SOME REMARKS ON $\mathcal{P}$-CONGRUENCES ON $\mathcal{P}$-REGULAR SEMIGROUPS I

- $\mathcal{P}$-congruence pairs -

Y. Okamoto (岡本 洋介)
T. Imaoka (今岡 輝男)

Yamada and Sen introduced the new concept of $\mathcal{P}$-regularity in the class of regular semigroups which is a generalization of both the concepts of "orthodox" and "(special) involution" (see [8],[9]). The purpose of this abstract is to characterize congruences on a $\mathcal{P}$-regular semigroups by using "$\mathcal{P}$-congruence pairs", which is a generalization of Petrich [7] for inverse semigroup and one of the authors [4] for regular $\ast$-semigroups. Also, for a given congruence $\rho$ on a $\mathcal{P}$-regular semigroup $S$, the maximum and the minimum congruences on $S$ whose traces coincide with the trace of $\rho$ ($= \rho \cap E(S) \times E(S)$) are determined.

1. Introduction. Let $S$ be a regular semigroup and $E$ the set of idempotents of $S$. Let $P \subseteq E$. If $S$ satisfies the following, it is called a $\mathcal{P}$-regular semigroup:

1. $P^2 \subseteq E$,
2. $qpq \subseteq P$ for any $q \in P$,
3. for any $a \in S$, there exists $a^+ \in V(a)$ (the set of all inverses of $a$) such that $a^+pa \subseteq P$ and $ap^1a^+ \subseteq P$.

In such a case, $S$ is denoted by $S(P)$ and $P$ is called a C-set in $S$. Throughout this paper, let $S(P)$ be a $\mathcal{P}$-regular semigroup.
with a C-set P such that the set of idempotents of S is E. Let 
\( a \in S(P) \) and \( a^+ \in V(a) \). If \( a^+ \) satisfies that \( a^+P^1a \subset P \) and 
\( aP^1a^+ \subset P \), it is called a \( P \)-inverse of a, and the set of \( P \) 
inverses of a is denoted by \( V_P(a) \). An element of a C-set P in 
S is called a projection. The class of \( P \)-regular semigroups 
contains both the classes of orthodox semigroups and regular 
\( * \)-semigroups. A good account of the concept of \( P \)-regularity 
can be seen in [8] and [9].

A congruence on S is sometimes called a \( P \)-congruence on 
\( S(P) \). Let \( \rho \) be a \( P \)-congruence on \( S(P) \), and put \( \overline{x} = x\rho \) for any 
\( x \in S, \overline{S} = \{ \overline{x} : x \in S \} \) and \( \overline{P} = \{ q : q \in P \} \). Then \( \overline{S}(\overline{P}) \) is also a 
\( P \)-regular semigroup with a C-set \( \overline{P} \). So \( \overline{S}(\overline{P}) \) is called the 
factor \( P \)-regular semigroup of \( S(P) \) mod. \( \rho \), and it is denoted by 
\( S(P)/(\rho) \).

Let \( \rho \) be a \( P \)-congruence on \( S(P) \). Then it is called an 
orthodox \( P \)-congruence on \( S(P) \) if \( S(P)/(\rho) \) is an orthodox 
semigroup, and it is called a strong \( P \)-congruence on \( S(P) \) if it 
satisfies that for \( a \in S(P) \) and \( e \in P \),

\[
a \rho e \text{ implies } a^+ \rho e \text{ for all } a^+ \in V_P(a). 
\]

As was seen in [8], if \( \rho \) is a strong \( P \)-congruence on \( S(P) \), then 
\( S(P)/(\rho) \) becomes a regular \( * \)-semigroup with the set \( \{ e\rho : e \in P \} \) of projections if the \( * \)-operation \( \# \) on \( S(P)/(\rho) \) is defined 
by \( (a\rho)\# = a^+\rho \) (\( a \in S(P) \), \( a^+ \in V_P(a) \)).

