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Kyoto University
The Global Weak Solutions of the Compressible Euler Equation with Spherical Symmetry

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1 Introduction

The compressible Euler equation for an isentropic gas in $\mathbb{R}^n$ is given by

\begin{align}
\rho_t + \nabla \cdot (\rho \vec{u}) &= 0, \\
(\rho \vec{u})_t + \nabla \cdot (\rho \vec{u} \otimes \vec{u} + p) &= 0,
\end{align}

with the equation of state

\begin{equation}
p = a^2 \rho^\gamma,
\end{equation}

where density $\rho$, velocity $\vec{u}$ and pressure $p$ are functions of $x \in \mathbb{R}^n$ and $t \geq 0$, while $a > 0$ and $\gamma \geq 1$ are given constants.

For one dimensional case ($n=1$), the Cauchy problem for (1.1) with (1.2) has been studied by many authors. Nishida [10] established the existence of global weak solutions, for the first time, for the case $\gamma = 1$ with arbitrary initial data, and Nishida and Smoller [11] for $\gamma \geq 1$ but with small initial data, both using Glimm's method. DiPerna [3] extended the latter result to the case of large initial data, using the theory of compensated compactness under the restriction $\gamma = 1 + 2/(2m + 1)$, $m \geq 2$ integers. Ding et al [1], [2] removed this restriction and established the existence of global weak solutions for $1 < \gamma \leq 5/3$. 
On the other hand, little is known for the case $n \geq 2$. No global solutions have been known to exist, but only local classical solutions ([5], [6], [8] and [9]).

In this paper, we will present global weak solutions first for the case $n \geq 2$. We will do this, however, only for the case of spherically symmetry with $\gamma = 1$. As will be seen below, our proof does not work without these restrictions.

Thus, we look for solutions of the form

\begin{equation}
\rho = \rho(t, |x|), \quad \bar{u} = \frac{x}{|x|} \cdot u(t, |x|).
\end{equation}

Then, denoting $r = |x|$, (1.1) becomes

\begin{equation}
\rho_t + \frac{1}{r^{n-1}} \left( r^{n-1} \rho u \right)_r = 0,
\rho \left( u_t + u u_r \right) + p_r = 0,
\end{equation}

This equation has a singularity at $r=0$. To avoid the difficulty caused by this singularity, we simply deal with the boundary value problem for (1.4) in the domain $1 \leq r < \infty$ (the exterior of a sphere) with the boundary condition $u(t, 1) = 0$, which is identical, under the assumption (1.3), to the usual boundary condition $\bar{n} \cdot \bar{u} = 0$ for (1.1) where $\bar{n}$ is the unit normal to the boundary.

Put $\tilde{\rho} = r^{n-1} \rho$. Then we get from (1.4)

\begin{equation}
\tilde{\rho}_t + (\tilde{\rho} u)_r = 0,
\end{equation}

\begin{equation}
u_t + u u_r + \frac{a^2 \gamma \tilde{\rho}_r}{\tilde{\rho}^{1-\gamma/2}} = \frac{a^2 \gamma (n-1) \tilde{\rho}^{\gamma-1}}{r^n \cdot r^{(n-1)(\gamma-2)}}.
\end{equation}

Introduce the Lagrangean mass coordinates

\begin{equation}
\tau = t, \quad \xi = \int_1^r \tilde{\rho}(t, r) \, dr.
\end{equation}

Then $\xi > 0$ as long as $\tilde{\rho} > 0$ for $r > 1$, and (1.5) is reformulated as

\begin{equation}
\tilde{\rho}_r + \tilde{\rho}^2 u_\xi = 0,
\end{equation}

\begin{equation}u_r + \frac{a^2 \gamma \tilde{\rho}_\xi}{\tilde{\rho}^{1-\gamma/2}} = \frac{a^2 \gamma (n-1) \tilde{\rho}^{\gamma-1}}{r^n \cdot r^{(n-1)(\gamma-2)}}.
\end{equation}
Put $v = 1/\tilde{\rho}$ and note that the inverse transformation to (1.6) is given by

$$t = \tau, \ r = 1 + \int_{0}^{\tau} v(\zeta, t) d\zeta. \ (1.8)$$

Then after changing $\tau$ to $t$ and $\xi$ to $x$, (1.7) is written as

$$v_t - u_x = 0, \ (1.9)$$

$$u_t + \left( \frac{a^2}{v^\gamma} \right)_x = \frac{a^2 \gamma(n-1)v^{1-\gamma}}{r^{n-1}(\gamma-2)},$$

where $r$ is now defined by $r = 1 + \int_{0}^{x} v(t, \zeta) d\zeta$.

Now we restrict ourselves to the case $\gamma = 1$. Then (1.7) becomes

$$v_t - u_x = 0, \ (1.10)$$

$$u_t + \left( \frac{a^2}{v} \right)_x = \frac{K}{1 + \int_{0}^{x} v(t, \zeta) d\zeta},$$

where $K = a^2(n-1)$.

Let us consider the initial boundary value problem for (1.10) in $t \geq 0, \ x \geq 0$ with the following boundary and initial conditions.

$$u(0, x) = u_0(x), \ v(0, x) = v_0(x), \ for \ x > 0, \ (1.11)$$

$$u(t, 0) = 0, \ for \ t > 0. \ (1.12)$$

Let $BV(R_+)$ denote the space of functions of bounded variation on $R_+ = (0, \infty)$. Our main result is as follows.

