SOME REMARKS ON $\mathcal{P}$-CONGRUENCES ON $\mathcal{P}$-REGULAR SEMIGROUPS I
- $\mathcal{P}$-congruence pairs -

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Yamada and Sen introduced the new concept of $\mathcal{P}$-regularity in the class of regular semigroups which is a generalization of both the concepts of "orthodox" and "(special) involution" (see [8],[9]). The purpose of this abstract is to characterize congruences on a $\mathcal{P}$-regular semigroups by using "$\mathcal{P}$-congruence pairs", which is a generalization of Petrich [7] for inverse semigroup and one of the authors [4] for regular $\star$-semigroups. Also, for a given congruence $\rho$ on a $\mathcal{P}$-regular semigroup $S$, the maximum and the minimum congruences on $S$ whose traces coincide with the trace of $\rho$ ($=\rho \cap E(S) \times E(S)$) are determined.

1. Introduction. Let $S$ be a regular semigroup and $E$ the set of idempotents of $S$. Let $P \subseteq E$. If $S$ satisfies the following, it is called a $\mathcal{P}$-regular semigroup:

(1) $P^2 \subseteq E$,
(2) $qPq \subseteq P$ for any $q \in P$,
(3) for any $a \in S$, there exists $a^+ \in V(a)$ (the set of all inverses of $a$) such that $a^+P^1a \subseteq P$ and $aP^1a^+ \subseteq P$.

In such a case, $S$ is denoted by $S(P)$ and $P$ is called a C-set in $S$. Throughout this paper, let $S(P)$ be a $\mathcal{P}$-regular semigroup
with a C-set $P$ such that the set of idempotents of $S$ is $E$. Let

$a \in S(P)$ and $a^+ \in V(a)$. If $a^+$ satisfies that $a^+P^1a \subseteq P$ and

$aP^1a^+ \subseteq P$, it is called a $P$-inverse of $a$, and the set of $P$

-inverses of $a$ is denoted by $V_P(a)$. An element of a C-set $P$ in

$S$ is called a projection. The class of $P$-regular semigroups

contains both the classes of orthodox semigroups and regular

$*$-semigroups. A good account of the concept of $P$-regularity

can be seen in [8] and [9].

A congruence on $S$ is sometimes called a $P$-congruence on

$S(P)$. Let $\rho$ be a $P$-congruence on $S(P)$, and put $\bar{x} = x\rho$ for any

$x \in S$, $\bar{S} = \{\bar{x}: x \in S\}$ and $\bar{P} = \{q: q \in P\}$. Then $\bar{S}(\bar{P})$ is also a

$P$-regular semigroup with a C-set $\bar{P}$. So $\bar{S}(\bar{P})$ is called the

factor $P$-regular semigroup of $S(P)$ mod. $\rho$, and it is denoted by

$S(P)/(\rho)_P$.

Let $\rho$ be a $P$-congruence on $S(P)$. Then it is called an

orthodox $P$-congruence on $S(P)$ if $S(P)/(\rho)_P$ is an orthodox

semigroup, and it is called a strong $P$-congruence on $S(P)$ if it

satisfies that for $a \in S(P)$ and $e \in P$,

$$a \rho e \text{ implies } a^+ \rho e \text{ for all } a^+ \in V_P(a).$$

As was seen in [8], if $\rho$ is a strong $P$-congruence on $S(P)$, then

$S(P)/(\rho)_P$ becomes a regular $*$-semigroup with the set \( \{e\rho: e \in P\} \)

of projections if the $*$-operation $#$ on $S(P)/(\rho)_P$ is defined

by $(a\rho)^# = a^+\rho (a \in S(P), a^+ \in V_P(a))$.

