The Global Weak Solutions of the Compressible Euler Equation with Spherical Symmetry

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1 Introduction

The compressible Euler equation for an isentropic gas in $\mathbb{R}^n$ is given by

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \vec{u}) &= 0, \\
(\rho \vec{u})_t + \nabla \cdot (\rho \vec{u} \otimes \vec{u} + p) &= 0,
\end{align*}
\]

(1.1)

with the equation of state

\[ p = a^2 \rho^\gamma, \]

(1.2)

where density $\rho$, velocity $\vec{u}$ and pressure $p$ are functions of $x \in \mathbb{R}^n$ and $t \geq 0$, while $a > 0$ and $\gamma \geq 1$ are given constants.

For one dimensional case ($n=1$), the Cauchy problem for (1.1) with (1.2) has been studied by many authors. Nishida [10] established the existence of global weak solutions, for the first time, for the case $\gamma = 1$ with arbitrary initial data, and Nishida and Smoller [11] for $\gamma \geq 1$ but with small initial data, both using Glimm's method. DiPerna [3] extended the latter result to the case of large initial data, using the theory of compensated compactness under the restriction $\gamma = 1 + 2/(2m + 1)$, $m \geq 2$ integers. Ding et al [1], [2] removed this restriction and established the existence of global weak solutions for $1 < \gamma \leq 5/3$. 
On the other hand, little is known for the case \( n \geq 2 \). No global solutions have been known to exist, but only local classical solutions ([5], [6], [8] and [9]).

In this paper, we will present global weak solutions first for the case \( n \geq 2 \). We will do this, however, only for the case of spherically symmetry with \( \gamma = 1 \). As will be seen below, our proof does not work without these restrictions.

Thus, we look for solutions of the form

\[
\rho(t, |x|), \quad \tilde{u} = \frac{x}{|x|} \cdot u(t, |x|). \tag{1.3}
\]

Then, denoting \( r = |x| \), (1.1) becomes

\[
\rho_t + \frac{1}{r^{n-1}} (r^{n-1} \rho u)_r = 0, \quad \rho (u_t + u u_r) + p_r = 0, \tag{1.4}
\]

This equation has a singularity at \( r=0 \). To avoid the difficulty caused by this singularity, we simply deal with the boundary value problem for (1.4) in the domain \( 1 \leq r < \infty \) (the exterior of a sphere) with the boundary condition \( u(t, 1) = 0 \), which is identical, under the assumption (1.3), to the usual boundary condition \( \tilde{n} \cdot \tilde{u} = 0 \) for (1.1) where \( \tilde{n} \) is the unit normal to the boundary.

Put \( \tilde{\rho} = r^{n-1} \rho \). Then we get from (1.4)

\[
\tilde{\rho}_t + (\tilde{\rho} u)_r = 0, \quad u_t + u u_r + \frac{a^2 \gamma \tilde{\rho}_r}{\tilde{\rho}^{\gamma-2} r^{(n-1)(\gamma-2)}} = \frac{a^2 \gamma (n-1) \tilde{\rho}^{\gamma-1}}{r^n}. \tag{1.5}
\]

Introduce the Lagrangean mass coordinates

\[
\tau = t, \quad \xi = \int_1^r \tilde{\rho}(t, r) \, dr. \tag{1.6}
\]

Then \( \xi > 0 \) as long as \( \tilde{\rho} > 0 \) for \( r > 1 \), and (1.5) is reformulated as

\[
\tilde{\rho}_r + \tilde{\rho}^2 u_\xi = 0, \quad u_r + \frac{a^2 \gamma \tilde{\rho}_\xi}{\tilde{\rho}^{\gamma-2} r^{(n-1)(\gamma-2)}} = \frac{a^2 \gamma (n-1) \tilde{\rho}^{\gamma-1}}{r^n}. \tag{1.7}
\]
Put $v = 1/\tilde{\rho}$ and note that the inverse transformation to (1.6) is given by

$$(1.8) \quad t = \tau, \quad r = 1 + \int_0^\tau v(\zeta, t) d\zeta.$$ 

Then after changing $\tau$ to $t$ and $\xi$ to $x$, (1.7) is written as

$$(1.9) \quad v_t - u_x = 0,$$

$$(1.10) \quad u_t + \left( \frac{a^2}{v^\gamma} \right)_x = \frac{K}{1 + \int_0^x v(t, \zeta) d\zeta},$$

where $r$ is now defined by $r = 1 + \int_0^x v(t, \zeta) d\zeta$.

Now we restrict ourselves to the case $\gamma = 1$. Then (1.7) becomes

$$(1.11) \quad v_t - u_x = 0,$$

$$(1.12) \quad u(t, 0) = 0, \quad \text{for } t > 0.$$

Let $BV(\mathbb{R}_+)$ denote the space of functions of bounded variation on $\mathbb{R}_+ = (0, \infty)$. Our main result is as follows.

**Theorem (Main Result)** Suppose that $u_0(x), v_0(x) \in BV(\mathbb{R}_+)$, and that $v_0(x) \geq \delta_0 > 0$ for all $x > 0$ with some positive constant $\delta_0$. Then (1.10), (1.11) and (1.12) have a global weak solution which belongs to the class

$$u, \ v \in L^\infty(0, T; BV(\mathbb{R}_+)) \cap \text{Lip}([0, T]; L^1_{loc}(\mathbb{R}_+))$$

for any $T > 0$.

