数理解析研究所講究録
786巻 1992年 32-42

SOME REMARKS ON $\mathcal{P}$-CONGRUENCES ON $\mathcal{P}$-REGULAR SEMIGROUPS I

- $\mathcal{P}$-congruence pairs -

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Yamada and Sen introduced the new concept of $\mathcal{P}$-regularity in the class of regular semigroups which is a generalization of both the concepts of "orthodox" and "(special) involution" (see [8],[9]). The purpose of this abstract is to characterize congruences on a $\mathcal{P}$-regular semigroups by using "$\mathcal{P}$-congruence pairs", which is a generalization of Petrich [7] for inverse semigroup and one of the authors [4] for regular $*$-semigroups. Also, for a given congruence ρ on a $\mathcal{P}$-regular semigroup S, the maximum and the minimum congruences on S whose traces coincide with the trace of ρ (= $\rho \cap E(S)\times E(S)$) are determined.

1. Introduction. Let S be a regular semigroup and E the set of idempotents of S. Let $P \subset E$. If S satisfies the following, it is called a $\mathcal{P}$-regular semigroup:

(1) $P^2 \subset E$,

(2) $qPq \subset P$ for any $q \in P$,

(3) for any $a \in S$, there exists $a^+ \in V(a)$ (the set of all inverses of a) such that $a^+P^1a \subset P$ and $aP^1a^+ \subset P$.

In such a case, S is denoted by S(P) and P is called a C-set in S. Throughout this paper, let S(P) be a $\mathcal{P}$-regular semigroup with a C-set P such that the set of idempotents of S is E. Let $a \in S(P)$ and $a^+ \in V(a)$. If a^+ satisfies that $a^+P^1a \subset P$ and $aP^1a^+ \subset P$, it is called a $\mathcal{P}$-inverse of a, and the set of $\mathcal{P}$-inverses of a is denoted by $V_P(a)$. An element of a C-set P in S is called a projection. The class of $\mathcal{P}$-regular semigroups contains both the classes of orthodox semigroups and regular $*$-semigroups. A good account of the concept of $\mathcal{P}$-regularity can be seen in [8] and [9].

A congruence on S is sometimes called a $\mathcal{P}$-congruence on S(P). Let ρ be a $\mathcal{P}$-congruence on S(P), and put $\overline{x} = x\rho$ for any $x \in S$, $\overline{S} = \{\overline{x}: x \in S\}$ and $\overline{P} = \{\overline{q}: q \in P\}$. Then $\overline{S}(\overline{P})$ is also a $\mathcal{P}$-regular semigroup with a C-set $\overline{P}$. So $\overline{S}(\overline{P})$ is called the factor $\mathcal{P}$-regular semigroup of S(P) mod. ρ, and it is denoted by $S(P)/(\rho)_{\mathcal{P}}$.

Let ρ be a $\mathcal{P}$-congruence on S(P). Then it is called an orthodox $\mathcal{P}$-congruence on S(P) if $S(P)/(\rho)_{\mathcal{P}}$ is an orthodox semigroup, and it is called a strong $\mathcal{P}$-congruence on S(P) if it satisfies that for $a \in S(P)$ and $e \in P$,

$$a \,\rho\, e \text{ implies } a^+ \,\rho\, e \text{ for all } a^+ \in V_P(a).$$

As was seen in [8], if ρ is a strong $\mathcal{P}$-congruence on S(P), then $S(P)/(\rho)_{\mathcal{P}}$ becomes a regular $*$-semigroup with the set $\{e\rho: e \in P\}$ of projections if the $*$-operation # on $S(P)/(\rho)_{\mathcal{P}}$ is defined by $(a\rho)^{\#} = a^+\rho$ $(a \in S(P),\ a^+ \in V_P(a))$.

The set $\{a \in S(P): a \,\rho\, e$ for some $e \in E$ $[e \in P]\}$ is called the $[\mathcal{P}$-$]$kernel of ρ, and it denoted by $[\mathcal{P}\text{-}]\ker\rho$. The restriction $\rho \cap (E \times E)$ $[\rho \cap (P \times P)]$ of ρ is called the $[\mathcal{P}$-$]$trace of ρ, and it is denoted by $[\mathcal{P}\text{-}]\mathrm{tr}\rho$.