The set \( \{ a \in S(P) : a \rho e \text{ for some } e \in E \} \) is called the 
\( [P] \)-kernel of \( \rho \), and it denoted by \( [P]\ker \rho \). The restriction 
\( \rho \cap (E \times E) \) \( [\rho \cap (P \times P)] \) of \( \rho \) is called the \( [P] \)-trace of \( \rho \), and it 
is denoted by \( [P]\tr \rho \).
For any subset $A$ of $S(P)$, define the terminology as follows:

- $A$ is $\mathcal{P}$-full if $E \subseteq A$ ($P \subseteq A$).
- $A$ is a $\mathcal{P}$-subset if $V_p(a) \subseteq A$ for any $a \in A$.
- $A$ is a $\mathcal{P}$-self-conjugate 
  if $x^*Ax \subseteq A$ for any $x \in S(P)$ and $x^* \in V_p(x)$.
- $A$ is weakly closed if $a^2 \in A$ for any $a \in A$.

The following results are fundamental and are used frequently in this abstract.

**Result 1.1** (due to [8] and [9]). Let $a, b \in S(P), e \in E$ and $q \in P$. Then

1. $V_p(b)V_p(a) \subseteq V_p(ab)$,
2. if $a^+ \in V_p(a)$, then $a \in V_p(a^+)$,
3. $V_p(e) \subseteq E$,
4. $q \in V_p(q)$.

**Result 1.2** (due to [2]). Let $\rho$ be a $\mathcal{P}$-congruence on $S(P)$ and $a, b \in S(P)$. Then $a \rho b$ if and only if 

$$ba' \in \ker\rho, \quad aa' \rho bb' aa', \quad b' b \rho b' ba' a$$

for some $a' \in V(a)$ and $b' \in V(b)$.

In section 2, for a given $\mathcal{P}$-congruence $\rho$ on a $\mathcal{P}$-regular semigroup $S(P)$, the maximum and the minimum $\mathcal{P}$-congruences on $S(P)$ whose traces coincide with $tr\rho$ are determined, and the properties for those $\mathcal{P}$-congruences are given.

The concept introduced in section 3 is "$\mathcal{P}$-congruence pairs". This concept is a characterization of the pair $(tr\rho, \ker\rho)$ associated with a given $\mathcal{P}$-congruence $\rho$ on $S(P)$, and the
pair uniquely determines the $\mathcal{P}$-congruence $\kappa$ such that $\text{tr}\kappa = \text{tr}\rho$ and $\ker\kappa = \ker\rho$.

We use the notation and terminology of [3] and [9] unless otherwise stated.

2. $\mathcal{P}$-congruences with the same trace. For any $\mathcal{P}$-congruence $\rho$ on $S(P)$, define a relation $\rho_{\text{max}}$ on $S(P)$ as follows:

$$\rho_{\text{max}} = \{(a,b) : \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \text{ such that } aea^+ \rho beb^+aea^+, \quad beb^+ \rho aea^+beb^+, \quad a^+ea \rho a^+eb^+eb \rho b^+eb^+ea \text{ for all } e \in P\}.$$ 

Then we can easily see that

$$\rho_{\text{max}} = \{(a,b) : aea^+ \rho beb^+aea^+, \quad beb^+ \rho aea^+beb^+, \quad a^+ea \rho a^+eb^+eb \rho b^+eb^+ea \text{ for all } a^+ \in V_P(a), \quad b^+ \in V_P(b) \text{ and } e \in P\}$$

Lemma 2.1. Let $\rho$ be a $\mathcal{P}$-congruence on $S(P)$ and $a, b \in S(P)$. If $a \rho_{\text{max}} b$, then

$$aa^+ \rho bb^+aa^+, \quad bb^+ \rho aa^+bb^+, \quad a^+a \rho a^+ab^+b, \quad b^+b \rho b^+ba^+a$$

for any $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$.

Theorem 2.2. For any $\mathcal{P}$-congruence $\rho$ on a $\mathcal{P}$-regular semigroup $S(P)$, $\rho_{\text{max}}$ is the greatest $\mathcal{P}$-congruence on $S(P)$ whose trace coincides with $\text{tr}\rho$.