The definition of the weak solution will be given in section 4. This theorem can be proved by following Nishida's argument [10] based on Glimm's
method. Indeed this can be seen from the following two simple observations. First, the homogeneous equation corresponding to (1.10),

\begin{align}
  v_t - u_x &= 0, \\
  u_t + \left( \frac{a^2}{v} \right)_x &= 0,
\end{align}

(1.13)

is just the same equation as solved by Nishida [10] using Glimm's method both on the Cauchy problem and the initial boundary value problem. Note that if \( \gamma > 1 \), the homogeneous equation for (1.9) has a variable coefficient and hence does not coincide with the one dimensional Euler equation.

The second observation is that, as long as \( v \geq 0 \), the right hand side of (1.10),

\begin{equation}
  \frac{K}{1 + \int_{0}^{x} v(t, \zeta) \, d\zeta},
\end{equation}

(1.14)

is monotone decreasing in \( x \) and has an a priori estimate

\begin{equation}
  T. V. \left( \frac{K}{1 + \int_{0}^{x} v(t, \zeta) \, d\zeta} \right) \leq K,
\end{equation}

(1.15)

independent of \( v \). The one dimensional inhomogeneous Euler equation has been studied in [12]. However, the conditions imposed therein on the inhomogeneous term are not applicable to our (1.14).

These observations allow us to use Nishida's argument [10] to construct global weak solutions to (1.10), (1.11) and (1.12). More precisely, we will first construct, in section 2, approximate solutions of the form

\[ \{ \text{solution of Riemann problem for (1.13)} \} + \{ \text{nonhomogeneous term} \} \times t. \]

This is the main idea of [12]. Then in section 3, we will estimate the total variation of the approximate solutions. Thanks to (1.15), this can be done with a slight modification of Nishida's argument [10]. In section 4, we will show that there exists a subsequence of approximate solutions which converges strongly in \( L^1_{\text{loc}} \) for any finite time interval. Finally, for the sake of completeness, we give in Appendix a detailed proof of two lemmas used in section 3. These lemmas are due to Nishida [10], but their proofs are not found in the literature.
2 The Difference Scheme

To construct the approximate solutions, we shall use the difference scheme developed in [10]. For $l, h > 0$, define

\[ Y = \{ (n, m); \; n = 1, 2, 3, \ldots, m = 1, 3, 5, \ldots \}, \]
\[ A = \prod_{(m,n) \in Y} [\{nh\} \times ((m-1)l, (m+1)l)] , \]

where $l/h$ will be determined later. Choose a point $\{a_{nm}\} \in A$ randomly, and write $a_{nm} = (nh, c_{nm})$. For $n = 0$, we put $c_{0m} = ml$. We denote approximate solutions by $u^l$ and $v^l$. Mesh lengths $l$ and $h$ are chosen so that $l/h > a/(\inf v^t)$, for any given $T > 0$. We shall show later that there exists a $\delta > 0$ such that $\inf v^l \geq \delta > 0$.

For $0 \leq t < h$, $ml \leq x < (m+2)l$, $m : \text{odd}$, we define

\[ u^l(t, x) = u^l_0(t, x) + U^l(t, x)t, \]
\[ v^l(t, x) = v^l_0(t, x), \]

where $u^l_0$ and $v^l_0$ are the solutions of

\[ v_t - u_x = 0, \]
\[ u_t + \left( \frac{a^2}{v} \right)_x = 0, \]

with initial data

\[ u^l_0(0, x) = \begin{cases} u_0(ml), & x < (m+1)l, \\ u_0((m+2)l), & x > (m+1)l, \end{cases} \]
\[ v^l_0(0, x) = \begin{cases} v_0(ml), & x < (m+1)l, \\ v_0((m+2)l), & x > (m+1)l, \end{cases} \]

and

\[ U^l(t, x) = \frac{K}{1 + \sum_{j=1}^{m+1} v_0((2j-1)l) \cdot 2l} . \]

For $0 \leq t < h$, $0 \leq x < l$, we define $u^l$ and $v^l$ by (2.2) where $u^l_0$ and $v^l_0$ are the solutions of (2.3) with initial boundary data

\[ u^l_0(0, x) = u_0(l), \; v^l_0(0, x) = v_0(l), \; x > 0, \]
(2.7) \quad u(t, 0) = 0, \quad t > 0,

and

(2.8) \quad U^l(t, x) = K.

Suppose that $u^l$ and $v^l$ are defined for $0 \leq t < nh$. For $nh \leq t < (n+1)h$, $ml \leq x < (m+2)l$, $m : \text{odd}$, we define

(2.9) \quad u^l(t, x) = u_0^l(t, x) + U^l(t, x) \cdot (t - nh),
\quad v^l(t, x) = v_0^l(t, x),

where $u_0^l$ and $v_0^l$ are the solutions of (2.3) with initial data $(t=nh)$

(2.10) \quad u_0^l(nh, x) = \begin{cases} 
    u^l(nh - 0, c_{nm}), & x < (m + 1)l, \\
    u^l(nh - 0, c_{nm+2}), & x > (m + 1)l,
\end{cases}
\quad v_0^l(nh, x) = \begin{cases} 
    v^l(nh - 0, c_{nm}), & x < (m + 1)l, \\
    v^l(nh - 0, c_{nm+2}), & x > (m + 1)l,
\end{cases}

and

(2.11) \quad U^l(t, x) = \frac{K}{1 + \sum_{j=1}^{m+1} v^l(nh - 0, c_{2j-1}) \cdot 2l}.

