ON THE CLOSED IMAGES OF A DEVELOPABLE SPACE

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ABSTRACT

We study the properties of the image of a developable space and an orthocompact developable space under a closed mapping, comparing with Lašnev spaces. Two classes $\mathcal{C}$ and $\mathcal{C}'$ are defined and their properties are given.

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1. Introduction.

Throughout this paper, all spaces are assumed to be $T_1$ topological ones and mappings to be continuous and onto. The letter $N$ always denotes natural numbers. The letter $Z$ always denotes a convergent sequence of points of a space such that $Z = \{z_n : n \in N\}$ and $Z + p$ implies that $Z$ converges to $p$ as $n \to \infty$. We denote the topology of $X$ by $\tau_X$. We use the brief expressions HCP and IP in place of "hereditarily closure-preserving" and "interior-preserving", respectively.

As a nice generalization of metric spaces, we have a class of developable spaces, which are defined to be ones $X$ having a sequence $\{U_n : n \in N\}$ of open covers of $X$ such that for each point $p \in X$, $\{S(p, U_n) : n \in N\}$ is a local base at $p$ in $X$. Until now, the image of a metric space under a closed mapping, called a Lašnev space, is widely studied. But the study of the image of a developable space, briefly called the closed image of a developable space, has not been published yet. In this paper, we begin on its study, especially using the notion of pair-networks. This is our aim of this paper.

To start with, we give the meanings to the special spaces used later. A space $X$ is called semi-stratifiable if there exists a function $O : \{\text{closed subsets of } X\} \times N \to \tau_X$, called the semi-stratification of $X$, satisfying the following conditions:

(1) For each closed subset $F$ of $X$,
$$F = \bigcap\{O(F, n) : n \in N\}$$
and \( O(F, n+1) \subseteq O(F, n) \) for each \( n \).

(2) If \( F, G \) are closed subsets of \( X \) such that \( F \subseteq G \), then \( O(F, n) \subseteq O(G, n) \) for each \( n \).

2. The closed image of a developable space.

**DEFINITION [2].** Let \( P = \{(F_\alpha, V_\alpha) : \alpha \in A\} \) be a collection of ordered pairs of subsets of a space \( X \).

\( P \) is called a **pair-network** for \( X \) if whenever \( p \in U \subseteq \tau_X \), there exists \( \alpha \in A \) such that \( p \in F_\alpha \subseteq V_\alpha \subseteq U \). \( P \) is called **discrete** (resp. **HCP**) if the family \( \{F_\alpha : \alpha \in A\} \) is discrete (resp. HCP) in \( X \). \( P \) is called **\( \sigma \)-discrete** (resp. **\( \sigma \)-HCP**) in \( X \) if \( P = \bigcup\{P_n : n \in \mathbb{N}\} \) with each \( P_n \) discrete (resp. HCP) in \( X \). The other terms for \( P \) are similar.

In this paper, we assume that every \( F_\alpha \) is **closed** in \( X \), but every \( V_\alpha \) is **not necessarily open** in \( X \). Unless otherwise is stated explicitly, we assume that \( P \) has the members \( \{(F_\alpha, V_\alpha) : \alpha \in A\} \) or \( \{(F_\alpha, V_\alpha) : \alpha \in A_n, n \in \mathbb{N}\} \).

**THEOREM 1.** For a Fréchet space \( X \), the following are equivalent:

1. \( X \) has a **\( \sigma \)-HCP pair-network** \( P \) such that if \( Z \to p \in U \subseteq \tau_X \), then there exists \( \alpha \in A \) such that \( p \in F_\alpha \subseteq V_\alpha \subseteq U \) and \( Z \) is cofinal in \( V_\alpha \), where \( Z \) is cofinal in \( V_\alpha \) means \( z_n \in V_\alpha \) for infinitely many \( n \).

