms a decreasing local network at $p$ in $X$. Then $p_n \rightarrow p$ as $n \rightarrow \infty$, implying

$$p \in \bigcup \{ Z(n) : n \in N \}.$$ 

Using Fréchet-ness of $X$, we can take a convergent sequence $Z$ of points of $\bigcup \{ Z(n) : n \in N \}$ such that $Z \rightarrow p$. Because

$$p_n \nrightarrow p, \ n \in N, \ Z \cap Z(n) \nrightarrow \phi$$

for infinitely many $n$. We can take a convergent subsequence $Z' = \{ z_n(k) : k \in N \}$ of $Z$ such that $z_n(k) \in Z(n_k)$ and $k \leq n(k) < n(k+1), \ k \in N$. By the property of $P$ stated in (1), there exists $\alpha \in A_n$,
n ∈ N, such that p ∈ Fₐ ⊂ Vₐ ⊂ U and Z' is cofinal in Vₐ. But this is a contradiction because Vₐ ⊂ V(δₖ) for every k ≥ n. Hence Claim 1 is established.

Claim 2: Z is residual in Int V(δₘ) for some m.

Assume the contrary, i.e., Z is cofinal in X − Int V(δₙ) for every n. Then there exists a subset {k(n) : n ∈ N} of N such that zₖ(n) ∈ X − Int V(δₙ) and k ≤ k(n) < k(n+1), n ∈ N. Using Fréchet-ness of X, for each n we can take a sequence Z(n) of points of X − V(δₙ) such that Z(n) → zₖ(n). Since zₖ(n) ∉ p for each n, we can use the same argument as above to get a contradiction, which implies the validity of Claim 2.

Now, let k be the maximum of n and m in Claims 1 and 2, respectively. Since F(δₙ) ⊂ F(δₜ) and V(δₜ) ⊂ V(δₛ) for every s, t with t ≤ s, and this k satisfies both claims. This completes the proof (1) → (3).

In the sequel, we denote by C the class of all Fréchet spaces satisfying one and hence all of (1) to (3) in Theorem 1. With respect to the properties of C, the following hold:

THEOREM 2. C has the following properties:

(1) C is closed under closed mappings.

(2) C is closed under subspaces.

(3) {closed images of a developable space} ⊂ C.

(4) C is not finitely productive.

All except (4) are easily seen from Theorem 1. (4) is a direct consequence of Theorem 9, stated later.

We give a characterization of developable spaces in terms of pair-networks some what different from the results of Burke [2, Theorem 2.1].
THEOREM 3. For a space $X$, the following are equivalent:

(1) $X$ is a developable space.

(2) $X$ is first countable and $X \in C$.

(3) $X$ is a strongly Fréchet space having a $\sigma$-locally finite pair-network $P$ satisfying the same condition as in Theorem 1, (1).

(4) $X$ has a $\sigma$-locally finite pair-network $P$ such that each $V_\alpha$ is open in $X$.

PROOF. As well-known, a space $X$ is developable if and only if $X$ has a $\sigma$-discrete pair-network $P$ such that each $V_\alpha$ is open in $X$, [4]. So, (1) $\Rightarrow$ (4) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are obvious. (2) $\Rightarrow$ (1): We shall show that $X$ has a $\sigma$-discrete pair-network $P$ such that all $V_\alpha$ are open in $X$. Let $P$ be a $\sigma$-HCP pair-network for $X$ satisfying the same condition as in Theorem 1, (1). For each $n \in N$, let

$$X_n = \{ p \in X : \text{ord} (p, F_n) \geq n \},$$

where $F_n = \{ F_\alpha : \alpha \in A_n \}$. Since $X$ is Fréchet and each $F_n$ is HCP in $X$, each $X_n$ is a discrete closed subset of $X$. Let

$$X_{0n} = \{ p \in X : F_\alpha - \text{Int } V_\alpha = \{ p \} \text{ for some } \alpha \in A_n, \ n \in N \}.$$