For $nh \leq t < (n + 1)h$, $0 \leq x < l$, we define $u^l$ and $v^l$ as (2.9) where $u_0^l$ and $v_0^l$ are the solutions of (2.3) with initial $(t=nh)$ boundary data

(2.12) \quad u_0^l(nh, x) = u^l(nh - 0, c_{n1}), \quad v_0^l(nh, x) = v^l(nh - 0, c_{n1}), \quad x > 0,
\quad u(t, 0) = 0, \quad t > nh,

and $U^l(t, x)$ is as (2.8).

3 Bounds for Approximate Solutions

System (1.6) is hyperbolic provided $v > 0$, with the characteristic roots and Riemann invariants given by

(3.1) \quad \lambda = -\frac{a}{v}, \quad r = u + a \log v,
\quad \mu = \frac{a}{v}, \quad s = u - a \log v.
It is well-known, [10], that all shock wave curves in the \((r,s)\)-plane have the same figure. (See Figure 1.) The 1-shock wave curve \(S_1\), starting from \((r_0, s_0)\) can be expressed in the form

\[
\begin{align*}
    s - s_0 &= f(r - r_0) \quad \text{for } r \leq r_0, \\
    r - r_0 &= f(s - s_0) \quad \text{for } s \leq s_0,
\end{align*}
\]

where \(0 \leq f'(x) < 1, f''(x) \leq 0, \lim_{x \to -\infty} f'(x) = 1\).
The 1-rarefaction wave curve $R_1$ can be expressed in the form

\begin{equation}
 s - s_0 = 0 \quad \text{for } r \geq r_0,
\end{equation}

and the corresponding expression for the 2-rarefaction wave curve $R_2$ is

\begin{equation}
 r - r_0 = 0 \quad \text{for } s \geq s_0.
\end{equation}

Now we must prepare some lemmas to estimate Riemann invariants. First, let us consider (2.3) with following initial data

\begin{equation}
 u_0(x) = \begin{cases} u_l & x < 0, \\ u_r & x > 0. \end{cases} \quad v_0(x) = \begin{cases} v_l & x < 0, \\ v_r & x > 0. \end{cases}
\end{equation}

**Lemma 3.1** Let $u$ and $v$ are the solutions of (2.3) and (3.6). Then,

\begin{equation}
 r(t,x) \equiv r(u(t,x),v(t,x)) \geq r_0 \equiv \min (r(u_r,v_r), r(u_l,v_l)), \\
 s(t,x) \equiv s(u(t,x),v(t,x)) \leq s_0 \equiv \max (s(u^r,v^r), s(u^l,v^l)).
\end{equation}

Next consider (2.3) in $t \geq 0$, $x \geq 0$ with following initial and boundary conditions

\begin{equation}
 u(0,x) = u_0^+, \quad v(0,x) = v_0^+, \quad \text{for } x > 0,
\end{equation}

\begin{equation}
 u(t,0) = 0, \quad \text{for } t > 0.
\end{equation}

**Lemma 3.2** Let $u$ and $v$ are the solutions of (2.3), (3.8) and (3.9). Then,

\begin{equation}
 r(t,x) \equiv r(u(t,x),s(t,x)) \geq r(u_0^+,v_0^+), \\
 s(t,x) \equiv s(u(t,x),s(t,x)) \leq \max (-r(u_0^+,v_0^+), s(u_0^+,v_0^+)).
\end{equation}

The above two lemmas were proved in [10]. Using these two lemmas, we can get the following lemma.

**Lemma 3.3** Let $u^l$ and $v^l$ be the approximate solutions defined in section 2 and put $r_0 = \min r(u_0(x),v_0(x))$ and $s_0 = \max s(u_0(x),v_0(x))$. Then, for $0 < t < T$,

\begin{equation}
 r^l(t,x) \equiv r(u^l(t,x),s^l(t,x)) \geq r_0, \\
 s^l(t,x) \equiv s(u^l(t,x),s^l(t,x)) \leq \max (-r_0,s_0) + KT
\end{equation}
Let us consider Riemann problem (2.3) and (3.6). Denote by $\Delta r$ (resp $\Delta s$) the absolute value of the variation of the Riemann invariant $r$ (resp $s$) in the first (resp second) shock wave.

**Definition 3.4** We denote

$$P(u_l, v_l, u_r, v_r) = \Delta r + \Delta s.$$  

Then we have the following lemma.

**Lemma 3.5**

(3.12)  

$$P(u_1, v_1, u_3, v_3) \leq P(u_1, v_1, u_2, v_2) + P(u_2, v_2, u_3, v_3),$$  

where $u_1$, $u_2$ and $u_3$ are arbitrary constants and $v_1$, $v_2$ and $v_3$ are arbitrary positive constants.