Theorem 2.3. For any orthodox $\mathcal{P}$-congruence $\rho$ on $S(P)$,
\( \rho_{\text{max}} \) is the greatest orthodox \( \mathcal{P} \)-congruence on \( S(P) \) whose trace coincides with \( \text{tr}\rho \).

From now on, denote the maximum idempotent-separating congruence on a semigroup \( T \) by \( \mu_T \).

**Corollary 2.4** (compare with [8, Proposition 4.1]). The maximum idempotent-separating \( \mathcal{P} \)-congruence \( \mu_{S(P)} \) on \( S(P) \) is given as follows:

\[
\mu_{S(P)} = \{(a,b) : \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \text{ such that } aea^+ = beb^+aea^+, \text{ beb}^+ = aea^+beb^+, \text{ a}^+ea = a^+eab^+eb \text{ and } b^+eb = b^+eba^+ea \text{ for all } e \in P \}. \\
= \{(a,b) : aea^+ = beb^+aea^+, \text{ beb}^+ = aea^+beb^+, \text{ a}^+ea = a^+eab^+eb \text{ and } b^+eb = b^+eba^+ea \text{ for all } a^+ \in V_P(a), b^+ \in V_P(b) \text{ and } e \in P \}
\]

Let \( S \) be an orthodox semigroup and \( E \) the band of idempotents of \( S \). Then it is easy to check that \( S(E) \) is a \( \mathcal{P} \)-regular semigroup with a C-set \( E \) in \( S \). So we have immediately

**Corollary 2.5** ([1, Theorem 4.2]). Let \( \rho \) be a congruence on an orthodox semigroup \( S \) with the band \( E \) of idempotents of \( S \). Then

\[
\rho_{\text{max}} = \{(a,b) : \text{there exist } a' \in V(a) \text{ and } b' \in V(b) \text{ such that } aea' \rho \text{ beb'aea}', \text{ beb'} \rho \text{ aea'beb'}, \text{ a'ea } \rho \\
\text{a'eab'eb, b'eb } \rho \text{ b'eba'ea for any } e \in E \} \\
= \{(a,b) : aea' \rho \text{ beb'aea}', \text{ beb'} \rho \text{ aea'beb'}, \text{ a'ea } \rho
\]
a'eab'eb, b'eb ρ b'eba'ea for any a' ∈ V(a), b' ∈ V(b) and e ∈ E}
is the greatest congruence on S whose trace coincides with trρ.

On the other hand, the minimum P-congruence on S(P) with the same trace is given as follows:

**Theorem 2.6.** For any P-congruence ρ on a P-regular semigroup S(P), define a relation ρ₀ on S(P) by

ρ₀ = {(a,b): there exist x, y ∈ S(P)¹ and e, f ∈ E
such that a = xey, b = xfy and e ρ f}

Then ρₘᵟₐₙ = ρ₀ᵗ, the transitive closure of ρ₀, is the least P-congruence on S(P) whose trace coincides with trρ. In other words, the least P-congruence on S(P) with trρ as its trace is the P-congruence on S(P) generated by trρ.

The following corollary gives us the characterization which is different from both [1, Theorem 4.1] and [7, Theorem 3.3], of the least congruence on an orthodox semigroup with the same trace.

**Corollary 2.7.** For any congruence ρ on an orthodox semigroup S, the congruence generated by trρ is the least congruence on S whose trace coincides with trρ.

**Proposition 2.9.** For any P-congruence ρ on S(P), ρ = ρₘᵟₐₓ if and only if S(P)/(ρ)ₚ is a fundamental P-regular
For any \( P \)-congruences \( \rho \) and \( \sigma \) on \( S(P) \) such that \( \rho \subseteq \sigma \), define a relation \( \sigma / \rho \) on \( S(P)/(\rho) \) by
\[
\sigma / \rho = \{(a \rho, b \rho) : (a, b) \in \sigma\}
\]

**Proposition 2.10.** For any \( P \)-congruence \( \rho \) on \( S(P) \), \( \rho_{\text{max}} / \rho \) is the maximum idempotent-separating \( P \)-congruence on \( S(P)/(\rho) \).