Then obviously $\bigcup \{ X_{0n} : n \in N \}$ is a $\sigma$-discrete closed subset of $X$. For each $n$, by the method of [10] we can construct a $\sigma$-discrete family $H_n$ of closed subsets of $X$ from

$$B_n = F_n \cup \{ \{ x \} : x \in X_{0n} \cup X_n \}$$

such that $H_n$ satisfying the following: For each subfamily $B_0 \subset B_n$, if $p \in \bigcap B_0 - \bigcup (B_n - B_0)$, then $p \in H \subset \bigcap B_0 - \bigcup (B_n - B_0)$ for some $H \in H_n$. For each $H \in H_n$, $n \in N$, with $H \cap (X_{0n} \cup X_n) = \phi$, choose an open subset $V(H)$ of $X$ such that

$$H \subset V(H) \subset \bigcap \{ \text{Int } V_\alpha : \alpha \in \delta \},$$
where \( \delta \) is a finite subset of \( A_n \) such that
\[
H \subseteq \bigcap \{F_\alpha : \alpha \in \delta\} - \bigcup \{F_\alpha : \alpha \notin A_n - \delta\}.
\]
For each point \( p \in X \), let \( (O_n(p) : n \in N) \) be a local base at \( p \) in \( X \). Construct the pair-collection
\[
P' = \{(p, O_n(p)) : p \in X_k, k, n \in N\}
\]
\[
\bigcup \{(p, O_n(p)) : p \in X_{0n}, k, n \in N\}
\]
\[
\bigcup \{(H, V(H)) : H \in H_n', n \in N\},
\]
where
\[
H_n' = \{H \in H_n : H \cap (X_{0n} \cup X_n) = \emptyset\}, n \in N.
\]
Then it is easy to see that \( P' \) is a \( \sigma \)-discrete pair-network for \( X \) such that the second subset of each pair of \( P' \) is open in \( X \), proving that \( X \) is developable.

(3) \( \Rightarrow \) (2): It suffices to show that \( X \) is first countable.

Let \( P = \bigcup \{P_n : n \in N\} \) be a pair-network for \( X \) satisfying the same condition as in Theorem 1, (1), where each \( P_n = \{(F_\alpha, V_\alpha) : \alpha \in A_n\} \) is locally finite in \( X \). Without loss of generality we can assume \( A_n \subseteq A_{n+1}, n \in N \). For each point \( p, A_n(p) = \{\alpha \in A_n : p \in F_\alpha\}, n \in N \), is finite. For each \( n \), set
\[
\Delta_n = \{\delta \subseteq A_n(p) : p \in \text{Int} V(\delta)\},
\]
where
\[
V(\delta) = \bigcup \{V_\alpha : \alpha \in \delta\}, \delta \in \Delta_n.
\]
We show that
\[
\{\text{Int} V(\delta) : \delta \in \bigcup \{\Delta_n : n \in N\}\}
\]
is a local base at \( p \) in \( X \). Let \( p \in U \in \tau_X \). For each \( n \), we take \( \delta_n \subseteq A_n(p) \) such that
\[
\delta_n = \{a \in A_n(p) : V_\alpha \subseteq U\}.
\]
Assume \( p \notin \text{Int} V(\delta_n) \) for each \( n \). Since \( X \) is strongly Fréchet, there exists a sequence \( (p_n : n \in N) \) of points of \( X \) such that \( p_n \to p \) and \( p_n \notin V(\delta_n), n \in N \). By the property of \( P \), there
exists $a \in A_n$, $n \in N$, such that $p \in F_a \subset V_a \subset U$ and $\{p_n\}$ is cofinal in $V_a$. But this is a contradiction. Hence we have $p \in \text{Int} V(\delta_n) \subset U$ for some $m$.

As the corollaries, we have two: The former is already known [9, Cor. to Proposition 4] and the latter is known for the case when $X$ is an Moore space [3, Corollary 1.1]. The proof of the latter is the same as that of (2) $\rightarrow$ (1).

**COROLLARY 1.** If a closed image of a developable space is first countable, then it is developable.

**COROLLARY 2.** If $X$ is a closed image of a developable space, then $X = X_0 \cup X_1$, where $X_0$ is a $\sigma$-discrete closed subset and $X_1$ is a developable space.