We shall prove Lemma 3.5 in the Appendix A.

Denote by $i_0^{n\pm}$ the straight line segments joining the points $(0, (n \pm \frac{1}{2})h)$ and $a_{1n}$. Let $F(i_0^{n\pm})$ be the absolute value of the variation of the Riemann invariants for all shocks on $i_0^{n\pm}$. Then we also have the following Lemma.

**Lemma 3.6**

(3.13)  

$$F(i_0^{n+}) \leq F(i_0^{n-}).$$  

This lemma 3.6 will be proved in the Appendix B.

We denote

$$Z_1 = \{ l - 0, l + 0, 3l - 0, \ldots, (2m - 1)l - 0, (2m - 1)l + 0, \ldots \},$$  

$$Z_2 = \{ 2l, 4l, 6l, \ldots 2ml, \ldots \}.$$  

Let $Z_{(n)} = Z_1 \cup Z_2 \cup \{c_{mn}\}$ and line up the elements $z_{n,i}$ of $Z_{(n)}$ so that $z_{n,i} \leq z_{n,i+1}$. (We regard $(2m - 1)l - 0 < (2m - 1)l + 0$ for $m$ : integer.)

Let

$$F(nh - 0, u', v') = \frac{1}{2} F(i_0^{n-})$$  

$$+ \sum_{z_{n,i} \in Z_{(n)}} P(u'(nh - 0, z_{n,i}), v'(nh - 0, z_{n,i}), u'(nh - 0, z_{n,i+1}), v'(nh - 0, z_{n,i+1})), $$
\[ F(nh+0, u', v') = \frac{1}{2} F(i_0^{n+}) + \sum_{m: odd} P(u'(a_{nm}), v'(a_{nm}), u'(a_{n+2}), v'(a_{m+2})). \]

Using Lemma 3.5 and Lemma 3.6, we get

(3.14) \[ F((n+1)h+0, u', v') \leq F((n+1)h-0, u', v'). \]

The following equality is obvious from the definition of \( F, u' \) and \( v' \).

(3.15) \[ F((n+1)h-0, u_0', v_0') = F(nh+0, u', v'). \]

We also get

\[
F((n+1)h-0, u', v') = F((n+1)h-0, u_0', v_0') \\
+ \sum_{m: odd} P(u'(n+1)h-0, ml-0), v'(n+1)h-0, ml-0), \\
u'(n+1)h-0, ml+0), v'(n+1)h-0, ml+0)).
\]

**Lemma 3.7**

\[
P(u'(n+1)h-0, ml-0), v'(n+1)h-0, ml-0), \\
u'(n+1)h-0, ml+0), v'(n+1)h-0, ml+0) \\
\leq 2h \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\}, \ m: odd.
\]

**Proof.** From the definition,

\[
u'(n+1)h-0, ml-0) = u_0'(nh, ml) + U^l(nh, (m-1)l) \cdot h, \\
u'(n+1)h-0, ml+0) = u_0'(nh, ml) + U^l(nh, (m+1)l) \cdot h, \\
v'(n+1)h-0, ml-0) = v_0'(nh, ml).
\]

Therefore we get

\[
r'(n+1)h-0, ml-0) - r'(n+1)h-0, ml+0) \\
= s'(n+1)h-0, ml-0) - s'(n+1)h-0, ml+0) l \\
= h \times \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\} \geq 0
\]

Thus the following inequality holds.

(3.18) \[ \Delta r, \Delta s \leq h \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\} \leq \Delta r + \Delta s. \]
From (3.18), we get (3.16).

Using Lemma 3.7, we get

\[ F((n+1)h-0, u^l, v^l) - F((n+1)h-0, u_0^l, v_0^l) \]
\[ \leq 2h \sum_{m: \text{odd}} \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\} \leq 2Kh \]  

(3.19)

From (3.14), (3.15) and (3.19), we get

\[ F((n+1)h+0, u^l, v^l) \leq F(nh+0, u^l, v^l) + 2Kh \]  

(3.20)

Thus we obtain the following lemma.

**Lemma 3.8**

\[ F(nh+0, u^l, v^l) \leq F(+0, u^l, v^l) + 2KT \equiv F_0 + 2KT \]  

(3.21)

Denote by \( G(\tau) \) the absolute value of the sum of negative variation of \( r^l \) and \( s^l \) for \( t = \tau \). Then for \( nh \leq \tau < (n+1)h \), we get

\[ G(\tau) \leq G(nh) + 2h \sum_{m: \text{odd}} \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\} \]
\[ \leq G(nh) + 2Kh. \]  

(3.22)

**Lemma 3.9**

\[ G(nh) \leq 2F(nh+0, u^l, v^l). \]  

(3.23)

*Proof.* Denote by \( \delta s \) (resp \( \delta r \)) the absolute value of the Riemann invariant \( s \) (resp \( r \)) in the first (resp second) shock wave. By (3.2) and (3.3), \( \Delta r + \delta s < 2\Delta r \) on the first shock and \( \delta r + \Delta s < 2\Delta s \) on the second shock. So from (3.17), (3.18) and above arguments, we get (3.23). \( \square \)

