Parabolic Variational Inequality for the Cahn-Hilliard Equation with Constraint

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1. Introduction

In this paper we study the Cahn-Hilliard equation with constraint by means of subdifferential operator techniques. Such a state constraint problem was recently proposed by Blowey-Elliott [1] as a model of diffusive phase separation. The questions of the existence, uniqueness and asymptotic behaviour of solutions, treated in [1] for the special case of the deep quench limit, are considered in our paper without such a restriction.

The standard Cahn-Hilliard equation is a model of diffusive phase separation in isothermal binary systems, and in terms of the concentration $u$ of one of the components it has the form

$$u_t + \nu \Delta^2 u - \Delta f(u) = 0 \quad \text{in} \quad Q_T = (0, T) \times \Omega.$$  \hspace{1cm} (1.1)

Here $\Omega$ is a bounded domain in $\mathbb{R}^N, N \geq 1$, with a smooth boundary $\Gamma = \partial \Omega$, $\nu$ is a positive constant related to the surface tension, $f(u)$ corresponds to the volumetric part of the chemical potential difference between components and is given by

$$f(u) = F'(u),$$  \hspace{1cm} (1.2)

where $F(u)$ is a homogeneous (volumetric) free energy parametrized by temperature $\theta$, with the characteristic double-well form for $\theta$ below the critical temperature $\theta_c$. Usually the free energy is approximated by polynomials $F: \mathbb{R} \to \mathbb{R}$, e.g. in the simplest case by quartic polynomial

$$F(u) = F_o(\theta) + \alpha_2(\theta - \theta_c)u^2 + \alpha_4u^4$$  \hspace{1cm} (1.3)

with constants $\alpha_2, \alpha_4 > 0$ and a given function $F_o(\theta)$ of temperature. To preserve an explicit physical sense, the state variable $u$ often is subject to some constraints, e.g. in the case of concentration natural limitation is

$$0 \leq u \leq 1.$$  \hspace{1cm} (1.4)

Then the free energy $F(u)$ can be assumed in the form of the so-called regular solution model

$$F(u) = F_o(\theta) + \alpha_0[\log u + (1 - u)\log(1 - u)] + \alpha_1(\theta - \theta_c)u(u - 1)$$  \hspace{1cm} (1.5)

with a function $F_o(\theta)$ and positive constants $\alpha_0, \alpha_1$. The corresponding form of the chemical potential $f(u)$ is shown in Fig. 1. Moreover, as the deep quench limit of (1.5), i.e. as the
In particular, if \( v_0 \in D \), then (2.1) holds for \( 0 = s < t \), too.

The third theorem is concerned with the large time behaviour of the solution \( v(t) \) of (VI).

**Theorem 2.3.** In addition to the assumptions \((\varphi 1)-(\varphi 3)\) and \((p)\) suppose that \( \alpha' \in L^1(\mathbb{R}_+) \), and

\[(\varphi 4) \varphi^t \text{ converges to a proper l.s.c. convex function } \varphi^\infty \text{ on } H \text{ in the sense of Mosco [11] as } t \to \infty, \text{ i.e.} \]

\( (M1) \) for any \( z \in D(\varphi^\infty) \) there exists a function \( w : \mathbb{R}_+ \to H \) such that \( w(t) \to z \) in \( H \) and \( \varphi^t(w(t)) \to \varphi^\infty(z) \) as \( t \to \infty; \)

\( (M2) \) if \( w : \mathbb{R}_+ \to H \) and \( w(t) \to z \) weakly in \( H \) as \( t \to \infty \), then \( \lim \inf_{t \to \infty} \varphi^t(w(t)) \geq \varphi^\infty(z). \)

Let \( v \) be the solution of (VI) on \( \mathbb{R}_+ \) associated with initial datum \( v_0 \in D \), and denote by \( \omega(v_0) \) the \( \omega \)-limit set of \( v(t) \) in \( H \) as \( t \to \infty \), i.e. \( \omega(v_0) := \{ z \in H; v(t_n) \to z \text{ in } H \} \) for some \( t_n \) with \( t_n \to \infty \). Then \( \omega(v_0) \neq \emptyset \) and

\[ \partial \varphi^\infty(v_\infty) + p(v_\infty) \ni 0 \quad \text{for all } v_\infty \in \omega(v_0). \]

Finally we give a result on the continuous dependence of solutions of (VI) upon the data \( v_\infty, \{ \varphi^t \} \) and \( p(\cdot) \).

