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<th>SOME REMARKS ON $\mathcal{P}$-CONGRUENCES ON $\mathcal{P}$-REGULAR SEMIGROUPS I: $\mathcal{P}$-congruence pairs (Algebraic Theory of Codes and Combinatorics on Words)</th>
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<tr>
<td>Author(s)</td>
<td>Okamoto, Y.; Imaoka, T.</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1992), 786: 32-42</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1992-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/82589">http://hdl.handle.net/2433/82589</a></td>
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<td>Rights</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
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<td>Textversion</td>
<td>publisher</td>
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SOME REMARKS ON $\mathcal{P}$-CONGRUENCES ON $\mathcal{P}$-REGULAR SEMIGROUPS I

- $\mathcal{P}$-congruence pairs -

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Yamada and Sen introduced the new concept of $\mathcal{P}$-regularity in the class of regular semigroups which is a generalization of both the concepts of "orthodox" and "(special) involution" (see [8],[9]). The purpose of this abstract is to characterize congruences on a $\mathcal{P}$-regular semigroups by using "$\mathcal{P}$-congruence pairs", which is a generalization of Petrich [7] for inverse semigroup and one of the authors [4] for regular $\star$-semigroups.

Also, for a given congruence $\rho$ on a $\mathcal{P}$-regular semigroup $S$, the maximum and the minimum congruences on $S$ whose traces coincide with the trace of $\rho$ ($= \rho \cap E(S) \times E(S)$) are determined.

1. Introduction. Let $S$ be a regular semigroup and $E$ the set of idempotents of $S$. Let $P \subseteq E$. If $S$ satisfies the following, it is called a $\mathcal{P}$-regular semigroup:

1. $P^2 \subseteq E$,
2. $qPq \subseteq P$ for any $q \in P$,
3. for any $a \in S$, there exists $a^+ \in V(a)$ (the set of all inverses of $a$) such that $a^+P^1a \subseteq P$ and $aP^1a^+ \subseteq P$.

In such a case, $S$ is denoted by $S(P)$ and $P$ is called a C-set in $S$. Throughout this paper, let $S(P)$ be a $\mathcal{P}$-regular semigroup
with a C-set $P$ such that the set of idempotents of $S$ is $E$. Let
$a \in S(P)$ and $a^+ \in V(a)$. If $a^+$ satisfies that $a^+ P^1 a \subset P$ and $a P^1 a^+ \subset P$, it is called a \textit{$P$-inverse} of $a$, and the set of $P$-
inverses of $a$ is denoted by $V_P(a)$. An element of a C-set $P$ in $S$ is called a \textit{projection}. The class of $P$-regular semigroups contains both the classes of orthodox semigroups and regular $*$-semigroups. A good account of the concept of $P$-regularity can be seen in [8] and [9].

A congruence on $S$ is sometimes called a \textit{$P$-congruence} on $S(P)$. Let $\rho$ be a $P$-congruence on $S(P)$, and put $\overline{x} = x\rho$ for any $x \in S$, $\overline{S} = \{\overline{x}: x \in S\}$ and $\overline{P} = \{q: q \in P\}$. Then $\overline{S(P)}$ is also a $P$-regular semigroup with a C-set $\overline{P}$. So $\overline{S(P)}$ is called the factor \textit{$P$-regular semigroup} of $S(P)$ mod. $\rho$, and it is denoted by $S(P)/(\rho)_P$.

Let $\rho$ be a $P$-congruence on $S(P)$. Then it is called an \textit{orthodox $P$-congruence} on $S(P)$ if $S(P)/(\rho)_P$ is an orthodox semigroup, and it is called a \textit{strong $P$-congruence} on $S(P)$ if it satisfies that for $a \in S(P)$ and $e \in P$,

$$a \rho e \text{ implies } a^+ \rho e \text{ for all } a^+ \in V_P(a).$$

As was seen in [8], if $\rho$ is a strong $P$-congruence on $S(P)$, then $S(P)/(\rho)_P$ becomes a regular $*$-semigroup with the set $\{e\rho: e \in P\}$ of projections if the $*$-operation $#$ on $S(P)/(\rho)_P$ is defined by $(a\rho)^# = a^+\rho$ ($a \in S(P)$, $a^+ \in V_P(a)$).