**Theorem (Main Result)** Suppose that $u_0(x), \ v_0(x) \in BV(R_+)$, and that $v_0(x) \geq \delta_0 > 0$ for all $x > 0$ with some positive constant $\delta_0$. Then (1.10), (1.11) and (1.12) have a global weak solution which belongs to the class

$$u, \ v \in L^\infty(0, T; BV(R_+)) \cap Lip([0, T]; L_{loc}^1(R_+))$$

for any $T > 0$.

The definition of the weak solution will be given in section 4. This theorem can be proved by following Nishida’s argument [10] based on Glimm’s
method. Indeed this can be seen from the following two simple observations. First, the homogeneous equation corresponding to (1.10),

\[
\begin{align*}
  v_t - u_x &= 0, \\
  u_t + \left(\frac{a^2}{v}\right)_x &= 0,
\end{align*}
\]

is just the same equation as solved by Nishida [10] using Glimm’s method both on the Cauchy problem and the initial boundary value problem. Note that if $\gamma > 1$, the homogeneous equation for (1.9) has a variable coefficient and hence does not coincide with the one dimensional Euler equation.

The second observation is that, as long as $v \geq 0$, the right hand side of (1.10),

\[
\frac{K}{1 + \int_0^x v(t,\zeta)d\zeta},
\]

is monotone decreasing in $x$ and has an a priori estimate

\[
T. V. \left(\frac{K}{1 + \int_0^x v(t,\zeta)d\zeta}\right) \leq K,
\]

independent of $v$. The one dimensional inhomogeneous Euler equation has been studied in [12]. However, the conditions imposed therein on the inhomogeneous term are not applicable to our (1.14).

These observations allow us to use Nishida’s argument [10] to construct global weak solutions to (1.10), (1.11) and (1.12). More precisely, we will first construct, in section 2, approximate solutions of the form

\[
\{\text{solution of Riemann problem for (1.13)}\} + \{\text{nonhomogeneous term}\} \times t.
\]

This is the main idea of [12]. Then in section 3, we will estimate the total variation of the approximate solutions. Thanks to (1.15), this can be done with a slight modification of Nishida’s argument [10]. In section 4, we will show that there exists a subsequence of approximate solutions which converges strongly in $L^1_{loc}$ for any finite time interval. Finally, for the sake of completeness, we give in Appendix a detailed proof of two lemmas used in section 3. These lemmas are due to Nishida [10], but their proofs are not found in the literature.
2 The Difference Scheme

To construct the approximate solutions, we shall use the difference scheme developed in [10]. For $l, h > 0$, define

$$Y = \{(n, m); \ n = 1, 2, 3, \ldots, m = 1, 3, 5, \ldots\},$$

(2.1) $$A = \prod_{(m,n)\in Y} \{nh\} \times ((m-1)l, (m+1)l),$$

where $l/h$ will be determined later. Choose a point $\{a_{nm}\} \in A$ randomly, and write $a_{nm} = (nh, c_{nm})$. For $n = 0$, we put $c_{0m} = ml$. We denote approximate solutions by $u^l$ and $v^l$. Mesh lengths $l$ and $h$ are chosen so that $l/h > a/(\inf v^l)$, for any given $T > 0$. We shall show later that there exists a $\delta > 0$ such that $\inf v^l \geq \delta > 0$.

For $0 \leq t < h$, $ml \leq x < (m+2)l$, $m$ : odd, we define

(2.2) $$u^l(t,x) = u^l_0(t, x) + U^l(t,x) t,$$

$$v^l(t,x) = v^l_0(t, x),$$

where $u^l_0$ and $v^l_0$ are the solutions of

(2.3) $$v_t - u_x = 0,$$

$$u_t + \left(\frac{a^2}{v}\right)_x = 0,$$

with initial data

(2.4) $$u^l_0(0,x) = \begin{cases} u_0(ml), & x < (m + 1)l, \\ u_0((m+2)l), & x > (m + 1)l, \end{cases}$$

$$v^l_0(0,x) = \begin{cases} v_0(ml), & x < (m + 1)l, \\ v_0((m+2)l), & x > (m + 1)l, \end{cases}$$

and

(2.5) $$U^l(t,x) = \frac{K}{1 + \sum_{j=1}^{m+1} v_0((2j-1)l) \cdot 2l}.$$ 

For $0 \leq t < h$, $0 \leq x < l$, we define $u^l$ and $v^l$ by (2.2) where $u^l_0$ and $v^l_0$ are the solutions of (2.3) with initial boundary data

(2.6) $$u^l_0(0,x) = u_0(l), \ v^l_0(0,x) = v_0(l), \ x > 0,$$
\[(2.7) \quad u(t, 0) = 0, \quad t > 0, \]

and

\[(2.8) \quad U^l(t, x) = K. \]

