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<tbody>
<tr>
<td>Author(s)</td>
<td>Okamoto, Y.; Imaoka, T.</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1992), 786: 32-42</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1992-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/82589">http://hdl.handle.net/2433/82589</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
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<td>Textversion</td>
<td>publisher</td>
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SOME REMARKS ON \(\mathcal{P}\)-CONGRUENCES ON \(\mathcal{P}\)-REGULAR SEMIGROUPS I

- \(\mathcal{P}\)-congruence pairs -

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Yamada and Sen introduced the new concept of \(\mathcal{P}\)-regularity in the class of regular semigroups which is a generalization of both the concepts of "orthodox" and "(special) involution" (see [8],[9]). The purpose of this abstract is to characterize congruences on a \(\mathcal{P}\)-regular semigroups by using "\(\mathcal{P}\)-congruence pairs", which is a generalization of Petrich [7] for inverse semigroup and one of the authors [4] for regular \(\ast\)-semigroups. Also, for a given congruence \(\rho\) on a \(\mathcal{P}\)-regular semigroup \(S\), the maximum and the minimum congruences on \(S\) whose traces coincide with the trace of \(\rho\) (= \(\rho \cap \mathcal{E}(S) \times \mathcal{E}(S)\)) are determined.

1. Introduction. Let \(S\) be a regular semigroup and \(\mathcal{E}\) the set of idempotents of \(S\). Let \(P \subseteq \mathcal{E}\). If \(S\) satisfies the following, it is called a \(\mathcal{P}\)-regular semigroup:

1. \(P^2 \subseteq \mathcal{E}\),
2. \(qPq \subseteq P\) for any \(q \in P\),
3. for any \(a \in S\), there exists \(a^+ \in V(a)\) (the set of all inverses of \(a\)) such that \(a^+P^1a \subseteq P\) and \(aP^1a^+ \subseteq P\).

In such a case, \(S\) is denoted by \(S(P)\) and \(P\) is called a \(C\)-set in \(S\). Throughout this paper, let \(S(P)\) be a \(\mathcal{P}\)-regular semigroup.
with a C-set $P$ such that the set of idempotents of $S$ is $E$. Let $a \in S(P)$ and $a^+ \in V(a)$. If $a^+$ satisfies that $a^+P^1a \subseteq P$ and $aP^1a^+ \subseteq P$, it is called a $P$-inverse of $a$, and the set of $P$-inverses of $a$ is denoted by $V_P(a)$. An element of a C-set $P$ in $S$ is called a projection. The class of $P$-regular semigroups contains both the classes of orthodox semigroups and regular $*$-semigroups. A good account of the concept of $P$-regularity can be seen in [8] and [9].

A congruence on $S$ is sometimes called a $P$-congruence on $S(P)$. Let $\rho$ be a $P$-congruence on $S(P)$, and put $\overline{x} = x\rho$ for any $x \in S$, $\overline{S} = \{\overline{x} : x \in S\}$ and $\overline{P} = \{\overline{q} : q \in P\}$. Then $\overline{S}(\overline{P})$ is also a $P$-regular semigroup with a C-set $\overline{P}$. So $\overline{S}(\overline{P})$ is called the factor $P$-regular semigroup of $S(P)$ mod. $\rho$, and it is denoted by $S(P)/(\rho)_P$.

Let $\rho$ be a $P$-congruence on $S(P)$. Then it is called an orthodox $P$-congruence on $S(P)$ if $S(P)/(\rho)_P$ is an orthodox semigroup, and it is called a strong $P$-congruence on $S(P)$ if it satisfies that for $a \in S(P)$ and $e \in P$,

$$a \rho e \text{ implies } a^+ \rho e \text{ for all } a^+ \in V_P(a).$$

As was seen in [8], if $\rho$ is a strong $P$-congruence on $S(P)$, then $S(P)/(\rho)_P$ becomes a regular $*$-semigroup with the set $\{e\rho : e \in P\}$ of projections if the $*$-operation $\#$ on $S(P)/(\rho)_P$ is defined by $(a\rho)^\# = a^+\rho$ ($a \in S(P), a^+ \in V_P(a)$).