**Theorem 2.4.** Let \( \{ \varphi^t_n \} \) be a sequence of families of proper l.s.c. convex functions on \( H \) such that conditions \((\varphi 1)-(\varphi 3)\) are satisfied for common positive constants \( C_0, C_1 \) and a common function \( \alpha \in W^{1,1}_{\text{l.o.c}}(\mathbb{R}_+) \). Also, let \( p_n \) be a sequence of Lipschitz continuous operators in \( H \) such that condition \((p)\) is satisfied for a common Lipschitz constant \( L_0 > 0 \) and a non-negative \( C^1 \)-function \( P_n \) on \( H \). Suppose that for each \( t \leq 0 \), \( \varphi^t_n \) converges to \( \varphi^t \) on \( H \) in the sense of Mosco as \( n \to \infty \), i.e.

\( (m1) \) for any \( z \in D \), there exists \( \{ z_n \} \subset H \) such that \( z_n \in D_n (= D(\varphi^t_n)) \), \( z_n \to z \) in \( H \) and \( \varphi^t_n(z_n) \to \varphi^t(z) \) as \( n \to \infty; \)

\( (m2) \) if \( z_n \in H \) and \( z_n \to z \) weakly in \( H \) as \( n \to \infty \), then \( \lim \inf_{n \to \infty} \varphi^t_n(z_n) \geq \varphi^t(z). \)

Furthermore suppose that for each \( z \in H , \)

\[ p_n(z) \to p(z) \quad \text{in } H, \quad P_n(z) \to P(z) \quad \text{as } n \to \infty. \]
The cases (1.3), (1.5) and (1.6) of free energies can be written in the form (1.7) with appropriate functions $\hat{\beta}$ and $\hat{g}$, and these special cases have been studied by Blowey-Elliot [1] and Elliott-Luckhaus [5].

2. Abstract results

We shall study evolution system (1.8)-(1.10) in an abstract framework. Let $H$ and $V$ be (real) Hilbert spaces such that $V$ is densely and compactly embedded in $H$. $V^*$ will be the dual of $V$. Then, identifying $H$ with its dual, we have

$$V \subset H \subset V^*$$

with dense and compact injections. Further, let $J^*$ be the duality mapping from $V^*$ onto $V$, and for $t \in \mathbb{R}_+$, let $\varphi^t(\cdot)$ be a proper, l.s.c., non-negative and convex function on $H$. We shall consider the following problem (VI):

$$\begin{cases} J^*(v'(t)) + \partial \varphi^t(v(t)) + p(v(t)) \ni 0 & \text{in } H, \ t > 0, \\ v(0) = v_0, \end{cases}$$

where $v' = (\frac{d}{dt})v$, $\partial \varphi^t$ is the subdifferential of $\varphi^t$ in $H$; $p(\cdot) : H \rightarrow H$ is a Lipschitz continuous operator and $v_0$ a given initial datum.

When it is necessary to indicate the data $\varphi^t, p$ and $v_0$ explicitly, (VI) is denoted by $(VI; \varphi^t, p, v_0)$.

Throughout this paper we use the following notations:

- $(\cdot, \cdot)$: the inner product in $H$;
- $\langle \cdot, \cdot \rangle$: the duality pairing between $V^*$ and $V$;
- $| \cdot |_W$: the norm in $W$ for any normed space $W$;
- $J$: the duality mapping from $V$ onto $V^*$, hence $J^* = J^{-1}$.