The set $\{a \in S(P): a \rho e \text{ for some } e \in E \text{ [e } \in P]\}$ is called the

$[P]$-kernel of $\rho$, and it denoted by $[P]$ker$\rho$. The restriction

$\rho \cap (ExE) \text{ [ } \rho \cap (P \times P) \text{ ] of } \rho$ is called the $[P]$-trace of $\rho$, and it

is denoted by $[P]$tr$\rho$. 
For any subset $A$ of $S(P)$, define the terminology as follows:

- $A$ is **full** if $E \subseteq A$ [$P \subseteq A$],

- $A$ is a **$P$-subset** if $V_P(a) \subseteq A$ for any $a \in A$,

- $A$ is a **$P$-self-conjugate** if $x^*Ax \subseteq A$ for any $x \in S(P)$ and $x^+ \in V_P(x)$.

- $A$ is **weakly closed** if $a^2 \in A$ for any $a \in A$.

The following results are fundamental and are used frequently in this abstract.

**Result 1.1** (due to [8] and [9]). Let $a, b \in S(P)$, $e \in E$ and $q \in P$. Then

1. $V_P(b)V_P(a) \subseteq V_P(ab),$

2. if $a^+ \in V_P(a)$, then $a \in V_P(a^+),$

3. $V_P(e) \subseteq E,$

4. $q \in V_P(q).$

**Result 1.2** (due to [2]). Let $\rho$ be a **$P$-congruence** on $S(P)$ and $a, b \in S(P)$. Then $a \rho b$ if and only if

$ba' \in \ker \rho$, $aa' \rho$, $bb'aa'$, $b'b \rho$, $b'ba' a$

for some $a' \in V(a)$ and $b' \in V(b)$.

In section 2, for a given $P$-congruence $\rho$ on a $P$-regular semigroup $S(P)$, the maximum and the minimum $P$-congruences on $S(P)$ whose traces coincide with $\tr \rho$ are determined, and the properties for those $P$-congruences are given.

The concept introduced in section 3 is "$P$-congruence pairs". This concept is a characterization of the pair $(\tr \rho, \ker \rho)$ associated with a given $P$-congruence $\rho$ on $S(P)$, and the
pair uniquely determines the $\mathcal{P}$-congruence $\kappa$ such that $\text{tr}_\kappa = \text{tr}_\rho$ and $\ker\kappa = \ker\rho$.

We use the notation and terminology of [3] and [9] unless otherwise stated.

2. $\mathcal{P}$-congruences with the same trace. For any $\mathcal{P}$-congruence $\rho$ on $S(P)$, define a relation $\rho_{\text{max}}$ on $S(P)$ as follows:

$$\rho_{\text{max}} = \{(a,b) : \text{ there exist } a^+ \in V_\mathcal{P}(a) \text{ and } b^+ \in V_\mathcal{P}(b) \text{ such that } aea^+ \rho beb^+aea^*, beb^+ \rho aea^*beb^*, a^+ea \rho a^+eab^*eb \text{ and } b^+eb \rho b^+eba^*ea \text{ for all } e \in \mathcal{P}\}.$$ 

Then we can easily see that

$$\rho_{\text{max}} = \{(a,b) : aea^+ \rho beb^*aea^*, beb^+ \rho aea^*beb^*, a^+ea \rho a^+eab^*eb \text{ and } b^+eb \rho b^+eba^*ea \text{ for all } a^+ \in V_\mathcal{P}(a), b^+ \in V_\mathcal{P}(b) \text{ and } e \in \mathcal{P}\}.$$ 

**Lemma 2.1.** Let $\rho$ be a $\mathcal{P}$-congruence on $S(P)$ and $a, b \in S(P)$. If $a \rho_{\text{max}} b$, then

$$aa^+ \rho bb^*aa^*, bb^+ \rho aa^*bb^*, a^+a \rho a^+ab^*b, b^+b \rho b^*ba^*a$$

for any $a^+ \in V_\mathcal{P}(a)$ and $b^+ \in V_\mathcal{P}(b)$.

**Theorem 2.2.** For any $\mathcal{P}$-congruence $\rho$ on a $\mathcal{P}$-regular semigroup $S(P)$, $\rho_{\text{max}}$ is the greatest $\mathcal{P}$-congruence on $S(P)$ whose trace coincides with $\text{tr}_\rho$.