The proof of (3) $\rightarrow$ (2) above assures the following theorem:

**THEOREM 4.** If $X$ is a strongly Fréchet space and $X \in C$, then $X$ has a $\sigma$-HCP pair-network such that all $V_a$ are open in $X$.

But we do not know whether such a space is developable.

**QUESTION 1.** If $X$ is a strongly Fréchet space and $X \in C$, then is $X$ developable?

The following gives another characterization of the class $C$, which is similar to that of Lašnev spaces in terms of $\sigma$-HCP $k$-networks by Boged.
THEOREM 5. A space X belongs to \( \mathcal{C} \) if and only if X is a Fréchet space which has a \( \sigma \)-HCP pair-network \( P \) such that if \( K \subseteq U \subseteq \tau_X \) with \( K \) compact in \( X \), then there exists a finite subcollection \( \{(F_a, V_a) : a \in \delta\} \) of \( P \) such that
\[
K \subseteq \bigcup \{V_a : a \in \delta\} \subseteq U
\]
and \( K \cap F_a \neq \emptyset \) for each \( a \in \delta \).

PROOF. If part is trivial. Only if part: Let \( P \) be a \( \sigma \)-HCP pair-network for \( X \) satisfying the condition of Theorem 1, (1). Assume \( A_n \subseteq A_{n+1} \) for each \( n \). For each \( \delta \subseteq A_n \), \( n \in \mathbb{N} \), set
\[
F(\delta) = \bigcap \{F_a : a \in \delta\}, \quad V(\delta) = \bigcup \{V_a : a \in \delta\}
\]
and
\[
Q = \{(F(\delta), V(\delta)) : \delta \subseteq A_n, \ n \in \mathbb{N}\}.
\]
Then \( Q \) is a \( \sigma \)-HCP pair-network for \( X \). We shall show that \( Q \) has the required property. Let \( K \subseteq U \subseteq \tau_X \) with \( K \) compact in \( X \). For each \( n \in \mathbb{N} \), let
\[
A_{0n} = \{a \in A_n : F_a \cap K \neq \emptyset \text{ and } V_a \subseteq U\}.
\]
Then HCP-ness of \( \{F_a : a \in A_n\} \) implies
\[
\{F_a : a \in A_{0n}\} \upharpoonright K = \{F_1, F_2, \ldots, F_{k(n)}\}
\]
with some \( k(n) \in \mathbb{N} \), [? Proposition 3.7]. For each \( i \) with \( 1 \leq i \leq k(n) \), choose \( (F(\delta_{ni}), V(\delta_{ni})) \in Q \) such that
\[
\delta_{ni} = \{a \in A_{0n} : F_a \cap K = F_i\}.
\]
Obviously \( \bigcup \{V(\delta_{ni}) : 1 \leq i \leq k(n)\} \subseteq U \). Assume
\[
K \subseteq \bigcup \{V(\delta_{ni}) : 1 \leq i \leq k(n)\}
\]
for each \( n \). Choose a sequence \( \{p_n : n \in \mathbb{N}\} \) of points of \( X \) such that
\[
p_n \in K \setminus \bigcup \{V(\delta_{ni}) : 1 \leq i \leq k(n)\}, \ n \in \mathbb{N}.
\]
Since \( K \) is metrizable, \( \{p_n\} \) has a convergent subsequence \( Z \) to some point \( p \in K \) in \( X \). By the property of \( P \), there exists \( a_0 \in A_m, m \in \mathbb{N}, \) such that \( p \in F_{a_0} \subseteq V_{a_0} \subseteq U \) and
Z is cofinal in $V_{\alpha_0}$. But this is a contradiction because

$$V_{\alpha_0} \subseteq \bigcup \{V(\delta_{mi}) : 1 \leq i \leq k(m)\}.$$ 

This completes the proof.

Viewing Theorem 1, (1), we can easily observe that

a space $X$ belongs to $C$ if and only if $X$ is a Fréchet space

having a $\sigma$-HCP pair-network $P$ such that the following

conditions:

(C1) For each $\alpha \in A$, there exists an open subset $W_\alpha$ of $X$

such that $V_\alpha = F_\alpha \cup W_\alpha$.

