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CONTINUOUS FUNCTIONALS ON FUNCTION SPACES

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In this note, we assume that all spaces are Tychonoff. Let $C(X)$ be the set of all real-valued continuous functions on $X$. We call a real-valued function on $C(X)$ a functional. $C_p(X)$, $C_k(X)$ and $C_n(X)$ denote function spaces over $X$ with the pointwise convergent topology, the compact-open topology and the sup-norm topology respectively. For a family $\mathcal{A}$ of sets, we write $\bigcap \mathcal{A} = \bigcap \{A : A \in \mathcal{A}\}$. For a function $f$ on $X$ and a subset $M$ of $X$, the restriction of $f$ to $M$ is denoted by $f|_M$. The symbol $\pi_M : C_k(X) \to C_k(M)$ denotes the restriction map from $X$ to a subspace $M$. $\mathbb{R}$, $\omega$ and $\omega_1$ denote the real line, the first infinite ordinal and the first uncountable ordinal respectively.

First, we consider linear continuous functionals on $C_p(X)$. For any point $x$ in $X$, we can suppose that $x$ is a functional, which carries $f$ into $f(x)$ for any $f$ in $C(X)$, on $C(X)$. Obviously $x$ is a linear continuous functional on $C_p(X)$. The following fact is well-known.

**Fact 1.** Let $\lambda$ be a non-constant linear continuous functional on $C_p(X)$. There exist a finite subset $\{x_1, \ldots, x_n\}$ and non-zero numbers $\{\alpha_1, \ldots, \alpha_n\}$ such that $\lambda = \sum_{i=1}^{n} \alpha_i x_i$.

By Fact 1, we have;

- (1) For any pair $(f, g)$ of functions in $C_p(X)$, if $f|_{\{x_1, \ldots, x_n\}} = g|_{\{x_1, \ldots, x_n\}}$ holds, then $\lambda(f) = \lambda(g)$ holds,
• (2) There exists a real-valued continuous function $\tilde{\lambda}$ on $\mathbb{R}^{\{x_1, \ldots, x_n\}}$ such that $\lambda = \tilde{\lambda} \circ \pi_{\{x_1, \ldots, x_n\}}$.

In (2), the continuity of $\tilde{\lambda}$ is deduced by the following fact.

**Fact 2.** Let $F$ be a closed subset of $X$ and $\pi_F$ the restriction map from $C_p(X)$ into $C_p(F)$. Then $\pi_F$ is an open map onto $\pi_F(C_p(X))$.

Below, we shall deal with non-linear functionals in general. In view of (1), (2) and Fact 2, we define a notion.

**Definition.** Let $\xi$ be a functional on $C(X)$. A subset $S$ of $X$ is said to be a support for $\xi$ if $S$ is closed in $X$ and $\xi(f) = \xi(g)$ holds for any pair $(f, g)$ of functions in $C(X)$ such that $f|_{S} = g|_{S}$. $\text{Supp} \xi$ denotes the set of all supports for a functional $\xi$ on $C(X)$.

By Fact 2, if $\xi$ is a continuous functional on $C_p(X)$ and $S$ is a support for $\xi$, then there exists a real-valued continuous function $\tilde{\xi}$ on $\pi_S(C_p(X))$ such that $\xi = \tilde{\xi} \circ \pi_S$.

Moreover, we have a condition on the set $\{x_1, \ldots, x_n\}$ in Fact 1.

• (3) If $S$ is a support for $\lambda$, then $\{x_1, \ldots, x_n\} \subset S$ holds.

(3) says that the set $\{x_1, \ldots, x_n\}$ is minimal in supports for $\lambda$ in Fact 1. In general, we define a concept;

**Definition.** Let $\xi$ be a functional on $C(X)$ and $S$ a support for $\xi$. $S$ is said to be minimal if every support for $\xi$ contains $S$.

By (1) and (3), we have that every linear continuous functional on $C_p(X)$ has the finite minimal support. Generally, we have;
Theorem 3. ([1]) The minimal support $S$ for any continuous functional on $C_p(X)$ exists and $S$ is a separable subspace of $X$.

In the proof of Theorem 3, we show that, for any continuous functional $\xi$ on $C_p(X)$, $\bigcap \text{Supp} \xi$ is a support for $\xi$.

By Theorem 3, we have an operation from the set of all continuous functionals on $C_p(X)$ to the set of all closed separable subspaces of $X$. The following is remarkable.

Remark 4. For any countable subset $A$ of $X$, there exists a continuous functional $\xi_A$ on $C_p(X)$ such that $\bigcap \text{Supp} \xi_A = \overline{A}$.