From (3.23), (3.24) and (3.25), for any \( \tau \) (\( nh \leq \tau < (n+1)h \)),

\[ G(\tau) \leq G(nh) + 2Kh \leq 2F(nh+0, u^l, v^l) + 2Kh \]
\[ \leq 2F_0 + 6KT \equiv M_1. \]  

(3.24)

Now we can establish a priori estimates of \( u^l \) and \( v^l \). Denote by T.V.\( u \) the total variation of \( u \).
Theorem 3.10 For any $T > 0$, the variation of $u^l$ and $v^l$ is bounded uniformly for $h$ and $\{a_{mn}\}$. Their upper bound and lower bound, especially the positive lower bound of $v^l$, are also uniformly bounded.

Proof. Denote by $T.V^+u$ (resp $T.V^-u$) the absolute value of the positive (resp negative) variation of $u$. Put $f^l \equiv 2u^l = r^l + s^l$. Then $0 \leq f^l(t, 0) \leq Kh$. Without loss of generality, we assume that $u_0(x)$ and $v_0(x)$ are constant outside a bounded interval. Let

(3.25) \[ f^l(t, \infty) = r^l(t, \infty) + s^l(t, \infty) \equiv M_2. \]

Then from the definition,

\[ f^l(t, 0) + T.V^+ f^l - T.V^- f^l = f^l(t, \infty). \]

Since $T.V^- f^l(t, \cdot) \leq G(t)$ for any $t$, (3.24) yields

\[ T.V^+ f^l = f^l(t, \infty) + T.V^- f^l - f^l(t, 0) \leq M_1 + M_2. \]

Thus we get

(3.26) \[ T.V f^l = T.V 2u^l \leq 2M_1 + M_2. \]

From (3.26), we get

\[ |f^l| \leq Kh + 2M_1 + M_2 \leq KT + 2M_1 + M_2 \equiv 2M_3. \]

Therefore we get

(3.27) \[ |u_l| \leq M_3. \]

Using Lemma 3.2, we get

\[ 2a \log v^l = r^l - s^l \geq r_0 - (\max(-r_0, s_0) + KT). \]

Thus we get

(3.28) \[ v^l \geq \exp \frac{r_0 - (\max(-r_0, s_0) + KT)}{2a} \equiv \frac{1}{M_5}. \]

From the definition,

\[ r^l(t, 0) + T.V^+ r^l - T.V^- r^l = r^l(t, \infty). \]
Using Lemma 3.3 and (3.24),

\[
T.V^+, r^l = -r^l(0) + T.V^-, r^l + r(t, \infty) \leq -r_0 + M_1 + r(t, \infty).
\]

In view of (3.27) and (3.29), there exists a positive constant \(M_6\) such that

\[
v^l \leq M_6
\]

Theorem 3.11 For any interval \([x_1, x_2] \subset [0, \infty)\), we get

\[
\int_{x_1}^{x_2} |u'(t_2, x) - u'(t_1, x)| + |v'(t_2, x) - v'(t_1, x)| \, dx \\
\leq M \cdot (|t_2 - t_1| + h), \quad 0 \leq t_1, t_2 < T,
\]

where \(M\) depends on \(T, x_1,\) and \(x_2\), but not on \(l\) and \(h\).

Proof. Without loss of generality, we assume that

\[nh \leq t_1 < (n+1)h < \cdots < (n+k)h \leq t_2 < (n+k+1)h\]

Let

\[
\int_{x_1}^{x_2} |u'(t_2, x) - u'(t_1, x)| \, dx \\
\leq I_1 + I_2 + \int_{x_1}^{x_2} |u'(t_2, x) - u'((n+k)h+0, x)| + |u'(t_1, x) - u'((n+1)h-0, x)| \, dx
\]

where

\[I_1 = \int_{x_1}^{x_2} \sum_{i=1}^{k} |u'((n+i)h+0, x) - u'((n+i)h-0, x)| \, dx\]

\[I_2 = \int_{x_1}^{x_2} \sum_{i=1}^{k-1} |u'((n+i+1)h-0, x) - u'((n+i)h+0, x)| \, dx\]

and

\[k = \left\lfloor \frac{t_2 - t_1}{h} \right\rfloor\]
Denote by $1_{[\alpha,\beta]}$ the characteristic function of the interval $[\alpha, \beta]$. We regard $T.V.-l<x<l = T.V.0<x<l$. Then,

\begin{align*}
I_1 & \leq \sum_{i=0}^{k+1} \sum_{m: \text{integer}} \int_{x_1}^{x_2} T.V.2ml<x<(2m+2)l u^l((n+i)h-0, x) \cdot 1_{[2ml,(2m+2)l]} \ dx, \\
& \leq \left( \left[ \frac{t_2 - t_1}{h} \right] + 2 \right) \cdot \left( \sup_{0 \leq t \leq T} T.V.u^l(t, \cdot) \right) \cdot 2l.
\end{align*}

\begin{align*}
I_2 & \leq \sum_{i=0}^{k} \sum_{m} \int_{x_1}^{x_2} (T.V.(2m-1)l<x<(2m+1)l) u^l((n+i+1)h-0, x) \cdot 1_{[(2m-1)l,(2m+1)l]} + K h) \ dx, \\
& \leq \sum_{i=0}^{2l} 2l \cdot T.V.u^l((n+i+1)h-0, \cdot) + K(x_2 - x_1) h, \\
& \leq \left( \left[ \frac{t_2 - t_1}{h} \right] + 1 \right) \cdot \left( 2l \sup_{0 \leq t \leq T} T.V.u^l(t, \cdot) + K(x_2 - x_1) h \right).
\end{align*}