The set $\{a \in S(P) : a \rho e \text{ for some } e \in E [e \in P]\}$ is called the $[P]$-kernel of $\rho$, and it denoted by $[P]\ker\rho$. The restriction $\rho \cap (E \times E) [\rho \cap (P \times P)]$ of $\rho$ is called the $[P]$-trace of $\rho$, and it is denoted by $[P]\tr\rho$. 

For any subset $A$ of $S(P)$, define the terminology as follows:

- $A$ is $[P]$-full if $E \subseteq A$ ($P \subseteq A$).
- $A$ is a $P$-subset if $V_P(a) \subseteq A$ for any $a \in A$.
- $A$ is a $P$-self-conjugate if $x^*Ax \subseteq A$ for any $x \in S(P)$ and $x^+ \in V_P(x)$.
- $A$ is weakly closed if $a^2 \in A$ for any $a \in A$.

The following results are fundamental and are used frequently in this abstract.

**Result 1.1** (due to [8] and [9]). Let $a, b \in S(P), e \in E$ and $q \in P$. Then

1. $V_P(b)V_P(a) \subseteq V_P(ab),$
2. if $a^+ \in V_P(a)$, then $a \in V_P(a^+),$
3. $V_P(e) \subseteq E,$
4. $q \in V_P(q).$

**Result 1.2** (due to [2]). Let $\rho$ be a $P$-congruence on $S(P)$ and $a, b \in S(P)$. Then $a \rho b$ if and only if

$$ba' \in \ker \rho, \ aa' \rho \ bb'aa', \ b'b \rho \ b'ba'a$$

for some $a' \in V(a)$ and $b' \in V(b)$.

In section 2, for a given $P$-congruence $\rho$ on a $P$-regular semigroup $S(P)$, the maximum and the minimum $P$-congruences on $S(P)$ whose traces coincide with $\text{tr}\rho$ are determined, and the properties for those $P$-congruences are given.

The concept introduced in section 3 is "$P$-congruence pairs". This concept is a characterization of the pair $(\text{tr}\rho, \ker \rho)$ associated with a given $P$-congruence $\rho$ on $S(P)$, and the
pair uniquely determines the $\mathcal{P}$-congruence $\kappa$ such that $\text{tr}\kappa = \text{tr}\rho$ and $\ker\kappa = \ker\rho$.

We use the notation and terminology of [3] and [9] unless otherwise stated.

2. $\mathcal{P}$-congruences with the same trace. For any $\mathcal{P}$-congruence $\rho$ on $S(P)$, define a relation $\rho_{\text{max}}$ on $S(P)$ as follows:

$$
\rho_{\text{max}} = \{(a,b) : \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \text{ such that } aea^+ \rho beb^+aea^+, beb^+ \rho aea^+beb^+, a^+ea \rho a^+eab^+eb \text{ and } b^+eb \rho b^+ebea \text{ for all } e \in P\}.
$$

Then we can easily see that

$$
\rho_{\text{max}} = \{(a,b) : aea^+ \rho beb^+aea^+, beb^+ \rho aea^+beb^+, a^+ea \rho a^+eab^+eb \text{ and } b^+eb \rho b^+ebea \text{ for all } a^+ \in V_P(a), b^+ \in V_P(b) \text{ and } e \in P\}.
$$

Lemma 2.1. Let $\rho$ be a $\mathcal{P}$-congruence on $S(P)$ and $a, b \in S(P)$. If $a \rho_{\text{max}} b$, then

$aa^+ \rho bb^+aa^+, bb^+ \rho aa^+bb^+, a^+a \rho a^+ab^+b, b^+b \rho b^+ba^+$

for any $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$.