2. \( X \) has a **\( \sigma \)-HCP pair-network** \( P \) such that if \( Z \to p \in U \subseteq \tau_X \), then there exists \( \alpha \in A \) such that \( p \in F_\alpha \subseteq V_\alpha \subseteq U \) and \( Z \) is residual in \( V_\alpha \), where \( Z \) is residual in \( V_\alpha \) means \( \{z_n : n \geq m\} \subseteq V_\alpha \) for some \( m \in \mathbb{N} \).

3. \( X \) has a **\( \sigma \)-HCP pair-network** \( P \) such that if \( Z \to p \in U \subseteq \tau_X \) and \( Z \subseteq X - \{p\} \), then there exists \( \alpha \in A \) such that \( p \in F_\alpha \subseteq V_\alpha \subseteq U \), \( F_\alpha - \{p\} \subseteq \text{Int} \ V_\alpha \) and \( Z \) is residual in \( \text{Int} \ V_\alpha \).
PROOF. \((3) \rightarrow (2) \rightarrow (1)\) is trivial. \((1) \rightarrow (3)\): Let \(P\) be a \(\pi\)-HCP pair-network satisfying the condition of \((1)\). Without loss of generality we can assume \(A_n \subseteq A_{n+1}\) for each \(n \in \mathbb{N}\). For each \(\delta \subseteq A_n\), \(n \in \mathbb{N}\), let

\[
F(\delta) = \bigcap \{P_\alpha : \alpha \in \delta\}, \quad V(\delta) = \bigcup \{V_\alpha : \alpha \in \delta\}.
\]

Since the family of all intersections of members of a HCP family is also HCP in a Fréchet space [5, Remark 3.7], the pair-collection
\[
P' = \{(F(\delta), V(\delta)) : \delta \subseteq A_n, \ n \in \mathbb{N}\}
\]
is a \(\pi\)-HCP pair-network for \(X\). We show that \(P'\) has the required properties in \((3)\). Let \(Z \rightarrow p \in U \subseteq \tau_X\) and \(Z \subseteq X - \{p\}\).

Set for each \(n\)

\[
\delta_n = \{a \in A_n : p \in P_\alpha \subseteq V_\alpha \subseteq U\}.
\]

Then \(p \in P(\delta_n) \subseteq V(\delta_n) \subseteq U\) for each \(n\).

**Claim 1:** \(F(\delta_n) - \{p\} \subseteq \text{Int} \ V(\delta_n)\) for some \(n\).

Assume not. Take a sequence \(\{p_n : n \in \mathbb{N}\}\) of points such that

\[
p_n \in F(\delta_n) - \{p\} - \text{Int} \ V(\delta_n)\]

for each \(n\). Since \(X\) is Fréchet, for each \(n\) there exists a convergent sequence \(Z(n)\) of points of \(X - V(\delta_n)\) such that \(Z(n) \rightarrow p_n\). Note that \(\{F(\delta_n) : n \in \mathbb{N}\}\) forms a decreasing local network at \(p\) in \(X\). Then \(p_n \rightarrow p\) as \(n \rightarrow \infty\), implying

\[
p \in \bigcup \{Z(n) : n \in \mathbb{N}\}.
\]

Using Fréchet-ness of \(X\), we can take a convergent sequence \(Z\) of points of \(\bigcup \{Z(n) : n \in \mathbb{N}\}\) such that \(Z \rightarrow p\). Because

\[
p_n \not\rightarrow p, \ n \in \mathbb{N}, \ Z \cap Z(n) \not\rightarrow \phi \text{ for infinitely many } n
\]

we can take a convergent subsequence \(Z' = \{z_{n(k)} : k \in \mathbb{N}\}\) of \(Z\) such that \(z_{n(k)} \in Z(n_k)\) and \(k \leq n(k) < n(k+1), \ k \in \mathbb{N}\). By the property of \(P\) stated in \((1)\), there exists \(a \in A_n\),
\( n \in \mathbb{N}, \text{ such that } p \in F_a \subseteq V_a \subseteq U \) and \( Z' \) is cofinal in \( V_a \). But this is a contradiction because \( V_a \subseteq V(\delta_k) \) for every \( k \in n \).
Hence Claim 1 is established.