(C2) If $Z \to p \in U \in \tau_X$ and $Z \subseteq X - \{p\}$, then there exists

$\alpha \in A$ such that $p \in F_\alpha \subseteq V_\alpha \subseteq U$, $F_\alpha - \{p\} \subseteq W_\alpha$ and $Z$ is

residual in $W_\alpha$.

By setting one more additional condition to $P$, we define

a class $C'$ of spaces as follows: A space $X$ belongs to $C'$

if and only if $X$ is a Fréchet space having a $\sigma$-HCP pair-network

$P$ satisfying the following additional condition (IP) besides

(C1) and (C2):

(IP) For each $n$, $W_n = \{W_\alpha : \alpha \in A_n\}$ is an IP family

of open subsets of $X$.

With respect to the properties of $C'$, the following

holds and that corresponds to Theorem 2 for $C$.

THEOREM 6. $C'$ has the following properties:

(1) $C'$ is closed under closed mappings.

(2) $C'$ is closed under subspaces.

(3) A closed image of an orthocompact developable space

belongs to $C'$.

(4) $C'$ is not finitely productive.
PROOF. (2) is obvious and (4) is a direct consequence of Theorem 9. So, we state the proofs of (1) and (3) only. First, we show (1). Let \( f : X \to Y \) be a closed mapping of \( X \) onto a space \( Y \) and let \( X \in \mathcal{C}' \). Let \( P \) be a \( \sigma \)-HC P pair-network for \( X \) assured by the definition of \( X \in \mathcal{C}' \). Assume \( A_n \subset A_{n+1}, n \in N \). For each \( \delta \subset A_n, n \in N \), set
\[
F(\delta) = \bigcap \{ f(P_\alpha) : \alpha \in \delta \},
\]
\[
W(\delta) = Y - f(X - \bigcup \{ W_\alpha : \alpha \in \delta \}),
\]
\[
V(\delta) = F(\delta) \cup W(\delta).
\]
Obviously \( \{ F(\delta) : \delta \subset A_n \} \) is a HCP family of closed subsets and \( \{ W(\delta) : \delta \subset A_n \} \) is an IP family of open subsets of \( Y \).

Thus, the pair-collection
\[
P' = \{ (F(\delta), V(\delta)) : \delta \subset A_n, n \in N \}
\]
is a \( \sigma \)-HC P pair-network for \( Y \) satisfying (C1) and (IP).

We show that \( P' \) satisfies the condition (C2) in \( Y \). Let \( Z \to y \in U \in \tau_Y \) and \( Z \subset Y - \{ y \} \). For each \( n \), let
\[
\delta_n = \{ \alpha \subset A_n : P_\alpha \cap f^{-1}(y) \neq \phi \}
\]
and
\[
V_\alpha \subset f^{-1}(U).
\]

Then obviously, without loss of generality we can assume \( y \in F(\delta_n) \subset V(\delta_n) \subset U \) for each \( n \in N \).

Claim 1: \( F(\delta_n) - \{ y \} \subset W(\delta_n) \) for some \( m \).

To see it, assume the contrary. Then we can choose a point \( p_n \in F(\delta_n) - \{ y \} - W(\delta_n) \) for each \( n \). Since \( \{ F(\delta_n) : n \in N \} \) forms a decreasing local network at \( y \) in \( Y \), \( p_n \to y \) as \( n \to \infty \) in \( Y \).

Using the closedness of \( f \) and Fréchet-ness of \( X \), we can choose a sequence \( \{ q_n(k) : k \in N \} \) of points of \( X - f^{-1}(y) \) such that \( \{ q_n(k) \} \) converges to some point of \( f^{-1}(y) \), \( f(q_n(k)) = p_n(k) \).
and
\[ q_n(k) \in \bigcup \{ W_\alpha : \alpha \in \delta_n(k) \} \text{ for each } k \]
where \( k \leq n(k) < n(k+1), k \in \mathbb{N} \). By (C2) of \( C' \), there exists
\( \alpha \in A_n, n \in \mathbb{N}, \) such that \( \{ q_n(k) \} \) is residual in \( W_\alpha \) and
\( \alpha \in \delta_n \). But this is a contradiction. Hence Claim 1 is established.