The remaining terms can be evaluated similarly. For

$$\int_{x_1}^{x_2} |v^l(t_2, x) - v^l(t_1, x)| \ dx,$$

we also have a similar estimate. Combining these results gives (3.31). \hfill \square

4 Convergence of The Approximate Solution

Let $h_n = T/n$ and $h_n/l_n = \tilde{\delta} < \delta \equiv 1/M_5$. Consider the sequence $(u^{n}, v^{n})$ $(n = 1, 2, \cdots)$. Then from Theorem 3.9 and Theorem 3.10, there exists a subsequence which converges in $L^1_{loc}$ to functions $(u,v)$ uniformly for $t \in [0, T]$. Now we shall prove that $u(x,t)$ and $v(x,t)$ are the weak solutions of initial boundary value problem (1.6), (1.7) and (1.8) provided $\{a_{nm}\}$ is suitably chosen, namely, they satisfy the integral identity

\begin{align*}
& \int_0^T \int_0^\infty u \phi_t + \frac{a^2}{v} \phi_x + \frac{K}{1 + \int_0^\infty v(t, \zeta) d\zeta} \cdot \phi \ dx dt \\
& + \int_0^\infty u_0(x) \phi(0, x) dx = 0, \\
& \int_0^T \int_0^\infty v \psi_t - u \psi_x \ dx dt + \int_0^\infty v_0(x) \psi(0, x) dx = 0,
\end{align*}

(4.1) (4.2)
for any smooth functions $\phi$ and $\psi$ with compact support in the region 
\{$(t, x) : 0 \leq t < T, 0 \leq x < \infty$\} and $\phi(t, 0) = 0$. Now we know that $u_0^l$ and $v_0^l$ are weak solutions in each time strip $nh \leq t < (n+1)h$ so that for each test function $\phi$ satisfying $\phi(t, 0) = 0$,

\[
\int_{nh}^{(n+1)h} \int_0^\infty u^l \phi_t + \left(\frac{a^2}{v^l}\right) \phi_x + U^l(t, x) \cdot \phi \, dx \, dt 
+ \int_0^\infty u^l(nh + 0, x) \phi(nh, x) 
- \int_0^\infty u^l((n+1)h - 0, x) \phi((n+1)h, x) \, dx = 0
\quad (4.3)
\]

If we sum this over $n$, we get

\[
\int_0^T \int_0^\infty u^l \phi_t + \left(\frac{a^2}{v^l}\right) \phi_x + U^l(t, x) \cdot \phi \, dx \, dt 
+ \int_0^\infty u^l(0, x) \phi(0, x) 
= - \sum_{k=1}^{N} \int_0^\infty \{u^l(kh + 0, x) - u^l(kh - 0, x)\} \cdot \phi(kh, x) \, dx
\quad (4.4)
\]

where $N = T/h$. When $N \to \infty$, the right-hand side of the above equality tends to 0 for almost every $\{a_{nm}\} \in A$ (see [4]). It is immediate to see that

\[
\int_0^\infty u^l(0, x) \phi(0, x) \, dx \to \int_0^\infty u_0(x) \phi(0, x) \, dx \quad (N \to \infty).
\]

Lemma 4.1

\[
U^l(t, x) \to \frac{K}{1 + \int_0^\infty v(t, \zeta) \, d\zeta} \quad (N \to \infty).
\]

locally uniformly for $t$ and $x$.

Proof. Let $nh \leq t < (n+1)h$, $x \in ((m-1)l, (m+1)l)$, $m : odd$. Then

\[
\left| \int_0^x v^l(nh, \zeta) \, d\zeta - \sum_{j=1}^{m+1} v^l(nh, c_{2j-1}n) \right| \leq \|v^l\|_\infty \cdot l
\quad (4.6)
\]

On the other hand

\[
\int_0^x v^l(t, \zeta) \, d\zeta \to \int_0^x v(t, \zeta) \, d\zeta \quad (N \to \infty).
\quad (4.7)
\]
locally uniformly for $t$ and $x$.

We get

$$\left| \int_0^x v'(t, \zeta) d\zeta - \int_0^x v'(nh, \zeta) d\zeta \right| \leq \int_0^x T.V_{(m-1)l < \xi < (m+1)l} v'(nh, \cdot) \cdot 1_{([m-1]l, (m+1)l]} d\zeta$$

From (4.6), (4.7) and (4.8), we get (4.5).