For any subset A of S(P), define the terminology as follows:

A is [$\mathcal{P}$-]<u>full</u> if $E \subset A$ [$P \subset A$],

A is a $\mathcal{P}$-<u>subset</u> if $V_P(a) \subset A$ for any $a \in A$,

A is a $\mathcal{P}$-<u>self-conjugate</u>

if $x^+Ax \subset A$ for any $x \in S(P)$ and $x^+ \in V_P(x)$,

A is <u>weakly closed</u> if $a^2 \in A$ for any $a \in A$.

The following results are fundamental and are used frequently in this abstract.

Result 1.1 (due to [8] and [9]). <u>Let</u> $a, b \in S(P)$, $e \in E$ <u>and</u> $q \in P$. <u>Then</u>

(i) $V_P(b)V_P(a) \subset V_P(ab)$,

(ii) <u>if</u> $a^+ \in V_P(a)$, <u>then</u> $a \in V_P(a^+)$,

(iii) $V_P(e) \subset E$, (iv) $q \in V_P(q)$.

Result 1.2 (due to [2]). <u>Let</u> ρ <u>be a</u> $\mathcal{P}$-<u>congruence on</u> S(P) <u>and</u> $a, b \in S(P)$. <u>Then</u> $a \,\rho\, b$ <u>if and only if</u>

$$ba' \in \ker\rho,\ aa' \,\rho\, bb'aa',\ b'b \,\rho\, b'ba'a$$

<u>for some</u> $a' \in V(a)$ <u>and</u> $b' \in V(b)$.

In section 2, for a given $\mathcal{P}$-congruence ρ on a $\mathcal{P}$-regular semigroup S(P), the maximum and the minimum $\mathcal{P}$-congruences on S(P) whose traces coincide with $\mathrm{tr}\rho$ are determined, and the properties for those $\mathcal{P}$-congruences are given.

The concept introduced in section 3 is "$\mathcal{P}$-congruence pairs". This concept is a characterization of the pair ($\mathrm{tr}\rho$, $\ker\rho$) associated with a given $\mathcal{P}$-congruence ρ on S(P), and the pair uniquely determines the $\mathcal{P}$-congruence κ such that $\mathrm{tr}\kappa = \mathrm{tr}\rho$ and $\ker\kappa = \ker\rho$.

We use the notation and terminology of [3] and [9] unless otherwise stated.

2. $\mathcal{P}$-congruences with the same trace. For any $\mathcal{P}$-congruence ρ on S(P), define a relation ρ_{max} on S(P) as follows:

$$\rho_{max} = \{(a,b)\colon \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \text{ such that } aea^+ \,\rho\, beb^+aea^+,\ beb^+ \,\rho\, aea^+beb^+,\ a^+ea \,\rho\, a^+eab^+eb \text{ and } b^+eb \,\rho\, b^+eba^+ea \text{ for all } e \in P\}.$$

Then we can easily see that

$$\rho_{max} = \{(a,b)\colon aea^+ \,\rho\, beb^+aea^+,\ beb^+ \,\rho\, aea^+beb^+,\ a^+ea \,\rho\, a^+eab^+eb \text{ and } b^+eb \,\rho\, b^+eba^+ea \text{ for all } a^+ \in V_P(a),\ b^+ \in V_P(b) \text{ and } e \in P\}$$

Lemma 2.1. <u>Let</u> ρ <u>be</u> <u>a</u> $\mathcal{P}$-<u>congruence</u> <u>on</u> S(P) <u>and</u> a, b $\in$ S(P). <u>If</u> a ρ_{max} b, <u>then</u>

$$aa^+ \,\rho\, bb^+aa^+,\ bb^+ \,\rho\, aa^+bb^+,\ a^+a \,\rho\, a^+ab^+b,\ b^+b \,\rho\, b^+ba^+a$$

<u>for</u> <u>any</u> $a^+ \in V_P(a)$ <u>and</u> $b^+ \in V_P(b)$.