The set $\{a \in S(P): a \rho e \text{ for some } e \in E \text{ [e } \in P]\}$ is called the \textit{$[P]$-kernel} of $\rho$, and it denoted by $[P]\ker\rho$. The restriction $\rho \cap (E \times E)$ \textit{[} $\rho \cap (P \times P)$ \textit{]} of $\rho$ is called the \textit{$[P]$-trace} of $\rho$, and it is denoted by $[P]\tr\rho$. 
For any subset $A$ of $S(P)$, define the terminology as follows:

- $A$ is a $\mathcal{P}$-full if $E \subset A$ ($P \subset A$),
- $A$ is a $\mathcal{P}$-subset if $V_P(a) \subset A$ for any $a \in A$,
- $A$ is a $\mathcal{P}$-self-conjugate if $x^*Ax \subset A$ for any $x \in S(P)$ and $x^+ \in V_P(x)$,
- $A$ is weakly closed if $a^2 \in A$ for any $a \in A$.

The following results are fundamental and are used frequently in this abstract.

**Result 1.1** (due to [8] and [9]). Let $a, b \in S(P), e \in E$ and $q \in P$. Then

1. $V_P(b)V_P(a) \subset V_P(ab)$,
2. if $a^+ \in V_P(a)$, then $a \in V_P(a^+)$,
3. $V_P(e) \subset E$,
4. $q \in V_P(q)$.

**Result 1.2** (due to [2]). Let $\rho$ be a $\mathcal{P}$-congruence on $S(P)$ and $a, b \in S(P)$. Then $a \rho b$ if and only if $ba' \in \ker \rho$, $aa' \rho bb'aa'$, $b'b \rho b'ba'a$ for some $a' \in V(a)$ and $b' \in V(b)$.

In section 2, for a given $\mathcal{P}$-congruence $\rho$ on a $\mathcal{P}$-regular semigroup $S(P)$, the maximum and the minimum $\mathcal{P}$-congruences on $S(P)$ whose traces coincide with $\text{tr}\rho$ are determined, and the properties for those $\mathcal{P}$-congruences are given.

The concept introduced in section 3 is "$\mathcal{P}$-congruence pairs". This concept is a characterization of the pair $(\text{tr}\rho, \ker \rho)$ associated with a given $\mathcal{P}$-congruence $\rho$ on $S(P)$, and the
pair uniquely determines the $\mathcal{P}$-congruence $\kappa$ such that $\text{tr}\kappa = \text{tr}\rho$ and $\ker\kappa = \ker\rho$.

We use the notation and terminology of [3] and [9] unless otherwise stated.

2. $\mathcal{P}$-congruences with the same trace. For any $\mathcal{P}$-congruence $\rho$ on $S(P)$, define a relation $\rho_{\text{max}}$ on $S(P)$ as follows:

$$\rho_{\text{max}} = \{(a, b) : \text{there exist } a^+ \in V_\mathcal{P}(a) \text{ and } b^+ \in V_\mathcal{P}(b) \text{ such that } aea^+ \rho beb^+aea^+, beb^+ \rho aea^+beb^+, a^+ea \rho a^+eab^+eb \text{ and } b^+eb \rho b^+eba^+ea \text{ for all } e \in P\}.$$ 

Then we can easily see that

$$\rho_{\text{max}} = \{(a, b) : aea^+ \rho beb^+aea^+, beb^+ \rho aea^+beb^+, a^+ea \rho a^+eab^+eb \text{ and } b^+eb \rho b^+eba^+ea \text{ for all } a^+ \in V_\mathcal{P}(a), b^+ \in V_\mathcal{P}(b) \text{ and } e \in P\}$$

Lemma 2.1. Let $\rho$ be a $\mathcal{P}$-congruence on $S(P)$ and $a, b \in S(P)$. If $a \rho_{\text{max}} b$, then

$$aa^+ \rho bb^+aa^+, bb^+ \rho aa^+bb^+, a^+a \rho a^+ab^+b, b^+b \rho b^+ba^+a$$

for any $a^+ \in V_\mathcal{P}(a)$ and $b^+ \in V_\mathcal{P}(b)$.