Suppose that \(u^l\) and \(v^l\) are defined for \(0 \leq t < nh\). For \(nh \leq t < (n+1)h\), \(ml \leq x < (m+2)l\), \(m : \text{odd}\), we define

\[(2.9) \quad u^l(t, x) = u^l_0(t, x) + U^l(t, x) \cdot (t - nh), \]
\[v^l(t, x) = v^l_0(t, x), \]

where \(u^l_0\) and \(v^l_0\) are the solutions of (2.3) with initial data \((t=nh)\)

\[(2.10) \quad u^l_0(nh, x) = \begin{cases} u^l(nh - 0, c_{nm}), & x < (m + 1)l, \\ u^l(nh - 0, c_{nm+2}), & x > (m + 1)l, \end{cases} \]
\[v^l_0(nh, x) = \begin{cases} v^l(nh - 0, c_{nm}), & x < (m + 1)l, \\ v^l(nh - 0, c_{nm+2}), & x > (m + 1)l, \end{cases} \]

and

\[(2.11) \quad U^l(t, x) = \frac{K}{1 + \sum_{j=1}^{m+1} v^l(nh - 0, c_{2j-1}) \cdot 2l}. \]

For \(nh \leq t < (n+1)h\), \(0 \leq x < l\), we define \(u^l\) and \(v^l\) as (2.9) where \(u^l_0\) and \(v^l_0\) are the solutions of (2.3) with initial \((t=nh)\) boundary data

\[(2.12) \quad u^l_0(nh, x) = u^l(nh - 0, c_{n1}), \quad v^l_0(nh, x) = v^l(nh - 0, c_{n1}), \quad x > 0, \]
\[(2.13) \quad u(t, 0) = 0, \quad t > nh, \]

and \(U^l(t, x)\) is as (2.8).

3 \quad \textbf{Bounds for Approximate Solutions}

System (1.6) is hyperbolic provided \(v > 0\), with the characteristic roots and Riemann invariants given by

\[(3.1) \quad \lambda = \frac{a}{v}, \quad r = u + a \log v, \]
\[\mu = \frac{a}{v}, \quad s = u - a \log v. \]
It is well-known, [10], that all shock wave curves in the \((r,s)\)-plane have the same figure. (See Figure 1.) The 1-shock wave curve \(S_1\), starting from \((r_0, s_0)\) can be expressed in the form

\[
(3.2) \quad s - s_0 = f(r - r_0) \quad \text{for} \quad r \leq r_0,
\]

and the 2-shock wave curve \(S_2\) can also be expressed in the form

\[
(3.3) \quad r - r_0 = f(s - s_0) \quad \text{for} \quad s \leq s_0,
\]

where

\[
0 \leq f'(x) < 1, \quad f''(x) \leq 0, \quad \lim_{x \to -\infty} f'(x) = 1.
\]
The 1-rarefaction wave curve $R_1$ can be expressed in the form

\begin{equation}
\label{eq:3.4}
s - s_0 = 0 \quad \text{for } r \geq r_0,
\end{equation}

and the corresponding expression for the 2-rarefaction wave curve $R_2$ is

\begin{equation}
\label{eq:3.5}
r - r_0 = 0 \quad \text{for } s \geq s_0.
\end{equation}

Now we must prepare some lemmas to estimate Riemann invariants. First, let us consider (2.3) with following initial data

\begin{equation}
\label{eq:3.6}
\begin{aligned}
u_0(x) &= \begin{cases} u_l, & x \leq 0 \\ u_r, & x > 0 \end{cases} \\
v_0(x) &= \begin{cases} v_l, & x < 0 \\ v_r, & x > 0 \end{cases}
\end{aligned}
\end{equation}

**Lemma 3.1** Let $u$ and $v$ are the solutions of (2.3) and (3.6). Then,

\begin{equation}
\label{eq:3.7}
\begin{cases}
r(t, x) \equiv r(u(t, x), v(t, x)) \geq r_0 \equiv \min(r(u_r, v_r), r(u_l, v_l)), \\
s(t, x) \equiv s(u(t, x), v(t, x)) \leq s_0 \equiv \max(s(u_r, v_r), s(u_l, v_l)).
\end{cases}
\end{equation}

Next consider (2.3) in $t \geq 0$, $x \geq 0$ with following initial and boundary conditions

\begin{equation}
\label{eq:3.8}
u(0, x) = u_0^+, \quad v(0, x) = v_0^+, \quad \text{for } x > 0,
\end{equation}

\begin{equation}
\label{eq:3.9}
u(t, 0) = 0, \quad \text{for } t > 0.
\end{equation}

**Lemma 3.2** Let $u$ and $v$ are the solutions of (2.3), (3.8) and (3.9). Then,

\begin{equation}
\label{eq:3.10}
\begin{cases}
r(t, x) \equiv r(u(t, x), s(t, x)) \geq r(u_0^+, v_0^+), \\
s(t, x) \equiv s(u(t, x), s(t, x)) \leq \max(-r(u_0^+, v_0^+), s(u_0^+, v_0^+)).
\end{cases}
\end{equation}

The above two lemmas were proved in [10]. Using these two lemmas, we can get the following lemma.

**Lemma 3.3** Let $u^l$ and $v^l$ be the approximate solutions defined in section 2 and put $r_0 = \min r(u_0(x), v_0(x))$ and $s_0 = \max s(u_0(x), v_0(x))$. Then, for $0 < t < T$,

\begin{equation}
\label{eq:3.11}
\begin{cases}
r^l(t, x) \equiv r(u^l(t, x), s^l(t, x)) \geq r_0, \\
s^l(t, x) \equiv s(u^l(t, x), s^l(t, x)) \leq \max(-r_0, s_0) + KT
\end{cases}
\end{equation}
Let us consider Riemann problem (2.3) and (3.6). Denote by $\Delta r$ (resp $\Delta s$) the absolute value of the variation of the Riemann invariant $r$ (resp $s$) in the first (resp second) shock wave.