**Theorem 2.3.** For any orthodox $\mathcal{P}$-congruence $\rho$ on $S(P)$,
$\rho_{\text{max}}$ is the greatest orthodox $\mathcal{P}$-congruence on $S(P)$ whose trace coincides with $\text{tr}\rho$.

From now on, denote the maximum idempotent-separating congruence on a semigroup $T$ by $\mu_T$.

**Corollary 2.4** (compare with [8, Proposition 4.1]). The maximum idempotent-separating $\mathcal{P}$-congruence $\mu_{S(P)}$ on $S(P)$ is given as follows:

$$
\mu_{S(P)} = \{(a,b) : \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \text{ such that } aea^+ = beb^+a^+, \text{ beb}^+ = aea^+beb^+, a^+ea = a^+eab^+eb \text{ and } b^+eb = b^+eba^+ea \text{ for all } e \in P\}.
$$

$$
= \{(a,b) : aea^+ = beb^+a^+, \text{ beb}^+ = aea^+beb^+, a^+ea = a^+eab^+eb \text{ and } b^+eb = b^+eba^+ea \text{ for all } a^+ \in V_P(a), b^+ \in V_P(b) \text{ and } e \in P\}
$$

Let $S$ be an orthodox semigroup and $E$ the band of idempotents of $S$. Then it is easy to check that $S(E)$ is a $\mathcal{P}$-regular semigroup with a C-set $E$ in $S$. So we have immediately

**Corollary 2.5** ([1, Theorem 4.2]). Let $\rho$ be a congruence on an orthodox semigroup $S$ with the band $E$ of idempotents of $S$. Then

$$
\rho_{\text{max}} = \{(a,b) : \text{there exist } a' \in V(a) \text{ and } b' \in V(b) \text{ such that } aea' \rho b'eb, \text{ beb'} \rho aea'beb', \text{ a'ea } \rho a'eab'eb, \text{ b'eb } \rho b'eba'ea \text{ for any } e \in E\}
$$

$$
= \{(a,b) : aea' \rho b'eb, \text{ beb'} \rho aea'beb', \text{ a'ea } \rho a'eab'eb, \text{ b'eb } \rho b'eba'ea \text{ for any } e \in E\}
$$
a’eab’eb, b’eb \rho b’eba’ea for any a' \in V(a), b' \in V(b) and e \in E\}
is the greatest congruence on S whose trace coincides with tr\rho.

On the other hand, the minimum \mathcal{P}-congruence on S(P) with the same trace is given as follows:

**Theorem 2.6.** For any \mathcal{P}-congruence \rho on a \mathcal{P}-regular semigroup S(P), define a relation \rho_0 on S(P) by

\[
\rho_0 = \{(a, b) : \text{there exist } x, y \in S(P)^{\dagger} \text{ and } e, f \in E \text{ such that } a = xey, b = xfy \text{ and } e \rho f\}
\]

Then \rho_{\min} = \rho_0^t, the transitive closure of \rho_0, is the least \mathcal{P}-congruence on S(P) whose trace coincides with tr\rho. In other words, the least \mathcal{P}-congruence on S(P) with tr\rho as its trace is the \mathcal{P}-congruence on S(P) generated by tr\rho.

The following corollary gives us the characterization which is different from both [1, Theorem 4.1] and [7, Theorem 3.3], of the least congruence on an orthodox semigroup with the same trace.

**Corollary 2.7.** For any congruence \rho on an orthodox semigroup S, the congruence generated by tr\rho is the least congruence on S whose trace coincides with tr\rho.

**Proposition 2.9.** For any \mathcal{P}-congruence \rho on S(P), \rho = \rho_{\max} if and only if S(P)/(\rho) \mathcal{P} is a fundamental \mathcal{P}-regular
For any $\mathcal{P}$-congruences $\rho$ and $\sigma$ on $S(P)$ such that $\rho \subseteq \sigma$, define a relation $\sigma/\rho$ on $S(P)/(\rho)\mathcal{P}$ by
\[ \sigma/\rho = \{(a\rho, b\rho) : (a, b) \in \sigma\} \]

**Proposition 2.10.** For any $\mathcal{P}$-congruence $\rho$ on $S(P)$, $\rho_{\text{max}}/\rho$ is the maximum idempotent-separating $\mathcal{P}$-congruence on $S(P)/(\rho)\mathcal{P}$.