Theorem 2.2. For any $\mathcal{P}$-congruence $\rho$ on a $\mathcal{P}$-regular semigroup $S(P)$, $\rho_{\text{max}}$ is the greatest $\mathcal{P}$-congruence on $S(P)$ whose trace coincides with $\text{tr}\rho$.

Theorem 2.3. For any orthodox $\mathcal{P}$-congruence $\rho$ on $S(P)$,
\( \rho_{\text{max}} \) is the greatest orthodox \( \mathcal{P} \)-congruence on \( S(P) \) whose trace coincides with \( \text{tr} \rho \).

From now on, denote the maximum idempotent-separating congruence on a semigroup \( T \) by \( \mu_T \).

**Corollary 2.4** (compare with [8, Proposition 4.1]). The maximum idempotent-separating \( \mathcal{P} \)-congruence \( \mu_{S(P)} \) on \( S(P) \) is given as follows:

\[
\mu_{S(P)} = \{(a,b) : \text{there exist } a^+ \in V_p(a) \text{ and } b^+ \in V_p(b) \text{ such that } aea^+ = beb^+aea^+, \text{ beb}^+ = aea^+beb^+, \text{ a}^+ea = a^+eab^+eb \text{ and } b^+eb = b^+eba^+ea \text{ for all } e \in P\}.
\]

Let \( S \) be an orthodox semigroup and \( E \) the band of idempotents of \( S \). Then it is easy to check that \( S(E) \) is a \( \mathcal{P} \)-regular semigroup with a C-set \( E \) in \( S \). So we have immediately

**Corollary 2.5** ([1, Theorem 4.2]). Let \( \rho \) be a congruence on an orthodox semigroup \( S \) with the band \( E \) of idempotents of \( S \). Then

\[
\rho_{\text{max}} = \{(a,b) : \text{there exist } a' \in V(a) \text{ and } b' \in V(b) \text{ such that } aea' \rho \text{ beb}'aea', \text{ beb}' \rho \text{ aea}'be'b', \text{ a}'ea \rho \text{ a}'eab'e'b, b'eb \rho b'eba'e'a \text{ for any } e \in E\}
\]

= \{(a,b) : aea' \rho \text{ beb}'aea', \text{ beb}' \rho \text{ aea}'be'b', \text{ a}'ea \rho \text{ a}'eab'e'b, b'eb \rho b'eba'e'a \text{ for any } e \in E\}

= \{(a,b) : \text{there exist } a' \in V(a) \text{ and } b' \in V(b) \text{ such that } aea' \rho \text{ beb}'aea', \text{ beb}' \rho \text{ aea}'be'b', \text{ a}'ea \rho \text{ a}'eab'e'b, b'eb \rho b'eba'e'a \text{ for any } e \in E\}

a'eb'eb, b'eb ρ b'eba'ea for any a' ∈ V(a), b' ∈ V(b) and e ∈ E} is the greatest congruence on S whose trace coincides with \( tr_\rho \).

On the other hand, the minimum \( P \)-congruence on \( S(P) \) with the same trace is given as follows:

**Theorem 2.6.** For any \( P \)-congruence \( \rho \) on a \( P \)-regular semigroup \( S(P) \), define a relation \( \rho_0 \) on \( S(P) \) by

\[
\rho_0 = \{(a,b): \text{there exist } x, y \in S(P)^1 \text{ and } e, f \in E \text{ such that } a = xey, b = xfy \text{ and } e \rho f \}
\]

Then \( \rho_{\min} = \rho_0^t \), the transitive closure of \( \rho_0 \), is the least \( P \)-congruence on \( S(P) \) whose trace coincides with \( tr_\rho \). In other words, the least \( P \)-congruence on \( S(P) \) with \( tr_\rho \) as its trace is the \( P \)-congruence on \( S(P) \) generated by \( tr_\rho \).

The following corollary gives us the characterization which is different from both [1, Theorem 4.1] and [7, Theorem 3.3], of the least congruence on an orthodox semigroup with the same trace.

**Corollary 2.7.** For any congruence \( \rho \) on an orthodox semigroup \( S \), the congruence generated by \( tr_\rho \) is the least congruence on \( S \) whose trace coincides with \( tr_\rho \).