We use some basic notions and results about monotone operators and subdifferentials of convex functions; for details we refer to Brézis [2] and Lions [10].

We shall discuss $(VI) = (VI; \varphi^t, p, v_0)$ under the following additional hypotheses:

(\varphi 1) The effective domain $D(\varphi^t) = \{ z \in H; \varphi^t(z) < \infty \}$ of $\varphi^t$ is independent of $t \in \mathbb{R}_+$, $D := D(\varphi) \subset V$ and

$$\varphi^t(z) \geq C_0 |z|_V^2$$

for all $z \in V$ and all $t \in \mathbb{R}_+$,

where $C_0$ is a positive constant.

(\varphi 2) $(z_1^* - z_2^*, z_1 - z_2) \geq C_1 |z_1 - z_2|_V^2$ for all $z_i \in D$, $z_i^* \in \partial \varphi^t(z_i), i = 1, 2$, and all $t \in \mathbb{R}_+$, where $C_1$ is a positive constant.

(\varphi 3) There is a function $\alpha \in W_{\text{loc}}^{1,1}(\mathbb{R}_+)$ such that

$$\varphi^t(z) - \varphi^s(z) \leq |\alpha(t) - \alpha(s)|(1 + \varphi^s(z))$$

for all $z \in D$ and $s, t \in \mathbb{R}_+$ with $s \leq t$. 
(p) $p$ is a Lipschitz continuous operator in $H$ and there is a non-negative $C^1-$function $P : H \rightarrow \mathbb{R}$ whose gradient coincides with $p$, i.e. $p = \nabla P$; hence

$$\frac{d}{dt} P(w(t)) = (p(w(t)), w'(t)) \quad \text{for a.e. } t \in \mathbb{R}, \text{ if } w \in W^{1,2}_{\text{loc}} \mathbb{R}_{+}; H.$$ 

We now introduce a notion of the solution in a weak sense to problem (VI).

**Definition 2.1.** (i) Let $0 < T < \infty$. Then a function $v : [0, T] \rightarrow H$ is called a solution of (VI) on $[0, T]$, if $v \in L^2(0, T; V) \cap C([0, T]; V^*)$, $v' \in L^2_{\text{loc}}((0, T]; V^*)$, $v(0) = v_o$, $\varphi^t(v) \in L^1(0, T)$ and

$$-J'(v'(t)) - p(v(t)) \in \partial \varphi^t(v(t)) \quad \text{for a.e. } t \in [0, T].$$

(ii) A function $v : \mathbb{R}_{+} \rightarrow H$ is called a solution of (VI) on $\mathbb{R}_{+}$, if the restriction of $v$ to $[0, T]$ is a solution of (VI) on $[0, T]$ for every finite $T > 0$.

Our results for (VI) are given as follows.

**Theorem 2.1.** Assume that $(\varphi 1)-(\varphi 3)$ and (p) are satisfied. Let $T$ be any positive number. Then the following two statements (a) and (b) hold:

(a) If $v_o$ is given in the closure $D_\star$ of $D$ in $V^*$, then (VI) has one and only one solution $v$ on $[0, T]$ such that

$$t^\frac{1}{2} v' \in L^2(0, T; V^*), \quad \sup_{0 < t \leq T} t \varphi^t(v(t)) < \infty.$$ 

(b) If $v_o \in D$, then the solution $v$ of (VI) on $[0, T]$ satisfies that

$$v' \in L^2(0, T; V^*), \quad \sup_{0 \leq t \leq T} \varphi^t(v(t)) < \infty;$$

hence $v \in C([0, T]; H)$.

The second theorem is concerned with the energy inequality for (VI).