**Definition 3.4** We denote

$$P(u_i, v_i, u_r, v_r) = \Delta r + \Delta s.$$ 

Then we have the following lemma.

**Lemma 3.5**

(3.12) \[ P(u_1, v_1, u_3, v_3) \leq P(u_1, v_1, u_2, v_2) + P(u_2, v_2, u_3, v_3), \]

where $u_1$, $u_2$ and $u_3$ are arbitrary constants and $v_1$, $v_2$ and $v_3$ are arbitrary positive costants.

We shall prove Lemma 3.5 in the Appendix A.

Denote by $i_0^{n\pm}$ the straight line segments joining the points $(0, (n \pm \frac{1}{2})h)$ and $a_{1n}$. Let $F(i_0^{n\pm})$ be the absolute value of the variation of the Riemann invariants for all shocks on $i_0^{n\pm}$. Then we also have the following Lemma.

**Lemma 3.6**

(3.13) \[ F(i_0^{n+}) \leq F(i_0^{n-}). \]

This lemma 3.6 will be proved in the Appendix B.

We denote

\[ Z_1 = \{ l - 0, l + 0, 3l - 0, \ldots, (2m - 1)l - 0, (2m - 1)l + 0, \ldots \}, \]
\[ Z_2 = \{ 2l, 4l, 6l \ldots 2ml, \ldots \}. \]

Let $Z_{(n)} = Z_1 \cup Z_2 \cup \{ c_{nm} \}$ and line up the elements $z_{n,i}$ of $Z_{(n)}$ so that $z_{n,i} \leq z_{n,i+1}$. (We regard $(2m - 1)l - 0 < (2m - 1)l + 0$ for m : integer.

Let

\[ F(nh - 0, u', v') = \frac{1}{2} F(i_0^{n-}) \]

\[ + \sum_{z_{n,i} \in Z_{(n)}} P(u'(nh - 0, z_{n,i}), v'(nh - 0, z_{n,i}), u'(nh - 0, z_{n,i+1}), v'(nh - 0, z_{n,i+1})), \]
\[ F(nh+0, u^l, v^l) = \frac{1}{2} F(i_0^{n+}) + \sum_{m: \text{odd}} P(u^l(a_{nm}), v^l(a_{nm}), u^l(a_{nm+2}), v^l(a_{m+2})). \]

Using Lemma 3.5 and Lemma 3.6, we get

\[ F((n+1)h+0, u^l, v^l) \leq F((n+1)h-0, u^l, v^l). \]

The following equality is obvious from the definition of \( F, u^l \) and \( v^l \).

\[ F((n+1)h-0, u^l, v^l) = F(nh+0, u^l, v^l). \]

We also get

\[ F((n+1)h-0, u^l, v^l) = F((n+1)h-0, u^l_0, v^l_0) + \sum_{m: \text{odd}} P(u^l((n+1)h-0, ml-0), v^l((n+1)h-0, ml-0), u^l((n+1)h-0, ml+0), v^l((n+1)h-0, ml+0)). \]

**Lemma 3.7**

\[ P(u^l((n+1)h-0, ml-0), v^l((n+1)h-0, ml-0), u^l((n+1)h-0, ml+0), v^l((n+1)h-0, ml+0)) \leq 2h \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\}, \quad m: \text{odd}. \]

**Proof.** From the definition,

\[
\begin{align*}
u^l((n+1)h-0, ml-0) &= u^l_0(nh, ml) + U^l(nh, (m-1)l) \cdot h, \\
u^l((n+1)h-0, ml+0) &= u^l_0(nh, ml) + U^l(nh, (m+1)l) \cdot h, \\
v^l((n+1)h-0, ml-0) &= v^l((n+1)h-0, ml+0) = v^l_0(nh, ml).
\end{align*}
\]

Therefore we get

\[ r^l((n+1)h-0, ml-0) - r^l((n+1)h-0, ml+0) \]

\[ = s^l((n+1)h-0, ml-0) - s^l((n+1)h-0, ml+0) \cdot l \\
= h \times \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\} \geq 0
\]

Thus the following inequality holds.

\[ (3.18) \Delta r, \Delta s \leq h \left\{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \right\} \leq \Delta r + \Delta s. \]
From (3.18), we get (3.16).

Using Lemma 3.7, we get

\[
F((n+1)h-0, u^l, v^l) - F((n+1)h-0, u_0^l, v_0^l) \leq 2h \sum_{m: \text{odd}} \{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \} \leq 2Kh
\]

From (3.14), (3.15) and (3.19), we get

\[
F((n+1)h+0, u^l, v^l) \leq F(nh+0, u^l, v^l) + 2Kh
\]

Thus we obtain the following lemma.