Claim 2: \( Z \) is residual in \( \text{Int } V(\delta_m) \) for some \( m \).

Assume the contrary, i.e., \( Z \) is cofinal in \( X - \text{Int } V(\delta_n) \) for every \( n \). Then there exists a subset \( \{ k(n) : n \in \mathbb{N} \} \) of \( \mathbb{N} \) such that \( z_k(n) \in X - \text{Int } V(\delta_n) \) and \( k \prec k(n) < k(n+1) \), \( n \in \mathbb{N} \).
Using Fréchet-ness of \( X \), for each \( n \) we can take a sequence \( Z(n) \) of points of \( X - V(\delta_n) \) such that \( Z(n) \to z_k(n) \). Since \( z_k(n) \not\in p \) for each \( n \), we can use the same argument as above to get a contradiction, which implies the validity of Claim 2.
Now, let \( k \) be the maximum of \( n \) and \( m \) in Claims 1 and 2, respectively. Since \( F(\delta_s) \subseteq F(\delta_t) \) and \( V(\delta_t) \subseteq V(\delta_s) \) for every \( s, t \) with \( t \prec s \), and this \( k \) satisfies both claims. This completes the proof (1) \( \Rightarrow \) (3).

In the sequel, we denote by \( \mathbb{C} \) the class of all Fréchet spaces satisfying one and hence all of (1) to (3) in Theorem 1. With respect to the properties of \( \mathbb{C} \), the following hold:

THEOREM 2. \( \mathbb{C} \) has the following properties:

(1) \( \mathbb{C} \) is closed under closed mappings.
(2) \( \mathbb{C} \) is closed under subspaces.
(3) \( \{ \text{closed images of a developable space} \} \subseteq \mathbb{C} \).
(4) \( \mathbb{C} \) is not finitely productive.

All except (4) are easily seen from Theorem 1. (4) is a direct consequence of Theorem 9, stated later.

We give a characterization of developable spaces in terms of pair-networks some what different from the results of Burke [2, Theorem 2.1].
THEOREM 3. For a space $X$, the following are equivalent:

(1) $X$ is a developable space.

(2) $X$ is first countable and $X \in \mathcal{C}$.

(3) $X$ is a strongly Fréchet space having a $\sigma$-locally finite pair-network $P$ satisfying the same condition as in Theorem 1, (1).

(4) $X$ has a $\sigma$-locally finite pair-network $P$ such that each $V_\alpha$ is open in $X$.

PROOF. As well-known, a space $X$ is developable if and only if $X$ has a $\sigma$-discrete pair-network $P$ such that each $V_\alpha$ is open in $X$, [4]. So, (1) + (4) + (2) and (4) + (3) are obvious. (2) + (1): We shall show that $X$ has a $\sigma$-discrete pair-network $P$ such that all $V_\alpha$ are open in $X$. Let $P$ be a $\sigma$-HCP pair-network for $X$ satisfying the same condition as in Theorem 1, (1). For each $n \in \mathbb{N}$, let

$$X_n = \{ p \in X : \text{ord} \ (p, F_n) > \aleph_0 \},$$

where $F_n = \{ F_\alpha : \alpha \in A_n \}$. Since $X$ is Fréchet and each $F_n$ is HCP in $X$, each $X_n$ is a discrete closed subset of $X$. Let

$$X_{0n} = \{ p \in X : F_\alpha \cap \text{Int} \ V_\alpha = \{ p \} \text{ for some } \alpha \in A_n, \ n \in \mathbb{N} \}.$$