Using the same idea in the proof of Theorem 3, we can prove the following theorem.

Theorem 5. Let $\mathcal{F}$ be a non-empty proper closed subset of $C_p(X)$. We put

$$\text{Supp} \mathcal{F} = \{S \subset X : S \text{ is closed in } X, \pi_S^{-1} (\pi_S (\mathcal{F})) = \mathcal{F}\}.$$  

Then the set $\bigcap \text{Supp} \mathcal{F}$ belongs to $\text{Supp} \mathcal{F}$.

This theorem gives a result on the minimal support.

Theorem 6. ([1]) Let $\xi$ be a non-constant continuous functional on $C_p(X)$. For an $r \in \xi(C_p(X))$, we put $S_r = \bigcap \text{Supp} \xi^{-1}(r)$. Then we have

$$\bigcap \text{Supp} \xi = \overline{\cup \{S_r : r \in \xi(C_p(X))\}}.$$  

For function spaces with the compact-open topology, we have a similar result.

Theorem 7. ([2]) The minimal support for any continuous functional on $C_k(X)$ exists.
Making a comparison between Theorem 3 and Theorem 7, we have the following question naturally.

**Question.** Let $S$ be the minimal support in Theorem 7. Does $S$ have a dense $\sigma$-compact subset?

Below, we consider this question. For the proofs of the following results, see [2]. Let $\tau$ be a cardinal. A space $X$ is said to be *almost $\tau$-compact* if for any $\alpha < \tau$, there exists a non-empty compact subset $K_\alpha$ of $X$ such that $X = \bigcup \{K_\alpha : \alpha < \tau\}$. Almost $\omega$-compact spaces are said to be *almost $\sigma$-compact*. The smallest cardinal $\tau$ such that $X$ is almost $\tau$-compact, is denoted by $cd(X)$.

**Definition.** A space $X$ has the *property $(\sigma)$* if, for any continuous functional $\xi$ on $C_k(X)$, the closed subset $\cap Supp \xi$ of $X$ is almost $\sigma$-compact.

First, we give a sufficient condition of the property $(\sigma)$.

**Theorem 8.** If the space $C_k(X)$ satisfies the countable chain condition, then $X$ has the property $(\sigma)$.

Vidossich [4] and Nakhmanson [3] proved that $C_k(X)$ satisfies the countable chain condition if $X$ is submetrizable. We have the following corollary.

**Corollary 9.** If $X$ is submetrizable (in particular, metrizable), then $X$ has the property $(\sigma)$.

**Proposition 10.** The space $\omega_1$ has the property $(\sigma)$.

**Remark 11.** Nakhmanson [3] noted that $C_k(\omega_1)$ does not satisfy the countable chain condition.
In special cases, we have a condition that the property (σ) necessarily satisfies.

**Theorem 12.** Let $X$ be a space which has a closed-and-open subset $Y$ such that $\text{cd}(Y) = \omega_1$. If $X$ has the property (σ), then every compact subset of $X$ is metrizable.

Using Theorem 12, we have a space which does not have the property (σ).

**Example.** Let $D(\omega_1)$ be the discrete space whose cardinarity is $\omega_1$. The space $D(\omega_1) \oplus (\omega_1 + 1)$ does not have the property (σ).

**Remark 13.** The above example shows that the property (σ) is not preserved by topological sums in general. In fact, since $C_k(D(\omega_1)) = C_p(D(\omega_1))$ holds, every continuous functional on $C_k(D(\omega_1))$ has the countable minimal support. Obviously the space $\omega_1 + 1$ has the property (σ).

**Final Remarks.** Theorem 5 and Theorem 6 are valid for $C_k(X)$ (See [2]). Theorem 3 and Theorem 5 are not valid for $C_n(X)$. For any $f$ in $C_n(\omega_1)$, $\bar{f}$ denotes the unique extension of $f$ to $\omega_1 + 1$. We define a functional $\xi$ on $C_n(\omega_1)$ by the rule $\xi(f) = \bar{f}(\omega_1)$ for any $f$ in $C_n(\omega_1)$. Then $\xi$ is continuous obviously. Since $[\alpha, \omega_1) \in \text{Supp} \xi$ holds for any $\alpha < \omega_1$, we have $\cap \text{Supp} \xi = \emptyset$. Put $\mathcal{F} = \{f \in C_n(\omega_1) : \bar{f}(\omega_1) = 0\}$. Then $\mathcal{F}$ is a non-empty proper closed subset of $C_n(\omega_1)$. Similarly, we have $\cap \text{Supp} \mathcal{F} = \emptyset$ also.

**References**