Theorem 2.2. For any $\mathcal{P}$-congruence $\rho$ on a $\mathcal{P}$-regular semigroup $S(P)$, $\rho_{\text{max}}$ is the greatest $\mathcal{P}$-congruence on $S(P)$ whose trace coincides with $\text{tr}\rho$.

Theorem 2.3. For any orthodox $\mathcal{P}$-congruence $\rho$ on $S(P)$,
\( \rho_{\text{max}} \) is the greatest orthodox \( \mathcal{P} \)-congruence on \( S(P) \) whose trace coincides with \( \text{tr} \rho \).

From now on, denote the maximum idempotent-separating congruence on a semigroup \( T \) by \( \mu_T \).

**Corollary 2.4** (compare with [8, Proposition 4.1]). The maximum idempotent-separating \( \mathcal{P} \)-congruence \( \mu_{S(P)} \) on \( S(P) \) is given as follows:

\[
\mu_{S(P)} = \{(a,b): \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \text{ such that } aea^+ = beb^+a^+a^+, \quad beb^+ = aea^+beb^+, \quad a^+ea = a^+eab^+eb \text{ and } b^+eb = b^+eba^+ea \text{ for all } e \in P\}
\]

Let \( S \) be an orthodox semigroup and \( E \) the band of idempotents of \( S \). Then it is easy to check that \( S(E) \) is a \( \mathcal{P} \)-regular semigroup with a C-set \( E \) in \( S \). So we have immediately

**Corollary 2.5** ([1, Theorem 4.2]). Let \( \rho \) be a congruence on an orthodox semigroup \( S \) with the band \( E \) of idempotents of \( S \). Then

\[
\rho_{\text{max}} = \{(a,b): \text{there exist } a' \in V(a) \text{ and } b' \in V(b) \text{ such that } aea' \rho \text{ beb'aea', beb' } \rho \text{ aea'beb', a'ea } \rho \\
\quad a'eb'eb, b'eb \rho b'eba'ea \text{ for any } e \in E\}
\]

= \{(a,b): aea' \rho \text{ beb'aea', beb' } \rho \text{ aea'beb', a'ea } \rho \\
a'eb'eb, b'eb ρ b'eba'ea for any a' ∈ V(a), b' ∈ V(b) and e ∈ E

is the greatest congruence on S whose trace coincides with trρ.

On the other hand, the minimum $\mathcal{P}$-congruence on S(P) with the same trace is given as follows:

**Theorem 2.6.** For any $\mathcal{P}$-congruence $\rho$ on a $\mathcal{P}$-regular semigroup S(P), define a relation $\rho_0$ on S(P) by

$$\rho_0 = \{(a, b) : \text{there exist } x, y ∈ S(P) \text{ and } e, f ∈ E \text{ such that } a = xey, b = xfy \text{ and } e \rho f\}$$

Then $\rho_{min} = \rho_0^t$, the transitive closure of $\rho_0$, is the least $\mathcal{P}$-congruence on S(P) whose trace coincides with trρ. In other words, the least $\mathcal{P}$-congruence on S(P) with trρ as its trace is the $\mathcal{P}$-congruence on S(P) generated by trρ.

The following corollary gives us the characterization which is different from both [1, Theorem 4.1] and [7, Theorem 3.3], of the least congruence on an orthodox semigroup with the same trace.

**Corollary 2.7.** For any congruence $\rho$ on an orthodox semigroup S, the congruence generated by trρ is the least congruence on S whose trace coincides with trρ.

**Proposition 2.9.** For any $\mathcal{P}$-congruence $\rho$ on S(P), $\rho = \rho_{max}$ if and only if $(S(P)/\rho)_\mathcal{P}$ is a fundamental $\mathcal{P}$-regular
For any $\mathcal{P}$-congruences $\rho$ and $\sigma$ on $S(P)$ such that $\rho \subseteq \sigma$, define a relation $\sigma/\rho$ on $S(P)/(\rho)_{\mathcal{P}}$ by
\[
\sigma/\rho = \{(a\rho ,b\rho) : (a,b) \in \sigma\}
\]

**Proposition 2.10.** For any $\mathcal{P}$-congruence $\rho$ on $S(P)$, $\rho_{\max}/\rho$ is the maximum idempotent-separating $\mathcal{P}$-congruence on $S(P)/(\rho)_{\mathcal{P}}$.