Then obviously $\bigcup \{ X_{0n} : n \in \mathbb{N} \}$ is a $\sigma$-discrete closed subset of $X$. For each $n$, by the method of [10] we can construct a $\sigma$-discrete family $H_n$ of closed subsets of $X$ from

$$B_n = F_n \cup \{(x) : x \in X_{0n} \cup X_n \}$$

such that $H_n$ satisfying the following: For each subfamily $B_0 \subset B_n$, if $p \in \bigcap B_0 - \bigcup (B_n - B_0)$, then $p \in H \subset \bigcap B_0 - \bigcup (B_n - B_0)$ for some $H \in H_n$. For each $H \in H_n$, $n \in \mathbb{N}$, with $H \cap (X_{0n} \cup X_n) = \phi$, choose an open subset $V(H)$ of $X$ such that

$$H \subset V(H) \subset \bigcap \{ \text{Int} \ V_\alpha : \alpha \in \delta \},$$
where $\delta$ is a finite subset of $A_n$ such that

$$H \subset \bigcap \{F_\alpha : \alpha \in \delta\} \cap \bigcup \{F_\alpha : \alpha \notin A_n - \delta\}.$$  

For each point $p \in X$, let $(O_n(p) : n \in \mathbb{N})$ be a local base at $p$ in $X$. Construct the pair-collection

$$P' = \{((p), O_n(p)) : p \in X_k, k, n \in \mathbb{N}\}$$

$$\bigcup \{((p), O_n(p)) : p \in X_{On}, k, n \in \mathbb{N}\}$$

$$\bigcup \{(H, V(H)) : H \in H_n', n \in \mathbb{N}\},$$

where

$$H_n' = \{H \in H_n : H \cap (X_{On} \cup X_n) = \emptyset\}, n \in \mathbb{N}.$$  

Then it is easy to see that $P'$ is a $\sigma$-discrete pair-network for $X$ such that the second subset of each pair of $P'$ is open in $X$, proving that $X$ is developable.

(3) $\Rightarrow$ (2): It suffices to show that $X$ is first countable.

Let $P = \bigcup \{P_n : n \in \mathbb{N}\}$ be a pair-network for $X$ satisfying the same condition as in Theorem 1, (1), where each $P_n = ((F_\alpha, V_\alpha) : \alpha \in A_n)$ is locally finite in $X$. Without loss of generality we can assume $A_n \subset A_{n+1}$, $n \in \mathbb{N}$. For each point $p$, $A_n(p) = \{\alpha \in A_n : p \in F_\alpha\}$, $n \in \mathbb{N}$, is finite. For each $n$, set

$$\Delta_n = \{\delta \subset A_n(p) : p \in \text{Int} V(\delta)\},$$

where

$$V(\delta) = \bigcup \{V_\alpha : \alpha \in \delta\}, \delta \in \Delta_n.$$  

We show that

$$\{\text{Int} V(\delta) : \delta \in \bigcup \{\Delta_n : n \in \mathbb{N}\}\}$$

is a local base at $p$ in $X$. Let $p \in U \in \tau_X$. For each $n$, we take

$$\delta_n \subset A_n(p)$$

such that

$$\delta_n = \{\alpha \in A_n(p) : V_\alpha \subset U\}.$$  

Assume $p \notin \text{Int} V(\delta_n)$ for each $n$. Since $X$ is strongly Fréchet, there exists a sequence $(p_n : n \in \mathbb{N})$ of points of $X$ such that $p_n \rightarrow p$ and $p_n \notin V(\delta_n)$, $n \in \mathbb{N}$. By the property of $P$, there
exists $a \in A_n$, $n \in \mathbb{N}$, such that $p \in F_a \subseteq V_a \subseteq U$ and $\{p_n\}$ is cofinal in $V_a$. But this is a contradiction. Hence we have $p \in \operatorname{Int} V(\delta_n) \subseteq U$ for some $m$.

As the corollaries, we have two: The former is already known [9, Cor. to Proposition 4] and the latter is known for the case when $X$ is an Moore space [3, Corollary 1.1]. The proof of the latter is the same as that of (2) $\rightarrow$ (1).

**COROLLARY 1.** If a closed image of a developable space is first countable, then it is developable.

**COROLLARY 2.** If $X$ is a closed image of a developable space, then $X = X_0 \cup X_1$, where $X_0$ is a $\sigma$-discrete closed subset and $X_1$ is a developable space.

The proof of (3) $\rightarrow$ (2) above assures the following theorem:

**THEOREM 4.** If $X$ is a strongly Fréchet space and $X \in C$, then $X$ has a $\sigma$-HCP pair-network such that all $V_a$ are open in $X$.