Theorem 2.2. <u>For</u> <u>any</u> $\mathcal{P}$-<u>congruence</u> ρ <u>on</u> <u>a</u> $\mathcal{P}$-<u>regular</u> <u>semigroup</u> S(P), ρ_{max} <u>is</u> <u>the</u> <u>greatest</u> $\mathcal{P}$-<u>congruence</u> <u>on</u> S(P) <u>whose</u> <u>trace</u> <u>coincides</u> <u>with</u> $\mathrm{tr}\rho$.

Theorem 2.3. <u>For</u> <u>any</u> <u>orthodox</u> $\mathcal{P}$-<u>congruence</u> ρ <u>on</u> S(P),

ρ_{max} _is the greatest orthodox_ $\mathcal{P}$-_congruence on_ S(P) _whose trace coincides with_ trρ.

From now on, denote the maximum idempotent-separating congruence on a semigroup T by μ_T.

Corollary 2.4 (compare with [8, Proposition 4.1]). _The maximum idempotent-separating_ $\mathcal{P}$-_congruence_ $\mu_{S(P)}$ _on_ S(P) _is given as follows_:

$$\mu_{S(P)} = \{(a,b): \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \text{ such that } aea^+ = beb^+aea^+,\ beb^+ = aea^+beb^+,\ a^+ea = a^+eab^+eb \text{ and } b^+eb = b^+eba^+ea \text{ for all } e \in P\}.$$

$$= \{(a,b): aea^+ = beb^+aea^+,\ beb^+ = aea^+beb^+,\ a^+ea = a^+eab^+eb \text{ and } b^+eb = b^+eba^+ea \text{ for all } a^+ \in V_P(a),\ b^+ \in V_P(b) \text{ and } e \in P\}$$

Let S be an orthodox semigroup and E the band of idempotents of S. Then it is easy to check that S(E) is a $\mathcal{P}$-regular semigroup with a C-set E in S. So we have immediately

Corollary 2.5 ([1, Theorem 4.2]). _Let_ ρ _be a congruence on an othodox semigroup_ S _with the band_ E _of idempotents of_ S. _Then_

$$\rho_{max} = \{(a,b): \text{there exist } a' \in V(a) \text{ and } b' \in V(b) \text{ such that } aea'\ \rho\ beb'aea',\ beb'\ \rho\ aea'beb',\ a'ea\ \rho\ a'eab'eb,\ b'eb\ \rho\ b'eba'ea \text{ for any } e \in E\}$$

$$= \{(a,b): aea'\ \rho\ beb'aea',\ beb'\ \rho\ aea'beb',\ a'ea\ \rho$$

$a'eab'eb$, $b'eb\ \rho\ b'eba'ea$ for any $a' \in V(a)$, $b' \in V(b)$ and $e \in E\}$

is the greatest congruence on S whose trace coincides with $\mathrm{tr}\rho$.

On the other hand, the minimum $\mathcal{P}$-congruence on S(P) with the same trace is given as follows:

Theorem 2.6. For any $\mathcal{P}$-congruence ρ on a $\mathcal{P}$-regular semigroup S(P), define a relation ρ_0 on S(P) by

$$\rho_0 = \{(a,b):\ \text{there exist } x,\ y \in S(P)^1 \text{ and } e,\ f \in E \text{ such that } a = xey,\ b = xfy \text{ and } e\ \rho\ f\}$$

Then $\rho_{min} = \rho_0^{\,t}$, the transitive closure of ρ_0, is the least $\mathcal{P}$-congruence on S(P) whose trace coincides with $\mathrm{tr}\rho$. In other words, the least $\mathcal{P}$-congruence on S(P) with $\mathrm{tr}\rho$ as its trace is the $\mathcal{P}$-congruence on S(P) generated by $\mathrm{tr}\rho$.

The following corollary gives us the characterization which is defferent from both [1, Theorem 4.1] and [7, Theorem 3.3], of the least congruence on an orthodox semigroup with the same trace.

Corollary 2.7. For any congruence ρ on an orthodox semigroup S, the congruence generated by $\mathrm{tr}\rho$ is the least congruence on S whose trace coincides with $\mathrm{tr}\rho$.