Let $\Lambda$ be the lattice of all $\mathcal{P}$-congruences on $S(P)$. Define a relation $\Theta$ on $\Lambda$ as follows: for any $\rho$, $\sigma \in \Lambda$,
\[
\rho \Theta \sigma \quad \text{if and only if} \quad \text{tr} \rho = \text{tr} \sigma.
\]
It immediately follows from Theorems 2.2 and 2.6 that $\rho \Theta$, the $\Theta$-class containing $\rho \in \Lambda$, is the interval $[\rho_{\min}, \rho_{\max}]$ of $\Lambda$.

**Proposition 2.11 ([6, Theorem 5.1]).** If $\mathcal{P}$-congruences $\rho$ and $\sigma$ on $S(P)$ are $\Theta$-equivalent, then $\rho \sigma = \sigma \rho$. Therefore, for any $\rho \in \Lambda$, $\rho \Theta$ is a complete modular subsemilattice of $\Lambda$.

**Proposition 2.12.** Let $\xi \in \Lambda$, and let $\Gamma$ be the lattice of all idempotent-separating $\mathcal{P}$-congruences on $S(P)/(\xi_{\min})_{\mathcal{P}}$. Then the mapping $\rho \rightarrow \rho/\xi_{\min}$ is a complete isomorphism of $\xi \Theta$ onto $\Gamma$.

3. $\mathcal{P}$-congruence pairs. Let $\xi$ be an equivalence on $E$. Then $\xi$ is called a normal equivalence on $E$ if it satisfies the following conditions: for any $a \in S(P)$ and $e, f, g, h, i, j, k \in$
E,

(a) if $e \xi f$ and $aea^+ \in E$ for some $a^+ \in V_P(a)$, then $aea^+ \xi afa^+$.  

(b) if $e \xi f$, $g \xi h$ and $eg \in E$, then $eg \xi fh$.  

(c) if $\square \not\in (e\xi)(f\xi) \cap E \subset h\xi$, $\square \not\in (f\xi)(g\xi) \cap E \subset i\xi$ and $\square \not\in (e\xi)(i\xi) \cap E \subset j\xi$, then $\square \not\in (h\xi)(g\xi) \cap E \subset k\xi$ and $j \xi k$.

Let $\xi$ be a normal equivalence on $E$. Define a partial binary operation $\cdot$ on $E/\xi$ as follows: for any $e, f, g \in E$, 

$$e\xi \cdot f\xi = g\xi$$

where $\square \not\in (e\xi)(f\xi) \cap E \subset g\xi$.

It is easy to verify that the partial binary operation $\cdot$ is well-defined. The partial groupoid $E/\xi$ satisfies the following:

(w) if $e\xi \cdot f\xi, f\xi \cdot g\xi$ and $e\xi \cdot (f\xi \cdot g\xi) = [(e\xi \cdot f\xi) \cdot g\xi]$ are defined in $E/\xi$, then $(e\xi \cdot f\xi) \cdot g\xi = e\xi \cdot (f\xi \cdot g\xi)$ and $e\xi \cdot (f\xi \cdot g\xi) = e\xi \cdot (f\xi \cdot g\xi)$.