For each test function $\psi$, $v^l$ also satisfies,

$$\int_0^T \int_0^{\infty} \left( v^l \psi_t - u^l \psi_x \right) dx dt + \int_0^{\infty} v^l(0, x) \psi(O, x) dx$$

$$= - \sum_{k=1}^N \int_0^{\infty} \left\{ v^l(kh + 0, x) - v^l(kh - 0, x) \right\} \cdot \psi(kl, x) dx$$

$$- I_1 - I_2.$$

where

$$I_1 = \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} U^l(t, 0)(t - nh) \psi(t, 0) dt$$

and

$$I_2 = \sum_{n=0}^{N-1} \sum_{m:odd} \int_{nh}^{(n+1)h} \left\{ U^l(t, ml + 0) - U^l(t, ml - 0) \right\} (t - nh) \psi(t, ml) dt.$$
\[
\sum_{m: \text{odd}} \int_{nh}^{(n+1)h} \{ U^l(t, ml + 0) - U^l(t, ml - 0) \} (t-nh) \psi(t, ml) dt \leq K \| \psi \|_\infty h^2.
\]

Thus we get
\[
I_2 \leq \| \psi \|_\infty \sum_{n=0}^{N-1} K h^2 \leq K \| \psi \|_\infty hT.
\]

From above arguments, we can conclude that \(u\) and \(v\) satisfy (4.1) and (4.2). Thus we obtain our main result.

**Theorem 4.2 (Main Result)** Suppose that \(u_0(x), v_0(x) \in BV(\mathbb{R}_+)\), and that \(v_0(x) \geq \delta_0 > 0\) for all \(x > 0\) with some positive constant \(\delta_0\). Then (1.10), (1.11) and (1.12) have a global weak solution which belongs to the class

\[
\begin{align*}
&u, v \in L^\infty(0, T; BV(\mathbb{R}_+)) \cap Lip([0, T]; L^\infty_{loc}(\mathbb{R}_+))
\end{align*}
\]

for any \(T > 0\).
Appendix

A Proof of Lemma 3.5

Let $g(x) = -f(-x)$, and put

$$P(u_1, v_1, u_2, v_2) = \Delta r_1 + \Delta s_1$$
$$P(u_2, v_2, u_3, v_3) = \Delta r_2 + \Delta s_2$$
$$P(u_1, v_1, u_3, v_3) = \Delta r_3 + \Delta s_3$$

Then it is obvious that

$$\Delta r_3 + g(\Delta s_3) + \Delta s_3 + g(\Delta r_3)$$
$$\leq \Delta r_1 + \Delta r_2 + \Delta s_1 + \Delta s_2 + g(\Delta r_1) + g(\Delta r_2) + g(\Delta s_1) + g(\Delta s_2)$$

We notice that $f'' \leq 0$ and hence

$$\leq \Delta r_1 + \Delta r_2 + \Delta s_1 + \Delta s_2 + g(\Delta r_1 + \Delta r_2) + g(\Delta s_1 + \Delta s_2).$$

Let $x + g(x) = h(x)$, $\Delta r_3 = p'$, $\Delta s_3 = q'$, $\Delta r_1 + \Delta r_2 = p$ and $\Delta s_1 + \Delta s_2 = q$.

Then

(A.1) \hspace{1cm} h(p') + h(q') \leq h(p) + h(q).

Put $K = h(p') + h(q')$. We shall estimate $p + q$ from below under the restriction (A.1). To do this, as $h$ is monotone increasing function, we must estimate $p + q$ from below under the restriction

(A.2) \hspace{1cm} h(p) + h(q) = K.

We do this by using Lagrange's method of indeterminate coefficients.

Put $G(p, q, \lambda) = p + q + \lambda (h(p) + h(q) - K)$. Then

$$G_p = 1 + \lambda h'(p) = 0, \quad G_q = 1 + \lambda h'(q) = 0.$$

Because $h''(x) > 0$, we get $p = q$. So $p + q$ attains its extremum at $p = q$.

We can show that when $p = q$, $p + q$ is minimum under the restriction (A2). Therefore

$$h(p) = h(q) = \frac{K}{2} = \frac{h(p') + h(q')}{2} \geq h\left(\frac{p' + q'}{2}\right).$$
Hence it follows that
\[ p = q \geq \frac{p' + q'}{2}. \]
Thus we get
(A.3) \[ p + q \geq p' + q'. \]
which proves Lemma 3.5.

B Proof of Lemma 3.6

To prove Lemma 3.6, we must check the following 12 cases:

1) \( c_{1n} < l \),
   (1) \( S_2 \) crosses \( \iota_{0}^{n-} \),
   (2) \( R_2 \) crosses \( \iota_{0}^{n-} \),
   (3) no wave cross \( \iota_{0}^{n-} \).