Let $\Lambda$ be the lattice of all $\mathcal{P}$-congruences on $S(P)$. Define a relation $\Theta$ on $\Lambda$ as follows: for any $\rho, \sigma \in \Lambda$,
\[ \rho \Theta \sigma \quad \text{if and only if} \quad \text{tr}\rho = \text{tr}\sigma. \]
It immediately follows from Theorems 2.2 and 2.6 that $\rho \Theta$, the $\Theta$-class containing $\rho \in \Lambda$, is the interval $[\rho_{\text{min}}, \rho_{\text{max}}]$ of $\Lambda$.

**Proposition 2.11 ([6, Theorem 5.1]).** If $\mathcal{P}$-congruences $\rho$ and $\sigma$ on $S(P)$ are $\Theta$-equivalent, then $\rho \sigma = \sigma \rho$. Therefore, for any $\rho \in \Lambda$, $\rho \Theta$ is a complete modular subsemilattice of $\Lambda$.

**Proposition 2.12.** Let $\xi \in \Lambda$, and let $\Gamma$ be the lattice of all idempotent-separating $\mathcal{P}$-congruences on $S(P)/(\xi_{\text{min}})\mathcal{P}$. Then the mapping $\rho \mapsto \rho/\xi_{\text{min}}$ is a complete isomorphism of $\xi \Theta$ onto $\Gamma$.

### 3. $\mathcal{P}$-congruence pairs.
Let $\xi$ be an equivalence on $E$.
Then $\xi$ is called a normal equivalence on $E$ if it satisfies the following conditions: for any $a \in S(P)$ and $e, f, g, h, i, j, k \in E$, etc.
E,

(a) if \( e \in \xi \) f and \( a e a^+ \in E \) for some \( a^+ \in V_P(a) \),

then \( a e a^+ \in \xi \) afa^+.

(b) if \( e \in \xi \) f, g \( \in \xi \) h and \( e g \) fh \( \in E \), then \( e g \) fh.

(c) if \( \square \not\in (e \xi)(f \xi) \cap E \subset h \xi \), \( \square \not\in (f \xi)(g \xi) \cap E \subset i \xi \) and

\( \square \not\in (e \xi)(i \xi) \cap E \subset j \xi \) \( \square \not\in (h \xi)(g \xi) \cap E \subset k \xi \),

then \( \square \not\in (h \xi)(g \xi) \cap E \) \( \square \not\in (e \xi)(i \xi) \cap E \) and \( j \xi \subset k \xi \).

Let \( \xi \) be a normal equivalence on \( E \). Define a partial binary operation \( \ast \) on \( E/\xi \) as follows: for any \( e, f, g \in E \),

\[ e \xi \ast f \xi = g \xi, \text{ where } \square \not\in (e \xi)(f \xi) \cap E \subset g \xi. \]

It is easy to verify that the partial binary operation \( \ast \) is well-defined. The partial groupoid \( E/\xi \) satisfies the following:

(w) if \( e \xi \ast f \xi \), \( f \xi \ast g \xi \) and \( e \xi \ast (f \xi \ast g \xi) = (e \xi \ast f \xi) \ast g \xi \) are

defined in \( E/\xi \), then \( (e \xi \ast f \xi) \ast g \xi = e \xi \ast (f \xi \ast g \xi) \) is

defined in \( E/\xi \) and \( (e \xi \ast f \xi) \ast g \xi = e \xi \ast (f \xi \ast g \xi) \).