Let $K$ be a weakly closed full $\mathcal{P}$-subset of $S(P)$ and $\xi$ a normal equivalence on $E$. Then the pair $(\xi, K)$ is called a $\mathcal{P}$-congruence pair for $S(P)$ if it satisfies the following conditions: for any $a, b, c \in S(P), c^+ \in V_P(c), e, f, g \in E$ and $q \in P$,

(C1) $a \in K$ implies $a^+ a \xi a^+ a a^+$ for any $a^+ \in V_P(a)$,

(C2) $ae \xi f \in K$ and $e\xi \cdot f\xi = (a^+ a) \xi$ for some $a^+ \in V_P(a)$ imply $ab \in K$,

(C3) $ab^+ \in K$ and $aa^+ \xi b b^+ a a^+$, $b^+ b \xi b^+ b a^+ a$ for some $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$ imply $aq b^+ \in K$ and $a q a^+ \xi b q b^+ q a^+ a$, $b^+ q b \xi b^+ q b a^+ q a$. 
\[(C4) \quad a, b \in K, \quad aa^+ \xi ee^+aa^+, \quad ee^+ \xi aa^+ee^+, \quad a^+a \xi a^ae^+, \quad e^+e \xi e^+ea^+, \quad bb^+ \xi ff^+bb^+, \quad ff^+ \xi bb^+ff^+.
\]
\[
b^+b \xi b^+bf^+f, \quad f^+f \xi f^+fb^+b \text{ and } e\xi \cdot f \xi = g \xi \text{ for some}
\]
\[
a^+ \in \mathcal{V}_P(a), \quad b^+ \in \mathcal{V}_P(b), \quad e^+ \in \mathcal{V}_P(e) \text{ and } f^+ \in \mathcal{V}_P(f)
\]
\[
\text{imply } ab \in K,
\]
\[
(C5) \quad aq \in K \text{ and } aa^+ \xi qaa^+, \quad q \xi qa^+a \text{ for some } a^+ \in \mathcal{V}_P(a)
\]
\[
\text{imply } cac^+ \in K.
\]

For any \(\mathcal{P}\)-congruence pair \((\xi, K)\) for \(S(P)\), define a relation \(\kappa(\xi, K)\) on \(S(P)\) as follows:
\[
\kappa(\xi, K) = \{(a, b) : ab^+ \in K \text{ and } aa^+ \xi bb^+aa^+, \quad bb^+ \xi
\]
\[
aa^+bb^+, \quad a^+a \xi a^+ab^+b, \quad b^+b \xi b^+ba^+a \text{ for some}
\]
\[
\text{any } a^+ \in \mathcal{V}_P(a) \text{ and } b^+ \in \mathcal{V}_P(b)\}.
\]

Now we can determine \(\mathcal{P}\)-congruences on \(S(P)\) by \(\mathcal{P}\)-congruence pairs.

**Theorem 3.1.** For any \(\mathcal{P}\)-congruence pair \((\xi, K)\) for a \(\mathcal{P}\)

-regular semigroup \(S(P)\), \(\kappa(\xi, K)\) is a \(\mathcal{P}\)-congruence on \(S(P)\) such
that \(\text{tr}\kappa(\xi, K) = \xi\) and \(\ker\kappa(\xi, K) = K\). Conversely, for any \(\mathcal{P}\)
-congruence \(\rho\) on \(S(P)\), \((\text{tr}\rho, \ker\rho)\) is a \(\mathcal{P}\)-congruence pair for
\(S(P)\) and \(\rho = \kappa(\text{tr}\rho, \ker\rho)\).

Let \(\mathcal{A}\) be the set of \(\mathcal{P}\)-congruence pairs for \(S(P)\). Define
an order \(<\) on \(\mathcal{A}\) by
\[
(\xi_1, K_1) < (\xi_2, K_2) \text{ if and only if } \xi_1 \subset \xi_2, \quad K_1 \subset K_2.
\]

**Corollary 3.2.** The mappings
\[(\xi, K) \rightarrow \kappa(\xi, K), \quad \rho \rightarrow (\text{tr}\rho, \ker\rho)\]

are mutually inverse order-preserving mappings of \(A\) onto \(\Lambda\) and \(\Lambda\) onto \(A\), respectively. Therefore, \(A\) forms a complete lattice.

References


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