**Proposition 2.9.** For any \( P \)-congruence \( \rho \) on \( S(P) \), \( \rho = \rho_{\max} \) if and only if \( S(P)/(\rho) \) is a fundamental \( P \)-regular
For any \( P \)-congruences \( \rho \) and \( \sigma \) on \( S(P) \) such that \( \rho \subseteq \sigma \), define a relation \( \sigma / \rho \) on \( S(P)/(\rho)_P \) by

\[
\sigma / \rho = \{ (a\rho, b\rho) : (a, b) \in \sigma \}
\]

**Proposition 2.10.** For any \( P \)-congruence \( \rho \) on \( S(P) \), \( \rho_{\text{max}} / \rho \) is the maximum idempotent-separating \( P \)-congruence on \( S(P)/(\rho)_P \).

Let \( \Lambda \) be the lattice of all \( P \)-congruences on \( S(P) \). Define a relation \( \Theta \) on \( \Lambda \) as follows: for any \( \rho, \sigma \in \Lambda \),

\[
\rho \Theta \sigma \text{ if and only if } tr\rho = tr\sigma .
\]

It immediately follows from Theorems 2.2 and 2.6 that \( \rho \Theta \), the \( \Theta \)-class containing \( \rho \in \Lambda \), is the interval \([\rho_{\text{min}}, \rho_{\text{max}}]\) of \( \Lambda \).

**Proposition 2.11 ([6, Theorem 5.1]).** If \( P \)-congruences \( \rho \) and \( \sigma \) on \( S(P) \) are \( \Theta \)-equivalent, then \( \rho \sigma = \sigma \rho \). Therefore, for any \( \rho \in \Lambda \), \( \rho \Theta \) is a complete modular subsemilattice of \( \Lambda \).

**Proposition 2.12.** Let \( \xi \in \Lambda \), and let \( \Gamma \) be the lattice of all idempotent-separating \( P \)-congruences on \( S(P)/(\xi_{\text{min}})_P \). Then the mapping \( \rho \rightarrow \rho / \xi_{\text{min}} \) is a complete isomorphism of \( \xi \Theta \) onto \( \Gamma \).

3. \( P \)-congruence pairs. Let \( \xi \) be an equivalence on \( E \). Then \( \xi \) is called a normal equivalence on \( E \) if it satisfies the following conditions: for any \( a \in S(P) \) and \( e, f, g, h, i, j, k \in \)
E,

(a) if $e \xi f$ and $aea^+ \in E$ for some $a^+ \in V_p(a)$, then $aea^+ \xi afa^+$.

(b) if $e \xi f$, $g \xi h$ and $eg \in E$, then $eg \xi fh$.

(c) if $\square \neq (e \xi)(f \xi) \cap E \subset h \xi$, $\square \neq (f \xi)(g \xi) \cap E \subset i \xi$ and $\square \neq (e \xi)(i \xi) \cap E \subset j \xi$, $\square \neq (h \xi)(g \xi) \cap E \subset k \xi$, then $\square \neq (h \xi)(g \xi) \cap E \subset (e \xi)(i \xi) \cap E$ and $j \xi k$.

Let $\xi$ be a normal equivalence on $E$. Define a partial binary operation $\bullet$ on $E/\xi$ as follows: for any $e, f, g \in E$,

$$e\xi \bullet f\xi = g\xi,$$

where $\square \neq (e\xi)(f\xi) \cap E \subset g\xi$.

It is easy to verify that the partial binary operation $\bullet$ is well-defined. The partial groupoid $E/\xi$ satisfies the following:

(w) if $e\xi \bullet f\xi$, $f\xi \bullet g\xi$ and $e\xi \bullet (f\xi \bullet g\xi)$ $[(e\xi \bullet f\xi) \bullet g\xi]$ are defined in $E/\xi$, then $(e\xi \bullet f\xi) \bullet g\xi$ $[e\xi \bullet (f\xi \bullet g\xi)]$ is defined in $E/\xi$ and $(e\xi \bullet f\xi) \bullet g\xi = e\xi \bullet (f\xi \bullet g\xi)$.