Let \( K \) be a weakly closed full \( \mathcal{P} \)-subset of \( S(P) \) and \( \xi \) a

normal equivalence on \( E \). Then the pair \((\xi, K)\) is called a \( \mathcal{P} \)-congruence pair for \( S(P) \) if its satisfies the following conditions: for any \( a, b, c \in S(P), c^+ \in V_P(c) \), \( e, f, g \in E \) and \( q \in P \),

\[ (C1) \quad a \in K \text{ implies } a^+ a \xi a^+ a^+ a \text{ for any } a^+ \in V_P(a), \]

\[ (C2) \quad a e f b \in K \text{ and } e \xi \ast f \xi = (a^+ a) \xi \text{ for some } a^+ \in V_P(a) \]

imply \( a b \in K \),

\[ (C3) \quad a b^+ \in K \text{ and } a a^+ \xi b b^+ a^+, b^+ b \xi b^+ b a^+ \text{ for some } \]

\( a^+ \in V_P(a) \) and \( b^+ \in V_P(b) \) imply \( a q b^+ \in K \) and

\[ a q a^+ \xi b q b^+ a q a^+, b^+ q b \xi b^+ q b a^+ q a, \]
(C4) \( a, b \in K, \; aa^+ \xi \; ee^aa^+, \; ee^+ \xi \; aa^+ee^+, \; a^+a \xi \; a^ae^+e, \)
\[ e^e \xi \; e^eaa^+, \; bb^+ \xi \; ff^bb^+, \; ff^+ \xi \; bb^ff^+, \]
\[ b^+b \xi \; b^+bf^+f, \; f^+f \xi \; f^+fb^+b \text{ and } e^e \xi f^+f = g^e \text{ for some} \]
\[ a^+ \in V_p(a), \; b^+ \in V_p(b), \; e^+ \in V_p(e) \text{ and } f^+ \in V_p(f) \]
\[ \text{imply } ab \in K, \]
\[ (C5) \; aq \in K \text{ and } aa^+ \xi \; qaa^+, \; q \xi \; qa^a \text{ for some } a^+ \in V_p(a) \]
\[ \text{imply } cac^+ \in K. \]

For any \( \mathcal{P} \)-congruence pair \((\xi, K)\) for \( S(P) \), define a relation \( \kappa(\xi, K) \) on \( S(P) \) as follows:
\[
\kappa(\xi, K) = \{(a, b) : ab^+ \in K \text{ and } aa^+ \xi \; bb^+aa^+, \; bb^+ \xi \}
\[ \text{aa}^+bb^+, \; a^+a \xi \; a^+ab^+b, \; b^+b \xi \; b^+ba^+a \text{ for some} \]
\[ \text{any} \; a^+ \in V_p(a) \text{ and } b^+ \in V_p(b) \}. \]

Now we can determine \( \mathcal{P} \)-congruences on \( S(P) \) by \( \mathcal{P} \)-congruence pairs.

**Theorem 3.1.** For any \( \mathcal{P} \)-congruence pair \((\xi, K)\) for a \( \mathcal{P} \)
-regular semigroup \( S(P) \), \( \kappa(\xi, K) \) is a \( \mathcal{P} \)-congruence on \( S(P) \) such
that \( \text{tr} \kappa(\xi, K) = \xi \) and \( \ker \kappa(\xi, K) = K \). Conversely, for any \( \mathcal{P} \)
-congruence \( \rho \) on \( S(P) \), \( (\text{tr} \rho, \ker \rho) \) is a \( \mathcal{P} \)-congruence pair for \( S(P) \) and \( \rho = \kappa(\text{tr} \rho, \ker \rho) \).

Let \( \mathcal{A} \) be the set of \( \mathcal{P} \)-congruence pairs for \( S(P) \). Define
an order \( < \) on \( \mathcal{A} \) by
\[
(\xi_1, K_1) < (\xi_2, K_2) \text{ if and only if } \xi_1 \subset \xi_2, \; K_1 \subset K_2. \]

**Corollary 3.2.** The mappings
\[(\xi, K) \rightarrow \kappa(\xi, K), \quad \rho \rightarrow (\text{tr}\rho, \ker\rho)\]

are mutually inverse order-preserving mappings of \(\mathcal{A}\) onto \(\Lambda\) and of \(\Lambda\) onto \(\mathcal{A}\), respectively. Therefore, \(\mathcal{A}\) forms a complete lattice.

References


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