Let \( \Lambda \) be the lattice of all \( P \)-congruences on \( S(P) \). Define a relation \( \Theta \) on \( \Lambda \) as follows: for any \( \rho, \sigma \in \Lambda \),
\[
\rho \Theta \sigma \quad \text{if and only if} \quad tr \rho = tr \sigma.
\]
It immediately follows from Theorems 2.2 and 2.6 that \( \rho \Theta \), the \( \Theta \)-class containing \( \rho \in \Lambda \), is the interval \([ \rho_{\text{min}}, \rho_{\text{max}} ]\) of \( \Lambda \).

**Proposition 2.11 ([6, Theorem 5.1]).** If \( P \)-congruences \( \rho \) and \( \sigma \) on \( S(P) \) are \( \Theta \)-equivalent, then \( \rho \sigma = \sigma \rho \). Therefore, for any \( \rho \in \Lambda \), \( \rho \Theta \) is a complete modular subsemilattice of \( \Lambda \).

**Proposition 2.12.** Let \( \xi \in \Lambda \), and let \( \Gamma \) be the lattice of all idempotent-separating \( P \)-congruences on \( S(P)/(\xi_{\text{min}}) \). Then the mapping \( \rho \rightarrow \rho / \xi_{\text{min}} \) is a complete isomorphism of \( \xi \Theta \) onto \( \Gamma \).

3. **\( P \)-congruence pairs.** Let \( \xi \) be an equivalence on \( E \). Then \( \xi \) is called a normal equivalence on \( E \) if it satisfies the following conditions: for any \( a \in S(P) \) and \( e, f, g, h, i, j, k \in \)}
E,

(a) if \( e \not\in \xi f \) and \( aea^+ \in E \) for some \( a^+ \in V_p(a) \),
then \( aea^+ \in afa^+ \).

(b) if \( e \not\in \xi f \), \( g \not\in \xi h \) and \( eg, fh \in E \), then \( eg \not\in \xi fh \).

(c) if \( \Box \not\in (e\xi)(f\xi) \cap E \subset h\xi \), \( \Box \not\in (f\xi)(g\xi) \cap E \subset i\xi \) and
\( \Box \not\in (e\xi)(i\xi) \cap E \subset j\xi \) [\( \Box \not\in (h\xi)(g\xi) \cap E \subset k\xi \)],
then \( \Box \not\in (h\xi)(g\xi) \cap E \) [\( \Box \not\in (e\xi)(i\xi) \cap E \)] and \( j \not\in k \).

Let \( \xi \) be a normal equivalence on \( E \). Define a partial
binary operation \( \cdot \) on \( E/\xi \) as follows: for any \( e, f, g \in E \),
\( e\xi \cdot f\xi = g\xi \), where \( \Box \not\in (e\xi)(f\xi) \cap E \subset g\xi \).

It is easy to verify that the partial binary operation \( \cdot \) is
well-defined. The partial groupoid \( E/\xi \) satisfies the
following:

\( (w) \) if \( e\xi \cdot f\xi, f\xi \cdot g\xi \) and \( e\xi \cdot (f\xi \cdot g\xi) \)
\( [(e\xi \cdot f\xi) \cdot g\xi] \) are
defined in \( E/\xi \), then \( (e\xi \cdot f\xi) \cdot g\xi \) \( [e\xi \cdot (f\xi \cdot g\xi)] \) is
defined in \( E/\xi \) and \( (e\xi \cdot f\xi) \cdot g\xi = e\xi \cdot (f\xi \cdot g\xi) \).