**Theorem 2.2.** Assume that $(\varphi 1)-(\varphi 3)$ and (p) hold. Let $v$ be the solution of (VI) on $\mathbb{R}_{+}$ associated with initial datum $v_o \in D_\star$. Define

$$X(t, z) = \varphi^t(z) + P(z) \quad \text{for } z \in D \text{ and } t \in \mathbb{R}_{+}.$$ 

Then: (a)

$$\sup_{0 \leq r \leq t} |v(r)|^2_{V^*} + \int_0^t \varphi^r(v(r))dr \leq M_o\{ |v_o|^2_{V^*} + \int_0^t \varphi^r(z)dr + (|z|^2_{H} + 1)\}e^{M_o t}$$

for all $z \in D$ and $t > 0$,

where $M_o$ is a positive constant dependent only on $C_o$ in $(\varphi 1)$, the Lipschitz constant $L_p$ of $p(\cdot)$ and the value $|p(0)|_H$. 
limit of (1.5) as $\theta \to 0$, the non-smooth free energy

$$F(u) = \begin{cases} F_0(\theta) + \alpha_1 \theta_c u (1-u) & \text{if } 0 \leq u \leq 1, \\ \infty & \text{otherwise} \end{cases}$$

(1.6)

is obtained (see Fig. 2); the constraint (1.4) is included in formula (1.6). This type of free energy (1.6) was introduced by Oono-Puri [12], and the corresponding Cahn-Hilliard equation was numerically studied by them; subsequently this model was analyzed theoretically, too, by Blowey-Elliott [1].

For generality we propose in this paper the representation of (possibly non-smooth) free energy in the form

$$F(u) = \hat{\beta}(u) + \hat{g}(u),$$

(1.7)

where $\hat{\beta}$ is a proper, l.s.c. and convex function on $\mathbb{R}$ and $\hat{g}$ is a non-negative function of $C^1$-class on $\mathbb{R}$ with Lipschitz continuous derivative $g = \hat{g}'$ on $\mathbb{R}$. In such a non-smooth case of free energy functionals, the formula (1.2), giving the volumetric part $f(u)$ of the chemical potential difference, does not make sense any longer. Therefore, following the idea in [1], we introduce a generalized notion of chemical potential which is represented in terms of the multivalued function

$$F(u) = \{\xi + g(u); \xi \in \beta(u)\},$$

where $\beta$ is the subdifferential of $\hat{\beta}$ in $\mathbb{R}$. Then the Cahn-Hilliard equation (1.1) is extended to the general form

$$u_t + \nu \Delta^2 u - \Delta (\xi + g(u)) = 0, \quad \xi \in \beta(u) \quad \text{in } Q_T.$$  

(1.8)

Equation (1.8) is to be satisfied together with boundary conditions

$$\frac{\partial u}{\partial n} = 0, \quad \frac{\partial}{\partial n} (\nu \Delta u + \xi + g(u)) = 0 \quad \text{on } \Sigma_T := (0, T) \times \gamma$$

(1.9)

and initial condition

$$u(0, \cdot) = u_o \quad \text{in } \Omega,$$

(1.10)

where $u_o$ is a given initial datum, and $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\Gamma$. 
Let \( \{v_{\alpha}\} \) be a sequence in \( V^* \) such that \( v_{\alpha} \in D_{n\star} \) (=the closure of \( D_n \) in \( V^* \)), \( v_0 \in D_\star \) and \( v_{\alpha} \to v_0 \) in \( V^* \) as \( n \to \infty \). Then the solution \( v_n \) of \( (VI)_n := (VI; \varphi^t, p_n, v_{\alpha}) \) converges to the solution \( v \) of \( (VI) := (VI; \varphi^t, p, v_0) \) as \( n \to \infty \) in the following sense: for every finite \( T > 0 \) and every \( 0 < \delta < T \),

\[
\begin{align*}
     v_n & \to v \quad \text{in } C([0, T]; V^*), \\
     t^1 v_n \to t^1 v' & \quad \text{weakly in } L^2(0, T; V^*), \\
     v_n & \to v \quad \text{in } C([\delta, T]; H) \text{ and weakly}^* \text{ in } L^\infty(\delta, T; V),
\end{align*}
\]
as \( n \to \infty \).