Proposition 2.9. For any $\mathcal{P}$-congruence ρ on S(P), $\rho = \rho_{max}$ if and only if $S(P)/(\rho)_{\mathcal{P}}$ is a fundamental $\mathcal{P}$-regular semigroup.

For any $\mathcal{P}$-congruences ρ and σ on S(P) such that $\rho \subset \sigma$, define a relation σ/ρ on $S(P)/(\rho)_{\mathcal{P}}$ by

$$\sigma/\rho = \{(a\rho, b\rho): (a,b) \in \sigma\}$$

Proposition 2.10. For any $\mathcal{P}$-congruence ρ on S(P), ρ_{max}/ρ is the maximum idempotent-separating $\mathcal{P}$-congruence on $S(P)/(\rho)_{\mathcal{P}}$.

Let Λ be the lattice of all $\mathcal{P}$-congruences on S(P). Define a relaton Θ on Λ as follows: for any ρ, $\sigma \in \Lambda$,

$$\rho \,\Theta\, \sigma \quad \text{if and only if} \quad \mathrm{tr}\rho = \mathrm{tr}\sigma.$$

It immediately follows from Theorems 2.2 and 2.6 that $\rho\Theta$, the Θ-class containing $\rho \in \Lambda$, is the interval $[\rho_{min}, \rho_{max}]$ of Λ.

Proposition 2.11 ([6, Theorem 5.1]). If $\mathcal{P}$-congruences ρ and σ on S(P) are Θ-equivalent, then $\rho\sigma = \sigma\rho$. Therefore, for any $\rho \in \Lambda$, $\rho\Theta$ is a complete modular subsemilattice of Λ.

Proposition 2.12. Let $\xi \in \Lambda$, and let Γ be the lattice of all idempotent-separating $\mathcal{P}$-congruences on $S(P)/(\xi_{min})_{\mathcal{P}}$. Then the mapping $\rho \to \rho/\xi_{min}$ is a complete isomorphism of $\xi\Theta$ onto Γ.

3. $\mathcal{P}$-congruence pairs. Let ξ be an equivalence on E. Then ξ is called a normal equivalence on E if it satisfies the following conditions: for any $a \in S(P)$ and e, f, g, h, i, j, $k \in$

E,

(a) if $e \xi f$ and aea^+, $afa^+ \in E$ for some $a^+ \in V_P(a)$, then $aea^+ \xi afa^+$,

(b) if $e \xi f$, $g \xi h$ and eg, $fh \in E$, then $eg \xi fh$,

(c) if $\square \neq (e\xi)(f\xi)\cap E \subset h\xi$, $\square \neq (f\xi)(g\xi)\cap E \subset i\xi$ and $\square \neq (e\xi)(i\xi)\cap E \subset j\xi$ $[\square \neq (h\xi)(g\xi)\cap E \subset k\xi]$, then $\square \neq (h\xi)(g\xi)\cap E$ $[\square \neq (e\xi)(i\xi)\cap E]$ and $j \xi k$.

Let ξ be a normal equivalence on E. Define a partial binary operation $\bullet$ on E/ξ as follows: for any e, f, $g \in E$,

$$e\xi \bullet f\xi = g\xi, \text{ where } \square \neq (e\xi)(f\xi)\cap E \subset g\xi.$$

It is easy to verify that the partial binary operation $\bullet$ is well-defined. The partial groupoid E/ξ satisfies the following:

(w) if $e\xi \bullet f\xi$, $f\xi \bullet g\xi$ and $e\xi \bullet (f\xi \bullet g\xi)$ $[(e\xi \bullet f\xi)\bullet g\xi]$ are defined in E/ξ, then $(e\xi \bullet f\xi)\bullet g\xi$ $[e\xi \bullet (f\xi \bullet g\xi)]$ is defined in E/ξ and $(e\xi \bullet f\xi)\bullet g\xi = e\xi \bullet (f\xi \bullet g\xi)$.