But we do not know whether such a space is developable.

**QUESTION 1.** If $X$ is a strongly Fréchet space and $X \in C$, then is $X$ developable?

The following gives another characterization of the class $C$, which is similar to that of Lašnev spaces in terms of $\sigma$-HCP $k$-networks by Foged.
THEOREM 5. A space $X$ belongs to $C$ if and only if $X$ is a Fréchet space which has a $c$-HCP pair-network $P$ such that if $K \subseteq U \in \tau_X$ with $K$ compact in $X$, then there exists a finite subcollection $\{(F_a, V_a) : a \in \delta\}$ of $P$ such that $K \subseteq \bigcup\{V_a : a \in \delta\} \subseteq U$ and $K \cap F_a \neq \emptyset$ for each $a \in \delta$.

PROOF. If part is trivial. Only if part: Let $P$ be a $c$-HCP pair-network for $X$ satisfying the condition of Theorem 1, (1). Assume $A_n \subseteq A_{n+1}$ for each $n$. For each $\delta \subseteq A_n$, $n \in \mathbb{N}$, set $F(\delta) = \bigcap\{F_a : a \in \delta\}$, $V(\delta) = \bigcup\{V_a : a \in \delta\}$ and $Q = \{(F(\delta), V(\delta)) : \delta \subseteq A_n, n \in \mathbb{N}\}$. Then $Q$ is a $c$-HCP pair-network for $X$. We shall show that $Q$ has the required property. Let $K \subseteq U \in \tau_X$ with $K$ compact in $X$. For each $n \in \mathbb{N}$, let $A_{0n} = \{a \in A_n : F_a \cap K \neq \emptyset \text{ and } V_a \subseteq U\}$. Then HCP-ness of $\{F_a : a \in A_n\}$ implies $\{F_a : a \in A_{0n}\}|K = \{F_1, F_2, \ldots, F_{k(n)}\}$ with some $k(n) \in \mathbb{N}$, [7, Proposition 3.7]. For each $i$ with $1 \leq i \leq k(n)$, choose $(F(\delta_{n1}), V(\delta_{n1})) \in Q$ such that $\delta_{n1} = \{a \in A_{0n} : F_a \cap K = F_i\}$. Obviously $\bigcup\{V(\delta_{n1}) : 1 \leq i \leq k(n)\} \subseteq U$. Assume $K \notin \bigcup\{V(\delta_{n1}) : 1 \leq i \leq k(n)\}$ for each $n$. Choose a sequence $(p_n : n \in \mathbb{N})$ of points of $X$ such that $p_n \in K - \bigcup\{V(\delta_{n1}) : 1 \leq i \leq k(n)\}$, $n \in \mathbb{N}$. Since $K$ is metrizable, $(p_n)$ has a convergent subsequence $Z$ to some point $p \in K$ in $X$. By the property of $P$, there exists $a_0 \in A_m$, $m \in \mathbb{N}$, such that $p \in F_{a_0} \subseteq V_{a_0} \subseteq U$ and
Z is cofinal in $V_{\alpha_0}$. But this is a contradiction because

$$V_{\alpha_0} \subset \bigcup \{V(\delta_{m_1}) : 1 \leq 1 \leq k(m)\}.$$  

This completes the proof.

Viewing Theorem 1, (1), we can easily observe that

a space $X$ belongs to $\mathcal{C}$ if and only if $X$ is a Fréchet space

having a $\sigma$-HCP pair-network $P$ such that the following conditions:

(C1) For each $\alpha \in A$, there exists an open subset $W_\alpha$ of $X$ such that $V_\alpha = F_\alpha \cup W_\alpha$.

(C2) If $Z \to p \in U \in \tau_X$ and $Z \subset X - \{p\}$, then there exists $\alpha \in A$ such that $p \in F_\alpha \subset V_\alpha \subset U$, $F_\alpha - \{p\} \subset W_\alpha$ and $Z$ is residual in $W_\alpha$.