3. Sketch of the proofs

We sketch the proofs of the main theorems.

(1) (Uniqueness) Let \( v_1, i = 1, 2 \), be two solutions of \( (VI) \) on \([0, T]\) and put \( v := v_1 - v_2 \). Multiply the difference of two equations, which \( v_1 \) and \( v_2 \) satisfy, by \( v \), and then use the inequality

\[
|z|^2_H \leq \epsilon|z|^2_V + C(\epsilon)|z|^2_{V^*}
\]

for all \( z \in V \), where \( \epsilon \) is an arbitrary positive number and \( C(\epsilon) \) is a suitable positive constant dependent only on \( \epsilon \). Then we have an inequality of the form

\[
\frac{1}{2} \frac{d}{dt} |v(t)|^2_V + k_1 |v(t)|^2_V \leq k_2 |v(t)|^2_{V^*}
\]

for a.e. \( t \in [0, T] \), where \( k_1 \) and \( k_2 \) are some positive constants. Therefore, Gronwall's lemma implies that \( v = 0 \).

(2) (Approximate problems) Let \( v_0 \in D \) and \( \mu \) be any parameter in \((0, 1]\). Consider the following approximate problem \( (VI)_\mu \) for \( (VI) \):

\[
\begin{align*}
    \{ & (J^* + \mu I)(v_\mu(t)) + \partial \varphi^t(v_\mu(t)) + p(v_\mu(t)) \ni 0 \quad \text{in } H, \quad 0 < t < T, \\
    & v_\mu(0) = v_0.
\}
\]

By making use of the results in [9] this problem \( (VI)_\mu \) has one only one solution \( v_\mu \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V) \). Also, multiplying the equation of \( (VI)_\mu \) by \( v_\mu, v'_\mu \) and \( tv'_\mu \), we have similar estimates as those in Theorem 2.2.

(3) (Existence and estimates for \( (VI) \)) In the case when \( v_0 \in D \), by the standard monotonicity and compactness methods we can prove that the solution \( v_\mu \) tends to the solution \( v \) of \( (VI) \) as \( \mu \to 0 \) in the sense that

\[
\begin{align*}
    v_\mu & \to v \quad \text{in } C([0, T]; H) \text{ and weakly}^* \text{ in } L^\infty(0, T; V), \\
    v'_\mu & \to v' \quad \text{weakly in } L^2(0, T; V^*), \\
    \mu v'_\mu & \to 0 \quad \text{in } L^2(0, T; H).
\]

\]
Moreover we have the estimates in Theorem 2.2 for $v$. In the case when $v_0 \in D_*$, it is enough to approximate $v_0$ by a sequence $\{v_{0n}\} \subset D$ and to see the convergence of the solution $v_n$ associated with initial datum $v_{0n}$.

(4) (Proof of Theorem 2.3) From the energy estimates which were obtained in Theorem 2.2, it follows that $v' \in L^2(1, \infty; V^*)$ and $v \in L^\infty(1, \infty; V)$; hence Theorem 2.3 holds.

(5) (Proof of Theorem 2.4) Under the assumptions of Theorem 2.4, we see from the energy estimates for $v_n$ that $\{v_n\}$ is bounded in $C([0, T]; H) \cap L^2(0, T; V) \cap L^\infty_{loc}((0, T]; V) \cap W^{1,2}_{loc}((0, T]; V^*)$. Hence by the usual monotonicity and compactness argument we have the assertions of Theorem 2.4.

4. Application to the Cahn-Hilliard equation with constraint

We denote by (CHC) the Cahn-Hilliard equation with constraint (1.8)-(1.10). Here we suppose that

(A1) $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with a non-negative primitive $\hat{g}$ on $\mathbb{R}$.

(A2) $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that $0 \in R(\beta)$ and $int. D(\beta) \neq \emptyset$; we may assume that there is a non-negative proper l.s.c. convex function on $\mathbb{R}$ such that its subdifferential $\partial \hat{\beta}$ coincides with $\beta$ in $\mathbb{R}$.