Let K be a weakly closed full $\mathcal{P}$-subset of $S(P)$ and ξ a normal equivalence on E. Then the pair (ξ, K) is called a $\mathcal{P}$-<u>congruence</u> <u>pair</u> for $S(P)$ if itsatisfies the following conditions: for any a, b, $c \in S(P)$, $c^+ \in V_P(c)$, e, f, $g \in E$ and $q \in P$,

(C1) $a \in K$ implies $a^+a \, \xi \, a^+a^+aa$ for any $a^+ \in V_P(a)$,

(C2) $aefb \in K$ and $e\xi \bullet f\xi = (a^+a)\xi$ for some $a^+ \in V_P(a)$ imply $ab \in K$,

(C3) $ab^+ \in K$ and $aa^+ \, \xi \, bb^+aa^+$, $b^+b \, \xi \, b^+ba^+a$ for some $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$ imply $aqb^+ \in K$ and $aqa^+ \, \xi \, bqb^+aqa^+$, $b^+qb \, \xi \, b^+qba^+qa$, (C4) $a, b \in K$, $aa^+ \; \xi \; ee^+aa^+$, $ee^+ \; \xi \; aa^+ee^+$, $a^+a \; \xi \; a^+ae^+e$, $e^+e \; \xi \; e^+ea^+a$, $bb^+ \; \xi \; ff^+bb^+$, $ff^+ \; \xi \; bb^+ff^+$, $b^+b \; \xi \; b^+bf^+f$, $f^+f \; \xi \; f^+fb^+b$ and $e\xi \circ f\xi = g\xi$ for some $a^+ \in V_P(a)$, $b^+ \in V_P(b)$, $e^+ \in V_P(e)$ and $f^+ \in V_P(f)$ imply $ab \in K$,

(C5) $aq \in K$ and $aa^+ \; \xi \; qaa^+$, $q \; \xi \; qa^+a$ for some $a^+ \in V_P(a)$ imply $cac^+ \in K$.

For any $\mathcal{P}$-congruence pair (ξ,K) for S(P), define a relation $\kappa_{(\xi,K)}$ on S(P) as follows:

$$\kappa_{(\xi,K)} = \{(a,b)\colon ab^+ \in K \text{ and } aa^+ \; \xi \; bb^+aa^+,\ bb^+ \; \xi \; aa^+bb^+,\ a^+a \; \xi \; a^+ab^+b,\ b^+b \; \xi \; b^+ba^+a \text{ for some [any] } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b)\}.$$

Now we can determine $\mathcal{P}$-congruences on S(P) by $\mathcal{P}$-congruence pairs.

Theorem 3.1. <u>For any $\mathcal{P}$-congruence pair (ξ,K) for a $\mathcal{P}$-regular semigroup S(P), $\kappa_{(\xi,K)}$ is a $\mathcal{P}$-congruence on S(P) such that $\mathrm{tr}\,\kappa_{(\xi,K)} = \xi$ and $\ker\kappa_{(\xi,K)} = K$. Conversely, for any $\mathcal{P}$-congruence ρ on S(P), $(\mathrm{tr}\,\rho, \ker\rho)$ is a $\mathcal{P}$-congruence pair for S(P) and $\rho = \kappa_{(\mathrm{tr}\rho,\ker\rho)}$.</u>

Let $\mathcal{A}$ be the set of $\mathcal{P}$-congruence pairs for S(P). Define an order $\leqslant$ on $\mathcal{A}$ by

$$(\xi_1,K_1) \leqslant (\xi_2,K_2) \text{ if and only if } \xi_1 \subset \xi_2,\ K_1 \subset K_2.$$

Corollary 3.2. <u>The mappings</u>

$$(\xi, K) \to \kappa_{(\xi, K)}, \quad \rho \to (\mathrm{tr}\rho, \ker\rho)$$

<u>are</u> <u>mutually</u> <u>inverse</u> <u>order-preserving</u> <u>mappings</u> <u>of</u> $\mathscr{A}$ <u>onto</u> Λ <u>and</u> <u>of</u> Λ <u>onto</u> $\mathscr{A}$, <u>respectively</u>. <u>Therefore</u>, $\mathscr{A}$ <u>forms</u> <u>a</u> <u>complete</u> <u>lattice</u>.

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