By setting one more additional condition to $P$, we define

a class $\mathcal{C}'$ of spaces as follows: A space $X$ belongs to $\mathcal{C}'$ if and only $X$ is a Fréchet space having a $\sigma$-HCP pair-network $P$ satisfying the following additional condition (IP) besides (C1) and (C2):

(IP) For each $n$, $W_n = \{W_\alpha : \alpha \in A_n\}$ is an IP family of open subsets of $X$.

With respect to the properties of $\mathcal{C}'$, the following holds and that corresponds to Theorem 2 for $\mathcal{C}$.

THEOREM 6. $\mathcal{C}'$ has the following properties:

(1) $\mathcal{C}'$ is closed under closed mappings.

(2) $\mathcal{C}'$ is closed under subspaces.

(3) A closed image of an orthocompact developable space belongs to $\mathcal{C}'$.

(4) $\mathcal{C}'$ is not finitely productive.
PROOF. (2) is obvious and (4) is a direct consequence of Theorem 9. So, we state the proofs of (1) and (3) only. First, we show (1). Let \( f : X \to Y \) be a closed mapping of \( X \) onto a space \( Y \) and let \( X \subseteq C' \). Let \( P \) be a \( \sigma \)-HCP pair-network for \( X \) assured by the definition of \( X \subseteq C' \). Assume \( A_n \subseteq A_{n+1} \), \( n \in \mathbb{N} \). For each \( \delta \subseteq A_n \), \( n \in \mathbb{N} \), set
\[
F(\delta) = \bigcap \{ f(A') : a \in \delta \},
\]
\[
W(\delta) = Y \setminus f(X \setminus \bigcup W_a : a \in \delta),
\]
\[
V(\delta) = F(\delta) \cup W(\delta).
\]
Obviously \( \{ F(\delta) : \delta \subseteq A_n \} \) is a HCP family of closed subsets and \( \{ W(\delta) : \delta \subseteq A_n \} \) is an IP family of open subsets of \( Y \).

Thus, the pair-collection
\[
P' = \{(F(\delta), V(\delta)) : \delta \subseteq A_n, n \in \mathbb{N} \}
\]
is a \( \sigma \)-HCP pair-network for \( Y \) satisfying (C1) and (IP).

We show that \( P' \) satisfies the condition (C2) in \( Y \). Let \( Z \to y \in U \in \tau_Y \) and \( Z \subseteq Y \setminus \{y\} \). For each \( n \), let
\[
\delta_n = \{ a \subseteq A_n : \bar{f}_a \cap f^{-1}(y) \neq \emptyset \} \text{ and } V_a \subseteq f^{-1}(U).
\]
Then obviously, without loss of generality we can assume
\( y \in F(\delta_n) \subseteq V(\delta_n) \subseteq U \) for each \( n \in \mathbb{N} \).

Claim 1: \( F(\delta_n) - \{y\} \subseteq W(\delta_n) \) for some \( m \).

To see it, assume the contrary. Then we can choose a point \( p_n \in F(\delta_n) - \{y\} - W(\delta_n) \) for each \( n \). Since \( \{ F(\delta_n) : n \in \mathbb{N} \} \) forms a decreasing local network at \( y \) in \( Y \), \( p_n \to y \) as \( n \to \infty \) in \( Y \).

Using the closedness of \( f \) and Fréchet-ness of \( X \), we can choose a sequence \( \{ q_n(k) : k \in \mathbb{N} \} \) of points of \( X - f^{-1}(y) \) such that \( \{ q_n(k) \} \) converges to some point of \( f^{-1}(y) \), \( f(q_n(k)) = p_n(k) \).
and

\[ q_n(k) \in \bigcup \{ w_\alpha : \alpha \in \delta_n(k) \} \quad \text{for each } k \]

where \( k \leq n(k) < n(k+1), k \in \mathbb{N} \). By (C2) of \( C' \), there exists

\( \alpha \in A_n, n \in \mathbb{N} \), such that \( \{ q_n(k) \} \) is residual in \( W_\alpha \) and

\( \alpha \in \delta_n \). But this is a contradiction. Hence Claim 1 is established.