2) \( c_{1n} \geq l \),
   (1) \( S_2 \) and \( S_1 \) cross \( \iota_{0}^{n-} \),
   (2) \( R_2 \) and \( S_1 \) cross \( \iota_{0}^{n-} \),
   (3) \( S_2 \) and \( R_1 \) cross \( \iota_{0}^{n-} \),
   (4) \( R_2 \) and \( R_1 \) cross \( \iota_{0}^{n-} \),
   (5) \( S_1 \) crosses \( \iota_{0}^{n-} \),
   (6) \( R_1 \) crosses \( \iota_{0}^{n-} \),
   (7) \( S_2 \) crosses \( \iota_{0}^{n-} \),
   (8) \( R_2 \) crosses \( \iota_{0}^{n-} \),
   (9) no wave cross \( \iota_{0}^{n-} \).
Put $r_{-}^{n-1} = r^{l}(a_{1n-1})$, $s_{+}^{n-1} = s^{l}(a_{1n-1})$, $r_{+}^{n-1} = -s_{-}^{n-1}$

$$= r^{l}((n-1)h + 0,0), \text{ and } \delta_{n-1} = U^{l}(a_{1n-1}).$$

Put $r_{+}^{n-1'} = r^{l}((n-1)h + 0,2l)$ and $s_{+}^{n-1'} = s^{l}((n-1)h + 0,2l)$.

Put $A = (r_{-}^{n-1}, s_{-}^{n-1})$, $B = (r_{+}^{n-1}, s_{+}^{n-1})$ and $B' = (r_{+}^{n-1'}, s_{+}^{n-1'})$.

Put $C = (r_{+}^{n-1} + Kh, s_{+}^{n-1} + Kh)$, 
( resp $= (r_{+}^{n-1'} + \delta_{n-1}h, s_{+}^{n-1'} + \delta_{n-1}h,)$ ) if $c_{1n} < l$ (resp $c_{1n} \geq l$).

If $R_2$ crosses $i_{0}^{n+}$, $F(i_{0}^{n+}) = 0 \leq F(i_{0}^{n-})$, so that it is sufficient to consider the cases when $S_2$ crosses $i_{0}^{n+}$.

![Figure 2](image-url)

1) $c_{1n} < l$.

(1) $S_2$ crosses $i_{0}^{n-}$ (Figure 2). Denote by I (resp II) the halfspace

$$\{(r,s)|r+s<0\} \text{ (resp } \{(r,s)|r+s \geq 0\}).$$

i) $C \in I$.

In this case $S_2$ crosses $i_{0}^{n+}$. Denote by $V(PQ)$ the absolute value of the total variation of $r$ and $s$ by the line segment PQ. From Figure.3,

$$F(i_{0}^{n+}) = V(A'C) \leq V(A'C') = V(AB) = F(i_{0}^{n-}).$$
ii) $C \in II$.

In this case $R_2$ crosses $i_0^{n+}$. Then

\[(B.1) \quad F(i_0^{n-}) \geq F(i_0^{n+}) = 0.\]

(2) $R_2$ crosses $i_0^{-}$.

In this case $B \in II$ so that $R_2$ crosses $i_0^{n+}$. Then

\[(B.2) \quad F(i_0^{n-}) = F(i_0^{n+}) = 0.\]

(3) no wave crosses $i_0^{-}$.

In this case $(r_+^{n-1}, s_+^{n-1})$ is on the line $r + s = 0$. Hence $C \in II$. It is obvious that (B.3) also holds.
2) \( c_{1n} \geq l \).

(1) \( S_2 \) and \( S_1 \) cross \( i_0^{n-} \). (Figure.4)

Figure.4
i) $C \in I$.
From Figure 5,

$$F(i_0^{n+}) = V(A'C) \leq V(A'C') = V(A''B') = V(AB') = F(i_0^{n-}).$$

ii) $C \in II$ implies that
$R_2$ crosses $i_0^{n+}$. So we get (B2).
(2) $R_2$ and $S_1$ cross $i_0^{n-}$.

Figure 6

i) $C \in I$.

From Figure 6,

\[
F(i_0^{n+}) = V(A'C) \leq V(A'D) = V(A''E) = V(A''B'') \\
\leq V(BB'') = V(BB') = F(i_0^{n+})
\]

ii) $C \in II$.

Thus $R_2$ crosses $i_0^{n+}$, and we get (B2).
(3) \(S_2\) and \(R_1\) cross \(i_0^{n^-}\).

![Diagram](attachment:image_url)

**Figure.7**

Put \(G = (r_+^{n^-} + \delta_{n-1}h, s_+^{n^-} + \delta_{n-1}h)\) and \(II = (r^l(a_{1n}), s^l(a_{1n}))\).

Then \(H\) is on the line \(CG\).

i) \(H \in I\).

From Figure.7,

\[
F(i_0^{n^+}) = V(A'H) \leq V(A''G) \leq V(AB) = F(i_0^{n^-}).
\]

ii) \(H \in II\), so

\(R_2\) crosses \(i_0^{n^+}\), and we get (B2).

(4) \(R_2\) and \(R_1\) cross \(i_0^{n^-}\).

In this case, \(R_2\) crosses \(i_0^{n^+}\). So we get (B3).
(5) $S_1$ crosses $i_0^{-}$.

From Figure.8, $F(i_{0}^{n-}) = V(A'C) = V(AE) = V(AD) \leq V(AB') = F(i_{0}^{n-})$

Thus we get (B1).

ii) $C \in II$.

$R_2$ crosses $i_0^{n+}$. So we get (B2).
(6) $R_1$ crosses $i_0^{-}$.

In this case, it is obvious that $F(i_0^{n+}) = 0$. Hence we get (B3).

Cases (7), (8) and (9) are almost the same as cases (1), (2) and (3) in 1. Thus, we obtain Lemma 3.6.

References