(A3) $u_0 \in L^2(\Omega), u_0(x) \in \overline{D(\beta)}$ for a.e. $x \in \Omega$.

**Definition 4.1.** Let $0 < T < \infty$. Then $u : [0, T] \to H$ is called a (weak) solution of (CHC) on $[0, T]$, if $u$ satisfies the following properties (w1)-(w3):

(w1) $u \in L^2(0, T; H^1(\Omega)) \cap C([0, T]; (H^1(\Omega))^*) \cap L^\infty_{loc}((0, T]; H^1(\Omega) \cap W^{1,2}_{loc}((0, T]; (H^1(\Omega))^*)$ and $\hat{\beta}(u) \in L^1(Q_T)$; 

(w2) $u(0, \cdot) = u_0$ a.e. in $\Sigma_T$;

(w3) there is a function $\xi : [0, T] \to L^2(\Omega)$ such that

$$\xi \in L^1_{loc}((0, T]; L^2(\Omega)), \quad \xi \in \beta(u) \quad a.e. \ in \ Q_T$$

and

$$\frac{d}{dt}(u(t), \eta) + \nu(\Delta u(t), \Delta \eta) - (\xi(t) + g(u(t)), \Delta \eta) = 0$$

for all $\eta \in H^2(\Omega)$ with $\frac{\partial}{\partial n} a.e.$ on $\Gamma$, and for a.e. $t \in [0, T]$.

Applying Theorems 2.1-2.4 to (CHC) we have:

**Theorem 4.1.** Assume that (A1)-(A3) hold and

$$m := \frac{1}{|\Omega|} \int_{\Omega} u_0 dx \in int. D(\beta).$$
Then for every finite $T > 0$ problem (CHC) has one and only one solution $u$ on $[0, T]$, and the following statements (a) and (b) hold:

(a) $u \in L^\infty(\delta, \infty; H^1(\Omega))$, $u'(\delta, \infty; (H^1(\Omega))^*)$ for every $\delta > 0$, and hence the $\omega$-limit set $\omega(u_\delta) := \{ z \in L^2(\Omega); u(t_n) \to z \text{ in } L^2(\Omega) \text{ for some } t_n \text{ with } t_n \to \infty \}$ is non-empty;

(b) $\omega(u_\delta) \subset H^2(\Omega)$, and any $u_\infty \in \omega(u_\delta)$ with some $\mu_\infty \in \mathbb{R}$ and $\xi_\infty \in L^2(\Omega)$ solves the following stationary problem

$$-\nu \Delta u_\infty + \xi_\infty + g(u_\infty) = \mu_\infty \quad \text{in } \Omega, \quad \xi_\infty \in \beta(u_\infty) \quad \text{a.e. in } \Omega,$$

$$\frac{\partial u_\infty}{\partial n} = 0 \quad \text{a.e. on } \Gamma, \quad \frac{1}{|\Omega|} \int_\Omega u_\infty\, dx = m.$$  

Now, let us reformulate (CHC) as an evolution problem of the form (VI) in the space $H := \{ z \in L^2(\Omega); \int_\Omega z\, dx = 0 \}$ with $|z|_H = |z|_{L^2(\Omega)}$; put also $V := H \cap H^1(\Omega)$ with $|z|_V = |\nabla z|_{L^2(\Omega)}$.

For this purpose we consider the data $\varphi^t = \varphi, p(\cdot) \text{ and } v_\circ$ as follows:

$$\varphi(z) := \begin{cases} \frac{\nu}{2} |\nabla z|_{L^2(\Omega)}^2 + \int_\Omega \hat{\beta}(z + m)\, dx & \text{if } z \in V \\ \infty & \text{otherwise,} \end{cases}$$

where $m = \frac{1}{|\Omega|} \int_\Omega u_\circ\, dx$;

$$p(z) := \pi(g(z + m)), \quad P(z) := \int_\Omega \hat{g}(z + m)\, dx, \quad z \in H;$$

$$v_\circ := u_\circ - m.$$  

By virtue of the following lemma, problems (CHC) and (VI) associated with the data defined above are equivalent.