By the same argument as above, we can show that \( Z \) is residual in \( W(\delta_m) \) for some \( m \). This completes the proof of (1).

Since an orthocompact developable space \( X \) has a \( \sigma \)-discrete pair-network \( P \) such that for each \( n \) \( \{ V_\alpha : \alpha \in A_n \} \) is an IP family of open subsets of \( X \), obviously \( X \in C' \), which combined with (1) implies (3).

We give two lemmas used in the proof of Theorem 7.

**Lemma 1.** Let \( X \in C' \). Then for each discrete family

\( \{ F_\lambda : \lambda \in \Lambda \} \) of closed subsets of \( X \) there exist families

\( \{ W_\lambda : \lambda \in \Lambda \} \) of open subsets of \( X \) satisfying the following:

1. For each \( \lambda \), \( W_\lambda \) is an outer base of \( F_\lambda \) in \( X \).
2. \( \bigcup \{ W_\lambda : (X - F_\lambda) : \lambda \in \Lambda \} \) is IP in \( X \).

**Proof.** For each \( \lambda \in \Lambda \), there exists a sequence \( \{ O(\lambda, n) : n \in \mathbb{N} \} \) of open subsets of \( X \) such that

\[ F_\lambda = \bigcap \{ O(\lambda, n) : n \in \mathbb{N} \}, \]

\[ O(\lambda, n+1) \subseteq O(\lambda, n) \subseteq O(F_\lambda, n) \cap (X - \bigcup \{ F_\mu : \mu \notin \lambda \}) \].

Let \( P \) be a \( \sigma \)-HCP pair-network for \( X \) assured by \( X \in C' \). Let

\( \lambda \in \Lambda \) be fixed for a while. Set

\[ W_n = \{ W_\alpha : O(\lambda, n) : \alpha \in A_n \}, n \in \mathbb{N} \]

Let \( \{ W(\delta) : \delta \in \Delta(\lambda) \} \) be the totality of subfamilies of
\[ \bigcup \{w_n : n \in N\} \text{ such that} \]
\[ W(\delta) = F_\lambda \cup \left( \bigcup w(\delta) \right) \]
is an open neighborhood of \( F_\lambda \) in \( X \). We show that \( \{W(\delta) : \delta \in \Delta(\lambda)\} \) is an outer base of \( F_\lambda \) in \( X \). Let \( F_\lambda \subset O \subseteq \gamma_X \). Let
\[ w_n' = \{W \subseteq w_n : W \subset O\}, n \in N, \]
\[ w(\delta) = \bigcup \{w_n' : n \in N\}. \]
Then \( F_\lambda \subset W(\delta) \subset O \). To see that \( W(\delta) \) is open in \( X \), assume the contrary. Take a point \( p \in F_\lambda \setminus \text{Int } W(\delta) \). Since \( X \) is Préchet, there exists a sequence \( Z \) of points of \( X \setminus W(\delta) \) such that \( Z \to p \) in \( X \). By the property of \( P \), we can choose \( a \in A_n \), \( n \in N \), such that
\[ p \in F_a \subset V_a \subset O, F_a \setminus \{p\} \subset W_a \]
and \( Z \) is residual in \( W_a \). This implies also that \( Z \) is residual in \( W_a \cap O(\lambda, n) \). But this is a contradiction. From the property (IP) of \( P \), we can easily see that (2) is satisfied for thus constructed
\[ w_\lambda = \{W(\delta) : \delta \in \Delta(\lambda)\}, \lambda \in \Lambda. \]
This completes the proof.

A space \( X \) is called d-IP-expandable [6] if for each discrete family \( \{F_\lambda : \lambda \in \Lambda\} \) of closed subsets and each family \( \{U_\lambda : \lambda \in \Lambda\} \) of open subsets of \( X \) such that \( F_\lambda \subset U_\lambda \), \( \lambda \in \Lambda \), there exists an IP family \( \{V_\lambda : \lambda \in \Lambda\} \) of open subsets of \( X \) such that \( F_\lambda \subset V_\lambda \subset U_\lambda \), \( \lambda \in \Lambda \).