Lemma 3.8

\[
F(nh+0, u^l, v^l) \leq F(+0, u^l, v^l) + 2KT \equiv F_0 + 2KT
\]

Denote by \( G(\tau) \) the absolute value of the sum of negative variation of \( r^l \) and \( s^l \) for \( t = \tau \). Then for \( nh \leq \tau < (n+1)h \), we get

\[
G(\tau) \leq G(nh) + 2h \sum_{m: \text{odd}} \{ U^l(nh, (m-1)l) - U^l(nh, (m+1)l) \} \leq G(nh) + 2Kh.
\]

Lemma 3.9

\[
G(nh) \leq 2F(nh+0, u^l, v^l).
\]

**Proof.** Denote by \( \delta s \) (resp \( \delta r \)) the absolute value of the Riemann invariant \( s \) (resp \( r \)) in the first (resp second) shock wave. By (3.2) and (3.3), \( \Delta r + \delta s < 2\Delta r \) on the first shock and \( \delta r + \Delta s < 2\Delta s \) on the second shock. So from (3.17), (3.18) and above arguments, we get (3.23). \qed

From (3.23), (3.24) and (3.25), for any \( \tau \ (nh \leq \tau < (n+1)h) \),

\[
G(\tau) \leq G(nh) + 2Kh \leq 2F(nh+0, u^l, v^l) + 2Kh \leq 2F_0 + 6KT \equiv M_1.
\]

Now we can establish a priori estimates of \( u^l \) and \( v^l \). Denote by T.V.u the total variation of u.
**Theorem 3.10** For any $T > 0$, the variation of $u^l$ and $v^l$ is bounded uniformly for $h$ and $\{a_{mn}\}$. Their upper bound and lower bound, especially the positive lower bound of $v^l$, are also uniformly bounded.

**Proof.** Denote by $T.V^+.u$ (resp $T.V^-.u$) the absolute value of the positive (resp negative) variation of $u$. Put $f^l \equiv 2u^l = r^l + s^l$. Then $0 \leq f^l(t,0) \leq Kh$. Without loss of generality, we assume that $u_0(x)$ and $v_0(x)$ are constant outside a bounded interval. Let

$$f^l(t,\infty) = r^l(t,\infty) + s^l(t,\infty) \equiv M_2.$$ 

Then from the definition,

$$f^l(t,0) + T.V^+.f^l - T.V^-.f^l = f^l(t,\infty).$$

Since $T.V^-.f^l(t,\cdot) \leq G(t)$ for any $t$, (3.24) yields

$$T.V^+.f^l = f^l(t,\infty) + T.V^-.f^l - f^l(t,0) \leq M_1 + M_2.$$ 

Thus we get

(3.26) $$T.V.f^l = T.V.2u^l \leq 2M_1 + M_2.$$ 

From (3.26), we get

$$|f^l| \leq Kh + 2M_1 + M_2 \leq KT + 2M_1 + M_2 \equiv 2M_3.$$ 

Therefore we get

(3.27) $$|u^l| \leq M_3.$$ 

Using Lemma 3.2, we get

$$2a \log v^l = r^l - s^l \geq r_0 - (\max(-r_0,s_0) + KT).$$

Thus we get

(3.28) $$v^l \geq \exp \frac{r_0 - (\max(-r_0,s_0) + KT)}{2a} \equiv \frac{1}{M_5}.$$ 

From the definition,

$$r^l(t,0) + T.V^+.r^l - T.V^-.r^l = r^l(t,\infty).$$
Using Lemma 3.3 and (3.24),

\[(3.29)\quad T.V^+ r^l = -r^l(0) + T.V^- r^l + r(t, \infty) \leq -r_0 + M_1 + r(t, \infty).\]

In view of (3.27) and (3.29), there exists a positive constant $M_6$ such that

\[(3.30)\quad v^l \leq M_6\]

Theorem 3.11 For any interval $[x_1, x_2] \subset [0, \infty)$, we get

\[(3.31)\quad \int_{x_1}^{x_2} |u'(t_2, x) - u'(t_1, x)| + |v'(t_2, x) - v'(t_1, x)|\, dx \leq M \cdot (|t_2 - t_1| + h), \quad 0 \leq t_1, t_2 < T,
\]

where $M$ depends on $T$, $x_1$, and $x_2$, but not on $l$ and $h$.

**Proof.** Without loss of generality, we assume that

\[nh \leq t_1 < (n + 1)h < \cdots < (n + k)h \leq t_2 < (n + k + 1)h.\]

Let

\[\int_{x_1}^{x_2} |u'(t_2, x) - u'(t_1, x)|\, dx \leq I_1 + I_2 + \int_{x_1}^{x_2} |u'(t_2, x) - u'((n + k)h + 0, x)| + |u'(t_1, x) - u'((n + 1)h - 0, x)|\, dx\]

where

\[I_1 = \int_{x_1}^{x_2} \sum_{i=1}^{k} |u'((n + i)h + 0, x) - u'((n + i)h - 0, x)|\, dx\]

\[I_2 = \int_{x_1}^{x_2} \sum_{i=1}^{k-1} |u'((n + i + 1)h - 0, x) - u'((n + i)h + 0, x)|\, dx\]

and

\[k = \left\lceil \frac{t_2 - t_1}{h} \right\rceil\]
Denote by $1_{[\alpha, \beta]}$ the characteristic function of the interval $[\alpha, \beta]$. We regard $T.V.-l < x < l = T.V_{0 < x < l}$. Then,

\[ I_1 \leq \sum_{i=0}^{k+1} \sum_{m: integer} \int_{x_1}^{x_2} T.V_{2ml < x < (2m+2)l} u^l((n+i)h - 0, x) \cdot 1_{[2ml, (2m+2)l]} \, dx, \]

\[ \leq \left( \frac{t_2 - t_1}{h} + 2 \right) \cdot \left( \sup_{0 \leq t \leq T} T.V.u^l(t, \cdot) \right) \cdot 2l. \]

\[ I_2 \leq \sum_{i=0}^{k} \sum_{m} \int_{x_1}^{x_2} (T.V_{(2m-1)l < x < (2m+1)l} u_0^l((n+i+1)h - 0, x) \cdot 1_{[(2m-1)l, (2m+1)l]} + K h) \, dx, \]

\[ \leq 2l \cdot T.V.u_0^l((n+i+1)h - 0, \cdot) + K(x_2 - x_1) h, \]

\[ \leq \left( \frac{t_2 - t_1}{h} + 1 \right) \cdot \left( 2l \cdot \sup_{0 \leq t \leq T} T.V.u_0^l(t, \cdot) + K(x_2 - x_1) h \right). \]