**Lemma 4.1.** Let $\ell \in L^2(\Omega)$. Then $\pi(\ell) \in \partial \varphi(z)$ if and only if $z_m = z + m$ satisfies that there are $\mu_m \in \mathbb{R}$ and $\xi_m \in L^2(\Omega)$ such that

$$-\nu \Delta z_m + \xi_m = \ell + \mu_m \quad \text{in } L^2(\Omega), \quad \xi_m \in \beta(z_m) \quad \text{a.e. in } \Omega,$$

$$\frac{\partial z_m}{\partial n} = 0 \quad \text{a.e. on } \Gamma, \quad \frac{1}{|\Omega|} \int_\Omega z_m\, dx = m;$$

hence $z_m \in H^2(\Omega)$. Moreover, $\mu_m$ can be chosen so that

$$|\mu_m| \leq M(1 + |\ell|_{L^2(\Omega)}),$$

where $M > 0$ is a certain constant dependent only upon $\beta$ and $m$, and $z_m$ satisfies that

$$\nu |\Delta z_m|_{L^2(\Omega)} \leq |\ell|_{L^2(\Omega)} + |\mu_m||\Omega|^\frac{1}{2}.$$
By Theorem 2.1 problem (VI) has one and only one solution \( v \). Moreover we see from
the above lemma that the function \( u := v + m \) is the unique solution of (CHC), and from
Theorems 2.2 and 2.3 that (a) and (b) hold.

When the state constraint \( \xi \in \beta(u) \) is not imposed, the system (1.8)-(1.10) becomes
the standard Cahn-Hilliard problem. For such a problem various existence, uniqueness and
asymptotic results have been established; see e.g. Elliott [3], Elliott-Zheng [6] and Zheng [15].
For related results in abstract setting we refer to Temam [13] and von Wahl [14]. For the
Cahn-Hilliard models with non-smooth free energy functionals we refer to Elliott-Mikelic
[4]. The structure of stationary solutions corresponding to the Cahn-Hilliard equation was
studied by Gurtin-Matano [7]; their analysis covers also some cases of free energy \( F(u) \) with
infinite walls.

Finally we give examples of \( \beta \) and the corresponding Cahn-Hilliard equations.

**Example 4.1.** (i) (Logarithmic form) For constants \( \alpha_o > 0 \) and \( \theta > 0 \), \( \theta \) being a parameter,

\[
\beta(u) := \beta^\theta(u) = \begin{cases}
\alpha_o \theta \log \frac{u}{1-u} & \text{for } 0 < u < 1, \\
\emptyset & \text{otherwise.}
\end{cases}
\]

Given any Lipschitz continuous function \( \tilde{g} \) on \([0,1]\), we extend it to a Lipschitz continuous
function \( g \), with support in \([-1,2]\), on the whole line \( \mathbb{R} \).

(ii) (The limit of \( \beta^\theta \) as \( \theta \to 0 \))

\[
\beta(u) := \beta^0(u) = \begin{cases}
[0,\infty) & \text{if } u = 1, \\
\{0\} & \text{if } 0 < u < 1, \\
(-\infty,0] & \text{if } u = 0, \\
\emptyset & \text{otherwise,}
\end{cases}
\]

and \( g \) is the same as in (i).

**Example 4.2.** Denote by (CHC)_\( \theta \) and (CHC)_0 the Cahn-Hilliard equations (CHC) associated
with \( \beta = \beta^\theta \) and \( \beta = \beta^0 \), respectively. Then, by the theorems proved above, (CHC)_\( \theta \)
and (CHC)_0 have the unique solutions \( u^\theta \) and \( u^0 \), respectively, and moreover \( u^\theta \to u^0 \) as
\( \theta \to 0 \) in the similar sense as Theorem 2.4.

**References**


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