**Lemma 2.** If \( X \in C' \), then \( X \) is orthocompact.

**Proof.** By the lemma above, \( X \) is d-IP-expandable. Since a submetacompact, d-IP-expandable space is orthocompact, [6, Theorem 2.5], \( X \) is orthocompact.
From Lemmas 2 and 3, we have a characterization of
orthocompact developable spaces in terms of pair-networks as
follows:

THEOREM 7. For a space $X$, the following are equivalent:
(1) $X$ is an orthocompact developable space.
(2) $X$ is a first countable space and $X \in C'$.

A space $X$ is called $d$-paracompact \cite{[1]} if for each open
cover $U$ of $X$, there exists a $U$-mapping of $X$ onto a
developable space. A space $X$ is called subdevelopable
if $\tau_X$ contains a developable subtopology. With respect to
the notions, we have the following:

THEOREM 8. If $X \in C'$, then $X$ is both $d$-paracompact and
subdevelopable.

PROOF. If $X \in C'$, then by Lemma 1 $X$ is $D$-expandable
and hence $\text{id} d$-paracompact \cite{[1], Theorem 1}. Since a
d-paracompact space with a $G_\delta$-diagonal is subdevelopable
\cite{[8], Theorem 4}, $X$ is subdevelopable.

But we do not know whether the above holds for the class $C$.

QUESTION 2. If $X \in C$, then is $X$ $d$-paracompact or
subdevelopable?

It is well-known as Heyman's result that for any
non-discrete spaces $X, Y$, the product space $X \times Y$ being Lašnev
means both $X, Y$ are metrizable. This is true for the class $C'$.
we state it more generally.

**THEOREM 9.** Let $X$, $Y$ be non-discrete spaces. If $X \times Y \in \mathcal{C}'$, then $X \times Y$ is an orthocompact developable space.

**PROOF.** By the virtue of Theorem 7, it suffices to show that both $X$, $Y$ are first countable. Let $P$ be a $\sigma$-HCP pair-network for $X \times Y$ defining $X \times Y \in \mathcal{C}'$. Let $Z$ be a sequence of points of $X$ such that $Z \to x$ and $Z \subseteq X - \{x\}$. Let $y$ be an arbitrary point of $Y$. We show that $y$ has a countable local base in $Y$. Obviously

$$Z' = \{(z_k, y) : k \in \mathbb{N}\} \to (x, y)$$

in $X \times Y$. Since $\{W_\alpha : \alpha \in A_n\}$, $n \in \mathbb{N}$, is IP in $X \times Y$ by (IP), for each pair $(m, n) \in \mathbb{N}^2$ with

$$(z_m, y) \in W_\alpha \quad \text{for some } \alpha \in A_n,$$

there exists an open subset $O(m, n)$ of $X$ such that

$$(z_m, y) \in O(m, n) \subseteq \bigcap\{W_\alpha : \alpha \in A_n\},$$

$$(z_m, y) \not\in W_\alpha.$$ 

Let $N_0$ be the totality of such pairs $(m, n)$. Let $p : X \times Y \to Y$ be the projection. By the property of $P$, it is easily seen that $\{p(O(m, n)) : (m, n) \in N_0\}$ is a local base at $y$ in $Y$. This completes the proof.

**COROLLARY.** Let $X$, $Y$ be non-discrete spaces. If $X \times Y$ is the closed image of an orthocompact developable space, then $X \times Y$ is an orthocompact developable space.

But, we do not know whether Theorem 9 holds for the class $\mathcal{C}$:
QUESTION 3. For non-discrete spaces $X$, $Y$, does $X \times Y$ imply that $X \times Y$ is developable?

Finally, we pose the following question about the characterization of a closed image of a developable space:

QUESTION 4. If a space $X$ belongs to $C$, then is $X$ a closed image of a developable space?

REFERENCES

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