Let \( K \) be a weakly closed full \( \mathcal{P} \)-subset of \( S(P) \) and \( \xi \) a
normal equivalence on \( E \). Then the pair \( (\xi, K) \) is called a \( \mathcal{P} \)-
congruence pair for \( S(P) \) if its satisfies the following
conditions: for any \( a, b, c \in S(P), c^+ \in V_p(c), e, f, g \in E \) and \( q \in P \),

\( (C1) \) \( a \in K \) implies \( a^+ a \xi a^+ a a \) for any \( a^+ \in V_p(a) \),

\( (C2) \) \( aefb \in K \) and \( e\xi \cdot f\xi = (a^+ a)\xi \) for some \( a^+ \in V_p(a) \)
imply \( ab \in K \),

\( (C3) \) \( ab^+ \in K \) and \( ab^+ \xi b^+ a a^+ \), \( b^+ b \xi b^+ b a^+ a \) for some
\( a^+ \in V_p(a) \) and \( b^+ \in V_p(b) \) imply \( aqb^+ \in K \) and
\( aqa^+ \xi bqb^+ aq^+, b^+ qb \xi b^+ q b a^+ a q a \).
(C4) $a, b \in K, a^* \xi e^*a^*, e^* \xi a^*a^*, a^*a \xi a^*e^*a, e^*e \xi e^*a^*e, b^*b \xi f^*f^*b, f^*f \xi f^*f b^*b$ and $e^*f \xi f$ for some $a^* \in V_\mathcal{P}(a), b^* \in V_\mathcal{P}(b), e^* \in V_\mathcal{P}(e)$ and $f^* \in V_\mathcal{P}(f)$ imply $ab \in K$,

(C5) $aq \in K$ and $aa^* \xi qaa^*, q \xi qa^*a$ for some $a^* \in V_\mathcal{P}(a)$ imply $cac^* \in K$.

For any $\mathcal{P}$-congruence pair $(\xi, K)$ for $S(\mathcal{P})$, define a relation $\kappa(\xi, K)$ on $S(\mathcal{P})$ as follows:

$$\kappa(\xi, K) = \{(a, b) : ab^* \in K \text{ and } a^* \xi bb^*a^*, bb^* \xi a^*bb^*, a^*a \xi a^*ab^*b, b^*b \xi b^*ba^*a \text{ for some any } a^* \in V_\mathcal{P}(a) \text{ and } b^* \in V_\mathcal{P}(b)\}.$$ 

Now we can determine $\mathcal{P}$-congruences on $S(\mathcal{P})$ by $\mathcal{P}$-congruence pairs.

**Theorem 3.1.** For any $\mathcal{P}$-congruence pair $(\xi, K)$ for a $\mathcal{P}$-regular semigroup $S(P)$, $\kappa(\xi, K)$ is a $\mathcal{P}$-congruence on $S(P)$ such that $\text{tr}\kappa(\xi, K) = \xi$ and $\text{ker}\kappa(\xi, K) = K$. Conversely, for any $\mathcal{P}$-congruence $\rho$ on $S(P)$, $(\text{tr}\rho, \text{ker}\rho)$ is a $\mathcal{P}$-congruence pair for $S(P)$ and $\rho = \kappa(\text{tr}\rho, \text{ker}\rho)$.

Let $\mathcal{A}$ be the set of $\mathcal{P}$-congruence pairs for $S(P)$. Define an order $\prec$ on $\mathcal{A}$ by

$$(\xi_1, K_1) \prec (\xi_2, K_2) \text{ if and only if } \xi_1 \subset \xi_2, K_1 \subset K_2.$$

**Corollary 3.2.** The mappings
\[(\xi, K) \rightarrow \kappa(\xi, K), \quad \rho \rightarrow (\text{tr}_\rho, \ker \rho)\]
are mutually inverse order-preserving mappings of \(A\) onto \(A\) and of \(A\) onto \(A\), respectively. Therefore, \(A\) forms a complete lattice.

References

CODEC Co., Ltd.
2-11, Taira, 1-chome, Miyamae-ku, Kawasaki, 216  Japan
and
Department of Mathematics
Shimane University
Matsue, Shimane, 690  Japan