The remaining terms can be evaluated similarly. For

\[ \int_{x_1}^{x_2} |\psi(t_2, x) - \psi(t_1, x)| \, dx, \]

we also have a similar estimate. Combining these results gives (3.31).

\[ \square \]

4 Convergence of The Approximate Solution

Let $h_n = T/n$ and $h_n/l_n = \delta < \delta \equiv 1/M_5$. Consider the sequence $(u^n, v^n)$ ($n = 1, 2, \cdots$). Then from Theorem 3.9 and Theorem 3.10, there exists a subsequence which converges in $L_{loc}^1$ to functions $(u,v)$ uniformly for $t \in [0, T]$. Now we shall prove that $u(x,t)$ and $v(x,t)$ are the weak solutions of initial boundary value problem (1.6), (1.7) and (1.8) provided $\{a_{nm}\}$ is suitably chosen, namely, they satisfy the integral identity

\[ \int_0^T \int_0^\infty u \phi_t + \left( \frac{a^2}{v} \right) \phi_x + \frac{K}{1 + \int_0^x v(t, \zeta) \, d\zeta} \phi \, dx \, dt \]

\[ + \int_0^\infty u_0(x) \phi(0, x) \, dx = 0, \]

\[ (4.1) \]

\[ \int_0^T \int_0^\infty v \psi_t - u \psi_x \, dx \, dt + \int_0^\infty v_0(x) \psi(0, x) \, dx = 0, \]

\[ (4.2) \]
for any smooth functions $\phi$ and $\psi$ with compact support in the region
$\{(t, x) : 0 \leq t < T, 0 \leq x < \infty\}$ and $\phi(t, 0) = 0$. Now we know that $u_0^l$ and $v_0^l$ are weak solutions in each time strip $nh \leq t < (n+1)h$ so that for each test function $\phi$ satisfying $\phi(t, 0) = 0$,

$$
\int_{nh}^{(n+1)h} \int_0^\infty u^l \phi_t + \left( \frac{a^2}{v^l} \right) \phi_x + U^l(t, x) \cdot \phi \, dx \, dt
+ \int_0^\infty (nh+0, x) \phi(nh, x)
- \int_0^\infty ((n+1)h-0, x) \phi((n+1)h, x) \, dx = 0
$$

(4.3)

If we sum this over $n$, we get

$$
\int_0^T \int_0^\infty u^l \phi_t + \left( \frac{a^2}{v^l} \right) \phi_x + U^l(t, x) \cdot \phi \, dx \, dt + \int_0^\infty u^l(0, x) \phi(0, x)
= - \sum_{k=1}^N \int_0^\infty \{u^l(kh+0, x) - u^l(kh-0, x)\} \cdot \phi(kh, x) \, dx
$$

where $N=T/h$. When $N \to \infty$, the right-hand side of the above equality tends to 0 for almost every $\{a_{nm}\} \in A$ (see [4]). It is immediate to see that

$$
\int_0^\infty u^l(0, x) \phi(0, x) \, dx \to \int_0^\infty u_0(x) \phi(0, x) \, dx \quad (N \to \infty).
$$

Lemma 4.1

$$
U^l(t, x) \to \frac{K}{1 + \int_0^x v(t, \zeta) \, d\zeta} \quad (N \to \infty).
$$

locally uniformly for $t$ and $x$.

Proof. Let $nh \leq t < (n+1)h$, $x \in ((m-1)l, (m+1)l)$, $m : odd$. Then

$$
\left| \int_0^x v^l(nh, \zeta) \, d\zeta - \sum_{j=1}^{m+1} v^l(nh, c_{2j-1}n) \right| \leq \| v^l \|_\infty \cdot l.
$$

(4.6)

On the other hand

$$
\int_0^x v^l(t, \zeta) \, d\zeta \to \int_0^x v(t, \zeta) \, d\zeta \quad (N \to \infty).
$$

(4.7)
locally uniformly for $t$ and $x$.

We get

$$
\left| \int_0^x v'(t, \zeta) d\zeta - \int_0^x v'(nh, \zeta) d\zeta \right| 
\leq \int_0^x \sum_{m:odd} T.V_{(m-1)T \leq \zeta < (m+1)T} v'(nh, \cdot) \cdot 1_{[(m-1)T,(m+1)T]} d\zeta
\leq \sup_{0 \leq t \leq T} T.V.v' \cdot 2l.
$$

(4.8)

From (4.6), (4.7) and (4.8), we get (4.5).