Let $K$ be a weakly closed full $\mathcal{P}$-subset of $S(P)$ and $\xi$ a normal equivalence on $E$. Then the pair $(\xi, K)$ is called a $\mathcal{P}$-congruence pair for $S(P)$ if its satisfies the following conditions: for any $a, b, c \in S(P)$, $c^+ \in V_p(c)$, $e, f, g \in E$ and $q \in P$,

(C1) $a \in K$ implies $a^+ a \xi a^+ a a$ for any $a^+ \in V_p(a)$,

(C2) $aefb \in K$ and $e\xi \bullet f\xi = (a^+ a) \xi$ for some $a^+ \in V_p(a)$ imply $ab \in K$,

(C3) $ab^+ \in K$ and $aa^+ \xi b^+ b \xi b^+ b a a$, $b^+ b \xi b^+ b a a$ for some $a^+ \in V_p(a)$ and $b^+ \in V_p(b)$ imply $aqb^+ \in K$ and $aqa^+ \xi bqb^+ aqa^+$, $b^+ qb \xi b^+ qba^+ qa$. 


(C4) \( a, b \in K, \) \( aa^+ \xi ee^+aa^+, ee^+ \xi aa^+ee^+, a^+a \xi a^+ae^+e, \)
\( e^+e \xi e^+ea^+a, bb^+ \xi ff^+bb^+, ff^+ \xi bb^+ff^+, \)
\( b^+b \xi b^+bf^+f, f^+f \xi f^+fb^+b \) \( \text{and} \) \( e\xi f\xi = g\xi \) \( \text{for some} \)
\( a^+ \in V_p(a), b^+ \in V_p(b), e^+ \in V_p(e) \) \( \text{and} \) \( f^+ \in V_p(f) \)
\imply ab \in K,

(C5) \( aq \in K \) \( \text{and} \) \( aa^+ \xi qaa^+, q \xi qa^+a \) \( \text{for some} \) \( a^+ \in V_p(a) \)
\imply cac^+ \in K.

For any \( \mathcal{P} \)-congruence pair \((\xi, K)\) for \( S(P) \), define a relation
\( \kappa(\xi, K) \)
on \( S(P) \) as follows:

\[
\kappa(\xi, K) = \{(a, b) : ab^+ \in K \text{ and } aa^+ \xi bb^+aa^+, bb^+ \xi \\
aa^+bb^+, a^+a \xi a^+ab^+b, b^+b \xi b^+ba^+a \text{ for some} \}
\]
\( \text{any} \) \( a^+ \in V_p(a) \) \( \text{and} \) \( b^+ \in V_p(b) \} \).

Now we can determine \( \mathcal{P} \)-congruences on \( S(P) \) by \( \mathcal{P} \)-congruence pairs.

**Theorem 3.1.** For any \( \mathcal{P} \)-congruence pair \((\xi, K)\) for a \( \mathcal{P} \)
-regular semigroup \( S(P) \), \( \kappa(\xi, K) \) is a \( \mathcal{P} \)-congruence on \( S(P) \) such
that \( \text{tr}_\kappa(\xi, K) = \xi \) \( \text{and} \) \( \ker_\kappa(\xi, K) = K \). Conversely, for any \( \mathcal{P} \)
-congruence \( \rho \) on \( S(P) \), \( (\text{tr}_\rho, \ker_\rho) \) is a \( \mathcal{P} \)-congruence pair for
\( S(P) \) \( \text{and} \) \( \rho = \kappa(\text{tr}_\rho, \ker_\rho) \).

Let \( \mathcal{A} \) be the set of \( \mathcal{P} \)-congruence pairs for \( S(P) \). Define
an order \( \prec \) on \( \mathcal{A} \) by

\[
(\xi_1, K_1) \prec (\xi_2, K_2) \text{ if and only if } \xi_1 \subset \xi_2, K_1 \subset K_2.
\]

**Corollary 3.2.** The mappings
\((\xi, K) \rightarrow \kappa(\xi, K), \quad \rho \rightarrow (\text{tr}_\rho, \ker \rho)\)

are mutually inverse order-preserving mappings of \(\mathcal{A}\) onto \(\Lambda\) and of \(\Lambda\) onto \(\mathcal{A}\), respectively. Therefore, \(\mathcal{A}\) forms a complete lattice.

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