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A CONVERGENCE BALL FOR MULTISTEP SIMPLIFIED NEWTON-LIKE METHODS

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Abstract: In this paper, we give a convergence ball for multistep simplified Newton-like methods for solving nonlinear equations with nondifferentiable operator, which describes exactly the relation between the multistep simplified number and the convergence domain, and contains known convergence balls for several iterative methods as special cases.

Key words. A convergence ball, Newton-like methods

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Let f and g be operators in a domain D of a Banach space X, and consider the equation

$$F(x)=f(x)+g(x)=0, \tag{1}$$

where f is Frechet differentiable in an open convex set $D_0 \subset D$, while the differentiability of g is not assumed.

In this paper, we consider m-step simplified Newton-like method starting from $x^0 \in D_0$

$$x^{k,0}=x^k, \quad x^{k,i}=x^{k,i-1}-A(x^k)^{-1}F(x^{k,i-1}), 1\leqq i\leqq m,$$
$$x^{k+1}=x^{k,m}, k\geqq 0, \tag{2}$$

for solving the equation (1), where A(x) denotes a linear operator which approximates f'(x). Observe that if m=1, then (2) reduces to the usual Newton-like iteration

$$x^{k+1}=x^k-A(x^k)^{-1}F(x^k), \quad k\geqq 0, \tag{3}$$

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which includes the simplifed Newton-like method

$$x^{k+1}=x^k-A^{-1}F(x^k),\quad k\geqq 0, \tag{4}$$

with a constant linear operator A. The sequence $\{x^{0,i}\}_{i=1}^{\infty}$ $(m=+\infty)$ with $A=A(x^0)$ also defines the simplified Newton-like method.

Denoting by x^* a solution of (1), we say that the open ball $B(x^*,r)$ with center x^* and radius r is a convergence ball for (2), if the iteration (2) starting from any $x^0\in B(x^*,r)$ converges to the solution x^*.

The convergence for iteration (2) with g=0 has been considered in much literature, and the existence for local convergence domain was shown in [1;Theorem 10.2.4 and NR 10.2-2]. Here, we give a local convergence ball, which describes exactly the relation between the step-number m and the convergence domain, and contains known convergence balls(Rall[2] and Rheinboldt[3]) for several iterative methods as special cases.

We assume that there exist a solution $x^*\in D_0$ of the equation (1), constants $\bar{r}>0$, $q>0$, $w>0$ and a nonsingular linear operator P, such that for $x,y\in B(x^*,\bar{r})\subset D_0$, $A(x)^{-1}$ exists and

$$\|A(x)^{-1}P\|\leqq q$$

$$\|A(x)^{-1}F(x)\|\leqq \eta$$

$$\|P^{-1}(f'(x)-A(y))\|\leqq K\|x-y\|+c$$

$$\|P^{-1}(g(x)-g(y))\|\leqq e\|x-y\|.$$

Define the sequence $\{t_{k,i}\}$ by

$$t_{k,0}=0,\ t_{k,i}=q(\tfrac{K}{2}t_{k,i-1}+c+e)t_{k,i-1}+\eta_k,\ i=1,..,m+1, k=0,1,...$$

$$\eta_0=\eta,\quad \eta_k=t_{k-1,m+1}-t_{k-1,m}\ (k\geqq 1).$$

Obviously, the sequence $\{t_{k,i}\}$ satisfies

$$t_{k,i}\leqq t_{k,i+1}.$$

Theorem 1. Under the above notation and assumptions, we take a number t^* such that

$$t^* \geq \min \{\max_k t_{k,m-1},\ 2\bar{r}\}.$$

If $b=q(\frac{K}{2}t^*+c+e)<1$, and if we put $\hat{r}=\frac{2(1-b)}{qK}$, then the ball $B(x^*,r)$, with $r=\min\{\bar{r},\hat{r}\}$ is a convergence ball of the iteration (2) with any m and

$$\|x^{k+1}-x^*\|=\|x^{k,m}-x^*\|\leq(a\|x^k-x^*\|+b)^m\|x^k-x^*\|\leq\rho^m\|x^k-x^*\|, \quad (5)$$

where $a=\frac{qK}{2}$ and $\rho=a\|x^0-x^*\|+b<1$.

Corollary 1. Assume that there exists a solution $x^*\in D_0$ of the equation (1), $A(x^*)^{-1}$ exists and for any $x\in D_0$, the following hold:

$$\|A(x^*)^{-1}(f'(x)-A(y))\|\leq K\|x-y\|+c \quad (8)$$

$$\|A(x^*)^{-1}(A(x)-A(x^*))\|\leq L\|x-x^*\|+d$$

$$\|A(x^*)^{-1}(g(x)-g(x^*))\|\leq e\|x-x^*\|$$

$$p=c+d+e<1.$$

Then:

(i) The ball $B(x^*,r)\subset D_0$ with $r=2(1-p)/(3K+2L)$ is a convergence ball for the iteration (2) with any m and

$$\|x^{k+1}-x^*\|=\|x^{k,m}-x^*\|\leq(a\|x^k-x^*\|+b)^m\|x^k-x^*\|\leq\rho^m\|x^k-x^*\|,$$

where $a=(3K)/(2(1-Lr-d))$, $b=(c+e)/(1-Lr-d)$ and $\rho=a\|x^0-x^*\|+b<1$.