For each test function $\psi$, $v'$ also satisfies,

$$
\int_0^T \int_0^\infty (v' \psi_t - u' \psi_x) \, dx \, dt + \int_0^\infty v'(0, x) \psi(O, x) \, dx
= - \sum_{k=1}^{N} \int_0^\infty \left\{ v'(kh + 0, x) - v'(kh - 0, x) \right\} \cdot \psi(kl, x) \, dx
- I_1 - I_2.
$$

(4.9)

where

$$
I_1 = \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} U^l(t, 0)(t-nh) \psi(t, 0) \, dt
$$

and

$$
I_2 = \sum_{n=0}^{N-1} \sum_{m:odd} \int_{nh}^{(n+1)h} \{ U^l(t, ml + 0) - U^l(t, ml - 0) \} (t-nh) \psi(t, ml) \, dt.
$$

The first term of the right-hand side of equality (4.9) tends to 0 for almost every $\{a_{nm}\} \in A$ (see [4]). It is also immediate to see that

$$
\int_0^\infty v'(0, x) \psi(0, x) \, dx \to \int_0^\infty v_0(x) \psi(0, x) \, dx \quad (N \to \infty).
$$

We shall show that $I_1, I_2 \to 0$ as $N \to \infty$.

$$
I_1 \leq \| \psi \|_\infty \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} U^l(t, 0)(t-nh) \, dt
\leq \| \psi \|_\infty \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} K(t-nh) \, dt
\leq \| \psi \|_\infty h^2 a^2 T.
$$

(4.10)
\[
\sum_{m: \text{odd}} \int_{nh}^{(n+1)h} \{U^l(t, ml + 0) - U^l(t, ml - 0)\} (t-nh) \psi(t, ml) dt \leq K \| \psi \|_{\infty} h^2.
\]

Thus we get
\[(4.11) \quad I_2 \leq \| \psi \|_{\infty} \sum_{n=0}^{N-1} K h^2 \leq K \| \psi \|_{\infty} hT\]

From above arguments, we can conclude that \(u\) and \(v\) satisfy (4.1) and (4.2). Thus we obtain our main result.

**Theorem 4.2 (Main Result)** Suppose that \(u_0(x), v_0(x) \in BV(\mathbb{R}_+)\), and that \(v_0(x) \geq \delta_0 > 0\) for all \(x > 0\) with some positive constant \(\delta_0\). Then (1.10), (1.11) and (1.12) have a global weak solution which belongs to the class

\[u, \ v \in L^\infty(0, T; BV(\mathbb{R}_+)) \cap Lip([0, T]; L_{loc}^1(\mathbb{R}_+))\]

for any \(T > 0\).
Appendix

A Proof of Lemma 3.5

Let \( g(x) = -f(-x) \), and put
\[
\begin{align*}
P(u_1, v_1, u_2, v_2) &= \Delta r_1 + \Delta s_1 \\
P(u_2, v_2, u_3, v_3) &= \Delta r_2 + \Delta s_2 \\
P(u_1, v_1, u_3, v_3) &= \Delta r_3 + \Delta s_3
\end{align*}
\]
Then it is obvious that
\[
\Delta r_3 + g(\Delta s_3) + \Delta s_3 + g(\Delta r_3) \\
\leq \Delta r_1 + \Delta r_2 + \Delta s_1 + \Delta s_2 + g(\Delta r_1) + g(\Delta r_2) + g(\Delta s_1) + g(\Delta s_2)
\]
We notice that \( f'' \leq 0 \) and hence
\[
\leq \Delta r_1 + \Delta r_2 + \Delta s_1 + \Delta s_2 + g(\Delta r_1 + \Delta r_2) + g(\Delta s_1 + \Delta s_2).
\]
Let \( x + g(x) = h(x) \), \( \Delta r_3 = p' \), \( \Delta s_3 = q' \), \( \Delta r_1 + \Delta r_2 = p \) and \( \Delta s_1 + \Delta s_2 = q \). Then
(A.1) \[ h(p') + h(q') \leq h(p) + h(q). \]
Put \( K = h(p') + h(q') \). We shall estimate \( p + q \) from below under the restriction (A.1). To do this, as \( h \) is monotone increasing function, we must estimate \( p + q \) from below under the restriction
(A.2) \[ h(p) + h(q) = K. \]
We do this by using Lagrange's method of indeterminate coefficients.
Put \( G(p, q, \lambda) = p + q + \lambda (h(p) + h(q) - K) \). Then
\[
G_p = 1 + \lambda h'(p) = 0, \quad G_q = 1 + \lambda h'(q) = 0.
\]
Because \( h''(x) > 0 \), we get \( p = q \). So \( p + q \) attains its extremum at \( p = q \). We can show that when \( p = q \), \( p + q \) is minimum under the restriction (A2).
Therefore
\[
h(p) = h(q) = \frac{K}{2} = \frac{h(p') + h(q')}{2} \geq h\left(\frac{p' + q'}{2}\right).
\]
Hence it follows that
\[ p = q \geq \frac{p' + q'}{2}. \]
Thus we get
(A.3) \[ p + q \geq p' + q'. \]
which proves Lemma 3.5.

B Proof of Lemma 3.6

To prove Lemma 3.6, we must check the following 12 cases:

1) \(c_{1n} < l\),
   (1) \(S_2\) crosses \(i_0^{n-}\),
   (2) \(R_2\) crosses \(i_0^{n-}\),
   (3) no wave cross \(i_0^{n-}\).