By the same argument as above, we can show that \( Z \) is residual in \( W(\delta_m) \) for some \( m \). This completes the proof of (1).

Since an orthocompact developable space \( X \) has a \( \sigma \)-discrete pair-network \( P \) such that for each \( n \{ V_\alpha : \alpha \in A_n \} \) is an
IP family of open subsets of \( X \), obviously \( X \in C' \), which combined with (1) implies (3).

We give two lemmas used in the proof of Theorem 7.

**Lemma 1.** Let \( X \in C' \). Then for each discrete family
\( \{ F_\lambda : \lambda \in \Lambda \} \) of closed subsets of \( X \) there exist families
\( \{ W_\lambda : \lambda \in \Lambda \} \) of open subsets of \( X \) satisfying the following:
1. For each \( \lambda \), \( W_\lambda \) is an outer base of \( F_\lambda \) in \( X \).
2. \( \bigcup \{ W_\lambda \cap (X - F_\lambda) : \lambda \in \Lambda \} \) is IP in \( X \).

**Proof.** For each \( \lambda \in \Lambda \), there exists a sequence \( \{ O(\lambda, n) : n \in \mathbb{N} \} \) of open subsets of \( X \) such that
\[ F_\lambda = \bigcap \{ O(\lambda, n) : n \in \mathbb{N} \}, \]
\[ O(\lambda, n+1) \subset O(\lambda, n) \subset O(F_\lambda, n) \cap (X - \bigcup \{ F_\mu : \mu \uparrow \lambda \}). \]
Let \( P \) be a \( \sigma \)-HCP pair-network for \( X \) assured by \( X \in C' \). Let
\( \lambda \in \Lambda \) be fixed for a while. Set
\[ W_\lambda = \{ W_\alpha \cap O(\lambda, n) : \alpha \in A_n \}, n \in \mathbb{N}. \]
Let \( \{ W(\delta) : \delta \in \Delta(\lambda) \} \) be the totality of subfamilies of
\[ \bigcup \{ w_n : n \in \mathbb{N} \} \text{ such that} \]
\[ W(\delta) = \bigcup_{\lambda} F_{\lambda} \cup \bigcup w(\delta) \]
is an open neighborhood of \( F_{\lambda} \) in \( X \). We show that \( \{ W(\delta) : \delta \in \Delta(\lambda) \} \) is an outer base of \( F_{\lambda} \) in \( X \). Let \( F_{\lambda} \subseteq O \in \tau_X \). Let
\[ w_n' = \{ W \subseteq w_n : W \subseteq O \}, n \in \mathbb{N}, \]
\[ w(\delta) = \bigcup \{ w_n' : n \in \mathbb{N} \}. \]
Then \( F_{\lambda} \subseteq W(\delta) \subseteq O \). To see that \( W(\delta) \) is open in \( X \), assume the contrary. Take a point \( p \in F_{\lambda} \cap \text{Int} \ W(\delta) \). Since \( X \) is Fréchet, there exists a sequence \( Z \) of points of \( X \setminus W(\delta) \) such that \( Z \to p \in X \). By the property of \( P \), we can choose \( a \in A, n \in \mathbb{N}, \) such that
\[ p \in F_{\lambda} \subseteq V_{\lambda} \subseteq O, \quad F_{\lambda} \setminus \{ p \} \subseteq W_{\lambda} \]
and \( Z \) is residual in \( W_{\lambda} \). This implies also that \( Z \) is residual in \( W_{\lambda} \cap O(\lambda, n) \). But this is a contradiction. From the property (IP) of \( P \), we can easily see that (2) is satisfied for thus constructed
\[ w(\lambda) = \{ W(\delta) : \delta \in \Delta(\lambda) \}, \lambda \in \Lambda. \]
This completes the proof.