2) \(c_{1n} \geq l\),
   (1) \(S_2\) and \(S_1\) cross \(i_0^{n-}\),
   (2) \(R_2\) and \(S_1\) cross \(i_0^{n-}\),
   (3) \(S_2\) and \(R_1\) cross \(i_0^{n-}\),
   (4) \(R_2\) and \(R_1\) cross \(i_0^{n-}\),
   (5) \(S_1\) crosses \(i_0^{-}\),
   (6) \(R_1\) crosses \(i_0^{-}\),
   (7) \(S_2\) crosses \(i_0^{-}\),
   (8) \(R_2\) crosses \(i_0^{-}\),
   (9) no wave cross \(i_0^{-}\).
Put $r_{+}^{n-1} = r^l(a_{1n-1})$, $s_{+}^{n-1} = s^l(a_{1n-1})$, $r_{-}^{n-1} = -s_{-}^{n-1}$
$= r^l((n - 1)h + 0,0)$, and $\delta_{n-1} = U^l(a_{1n-1})$.
Put $r_{+}^{n-1'} = r^l((n - 1)h + 0, 2l)$ and $s_{+}^{n-1'} = s^l((n - 1)h + 0, 2l)$.
Put $A = (r_{-}^{n-1}, s_{-}^{n-1})$, $B = (r_{+}^{n-1}, s_{+}^{n-1})$ and $B' = (r_{+}^{n-1'}, s_{+}^{n-1'})$.
Put $C = (r_{+}^{n-1} + Kh, s_{+}^{n-1} + Kh)$, (resp $(r_{+}^{n-1'} + \delta_{n-1}h, s_{+}^{n-1'} + \delta_{n-1}h,)$) if $c_{1n} < l$ (resp $c_{1n} \geq l$).

If $R_2$ crosses $i_0^{n+}$, $F(i_0^{n+}) = 0 \leq F(i_0^{n-})$, so that it is sufficient to consider the cases when $S_2$ crosses $i_0^{n+}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Figure 2}
\end{figure}

1) $c_{1n} < l$.

(1) $S_2$ crosses $i_0^{n-}$ (Figure 2). Denote by I (resp II) the halfspace
\{(r, s)|r + s < 0\} (resp \{(r, s)|r + s \geq 0\}).

i) $C \in I$.

In this case $S_2$ crosses $i_0^{n+}$. Denote by $V(PQ)$ the absolute value of the total variation of $r$ and $s$ by the line segment PQ. From Figure 3,

\[ F(i_0^{n+}) = V(A'C) \leq V(A'C') = V(AB) = F(i_0^{n-}). \]
ii) $C \in II$.

In this case $R_2$ crosses $i_0^+$. Then

(B.1) \[ F(i_0^{n-}) \geq F(i_0^{n+}) = 0. \]

(2) $R_2$ crosses $i_0^-$.

In this case $B \in II$ so that $R_2$ crosses $i_0^+$. Then

(B.2) \[ F(i_0^{n-}) = F(i_0^{n+}) = 0. \]

(3) no wave crosses $i_0^-$. 

In this case $(r_+^{n-1}, s_+^{n-1})$ is on the line $r + s = 0$. Hence $C \in II$. It is obvious that (B.3) also holds.
2) $c_{1n} \geq l$.

(1) $S_2$ and $S_1$ cross $i_0^{-}$. (Figure.4)
i) \( C \in I \).

From Figure.5,

\[
F(i_0^{n+}) = V(A'C) \leq V(A'C') = V(A''B') = V(AB') = F(i_0^{n-}).
\]

ii) \( C \in II \) implies that

\( R_2 \) crosses \( i_0^{n+} \). So we get (B2).
(2) $R_2$ and $S_1$ cross $i_0^{n-}$.

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure6}
\caption{Figure 6}
\end{figure}

i) $C \in I$.
From Figure 6,
\[
F(i_0^{n+}) = V(A'C) \leq V(A'D) = V(A''E) = V(A''B'') \\
\leq V(BB'') = V(BB') = F(i_0^{n+})
\]

ii) $C \in II$.
Thus $R_2$ crosses $i_0^{n+}$, and we get (B2).
(3) $S_2$ and $R_1$ cross $i_0^{n-}$.

Put $G = (r_+^{n-1} + s_+^{n-1} + 2\delta_{n-1}h, s_+^{n-1} + 2\delta_{n-1}h)$ and $II = (r^l(a_{1n}), s^l(a_{1n}))$.

Then $H$ is on the line $CG$.

i) $H \in I$.

From Figure.7,

\[ F(i_0^{n+}) = V(A'H) \leq V(A''G) \leq V(AB) = F(i_0^{n-}). \]

ii) $H \in II$, so $R_2$ crosses $i_0^{n+}$, and we get (B2).

(4) $R_2$ and $R_1$ cross $i_0^{n-}$.

In this case, $R_2$ crosses $i_0^{n+}$. So we get (B3).
(5) $S_1$ crosses $i_0^{n-}$.

\[ r + s = 0 \]

\[ r + s = r_+^{n-1} + s_+^{n-1} + 2\delta_{n-1}h \]

Figure 8

i) $C \in I$.

From Figure 8,

\[ F(i_0^{n+}) = V(A'C) = V(AE) = V(AD) \leq V(AB') = F(i_0^{n-}) \]

Thus we get (B1).

ii) $C \in II$.

$R_2$ crosses $i_0^{n+}$. So we get (B2).
(6) $R_1$ crosses $i_0^{-}$.

In this case, it is obvious that $F(i_0^{n^+}) = 0$. Hence we get (B3).

Cases (7), (8) and (9) are almost the same as cases (1), (2) and (3) in 1). Thus, we obtain Lemma 3.6.

References