A space \( X \) is called d-IP-expandable [6] if for each discrete family \( \{ F_{\lambda} : \lambda \in \Lambda \} \) of closed subsets and each family \( \{ U_{\lambda} : \lambda \in \Lambda \} \) of open subsets of \( X \) such that \( F_{\lambda} \subseteq U_{\lambda}, \lambda \in \Lambda \), there exists an IP family \( \{ V_{\lambda} : \lambda \in \Lambda \} \) of open subsets of \( X \) such that \( F_{\lambda} \subseteq V_{\lambda} \subseteq U_{\lambda}, \lambda \in \Lambda \).

**LEMMA 2.** If \( X \in C' \), then \( X \) is orthocompact.

**PROOF.** By the lemma above, \( X \) is d-IP-expandable.
Since a submetacompact, d-IP-expandable space is orthocompact, [6, Theorem 2.5], \( X \) is orthocompact.
From Lemmas 2 and 3, we have a characterization of orthocompact developable spaces in terms of pair-networks as follows:

THEOREM 7. For a space X, the following are equivalent:
1. X is an orthocompact developable space.
2. X is a first countable space and $X \in \mathcal{C}'$.

A space X is called d-paracompact \[1\] if for each open cover $U$ of X, there exists a $U$-mapping of X onto a developable space. A space X is called subdevelopable if $\tau_X$ contains a developable subtopology. With respect to the notions, we have the following:

THEOREM 8. If $X \in \mathcal{C}'$, then X is both d-paracompact and subdevelopable.

PROOF. If $X \in \mathcal{C}'$, then by Lemma 1 X is D-expandable and hence id d-paracompact \[1, \text{Theorem 1}\]. Since a d-paracompact space with a $G_\delta$-diagonal is subdevelopable \[8, \text{Theorem 4}\], X is subdevelopable.

But we do not know whether the above holds for the class $\mathcal{C}$.

QUESTION 2. If $X \in \mathcal{C}$, then is X d-paracompact or subdevelopable?

It is well-known as Heyman's result that for any non-discrete spaces $X, Y$, the product space $X \times Y$ being $\Lambda$-nev means both $X, Y$ are metrizable. This is true for the class $\mathcal{C}'$. 
we state it more generally.

THEOREM 9. Let $X, Y$ be non-discrete spaces. If $X \times Y \in C'$, then $X \times Y$ is an orthocompact developable space.

PROOF. By the virtue of Theorem 7, it suffices to show that both $X, Y$ are first countable. Let $P$ be a $\sigma$-HCP pair-network for $X \times Y$ defining $X \times Y \in C'$. Let $Z$ be a sequence of points of $X$ such that $Z \to x$ and $Z \subseteq X - \{x\}$. Let $y$ be an arbitrary point of $Y$. We show that $y$ has a countable local base in $Y$. Obviously

$$Z' = \{(z_k, y) : k \in N\} \to (x, y)$$

in $X \times Y$. Since $\{W_\alpha : \alpha \in A_n\}, n \in N,$ is IP in $X \times Y$ by (IP), for each pair $(m, n) \in N^2$ with

$$(z_m, y) \in W_\alpha \text{ for some } \alpha \in A_n,$$

there exists an open subset $O(m, n)$ of $X$ such that

$$(z_m, y) \in O(m, n) \subseteq \bigcap\{W_\alpha : \alpha \in A_n, (z_m, y) \in W_\alpha\}.$$

Let $N_0$ be the totality of such pairs $(m, n)$. Let $p : X \times Y \to Y$ be the projection. By the property of $P$, it is easily seen that $\{p(O(m, n)) : (m, n) \in N_0\}$ is a local base at $y$ in $Y$. This completes the proof.

COROLLARY. Let $X, Y$ be non-discrete spaces. If $X \times Y$ is the closed image of an orthocompact developable space, then $X \times Y$ is an orthocompact developable space.

But, we do not know whether Theorem 9 holds for the class $C$:
QUESTION 3. For non-discrete spaces $X, Y$, does $X \times Y$ imply that $X \times Y$ is developable?

Finally, we pose the following question about the characterization of a closed image of a developable space:

QUESTION 4. If a space $X$ belongs to $C$, then is $X$ a closed image of a developable space?

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