(ii) The ball $B(x^*,r)\subset D$ with $r=\frac{2(1-p)}{K+2L}$ is a convergence ball for the iteration (3) and

$$\|x^{k+1}-x^*\|\leq\frac{1}{1-Lr-d}\left(\frac{K}{2}\|x^k-x^*\|+c+e\right)\|x^k-x^*\|.$$

Remark 1. If we replace the condition (8) by

$$\|A(x^*)^{-1}(f'(x)-A(x^*))\|\leq K\|x-x^*\|+c,$$

then the assertion (i) of Corollary 1 holds with $r=2(1-p-b)/(K+4L)$. In fact, we have

$$\|P^{-1}(f'(x^*+t(x^{k,i}-x^*))-A(x^k))\|$$
$$\leq\|P^{-1}(f'(x^*+t(x^{k,i}-x^*))-A(x^*))\|+\|P^{-1}(A(x^*)-A(x^k))\|,$$

so that, replacing $x^{0,i}$ and x^0 in the proof of Theorem 1 by $x^{k,i}$ and x^k, respectively, we obtain

$$\|x^{k,i+1}-x^*\|\leq q\{\frac{K}{2}\|x^{k,i}-x^*\|+c+L\|x^k-x^*\|+d+e\}\|x^{k,i}-x^*\|.$$

This means that the constants a and b in Corollary 1 may be replaced by $\bar{a}=q(K/2+L)$ and $\bar{b}=qp$, respectively. Then we have $\bar{a}r+\bar{b}<1$.

Now, we apply the result to the m-step simplified Newton method starting from $x^0\in D_0$

$$x^{k,0}=x^k,\quad x^{k,i}=x^{k,i-1}-f'(x^k)^{-1}f(x^{k,i-1}),\ 1\leq i\leq m, \tag{9}$$
$$x^{k+1}=x^{k,m},\ k\geq 0$$

for solving the nonlinear equation

$$f(x)=0. \tag{10}$$

Then we have the following result.

Corollary 2. (i)Let $f'(x^*)$ be nonsingular. If f' satisfies

$$\|f'(x^*)^{-1}(f'(x)-f'(x^*))\|\leq K\|x-x^*\|,\quad x\in D_0$$

then the ball $B(x^*,r)\subset D_0$ with $r=\frac{2}{5K}$ is a convergence ball for the iteration (9) with any m, and

$$\|x^{k+1}-x^*\|=\|x^{k,m}-x^*\|\leq a^m\|x^k-x^*\|^{m+1},$$

where $a=3K/(2(1-Kr))$.

(ii) (Rheinboldt[3]) If

$$\|f'(x^*)^{-1}(f'(x)-f'(y))\|\leq K\|x-y\|,\quad x,y\in D_0,$$

then the ball $B(x^*,r)\subset D_0$ with $r=2/(3K)$ is a convergence ball for Newton's method

$$x^{k+1}=x^k-f'(x^k)^{-1}f(x^k),\quad k\geq 0.$$

Remark 2. Rall[2] gave a convergence ball for Newton's method $B(x^*,r)\subset D_0$ with $r=(1-(\sqrt{2})^{-1})/K$, where Kantorovich's hypotheses are satisfied. This result can be extended to multistep simplified Newton method. In fact, if we assume that $f'(x^*)^{-1}$ exists and put $P=f'(x^*)$, $A(x)=f'(x)$, $g(x)=0$, in Theorem 1, then we can prove that for any $x\in B(x^*,r)$,

$$\|f'(x)^{-1}f'(x^*)\|\leqq q=1/(1-Kr),$$

$$\|f'(x)^{-1}f(x)\|\leqq \eta=(r-\tfrac{K}{2}r^2)/(1-Kr)$$

and $Kq\eta\leqq \frac{1}{2}$ (also see [2]). Furthermore, we define a sequence $\{t_{k,m}\}$ by

$$t_{0,0}=0,\quad t_{0,1}=\eta,\quad t_{k,i}=q\tfrac{K}{2}t_{k,i-1}^2+\eta,\quad i=2,\dots,m+1,$$

$$t_{k+1,0}=t_{k,m},\quad t_{k+1,1}=t_{k,m+1},\qquad k=0,1,\dots\ .$$

Then from the proof of Theorem 1 we see that the sequence $\{t_{k,i}\}$ is a majorant sequence for $\{x^{k,i}\}$, that is, (6) holds. Since $\{t_{k,m}\}$ converges to the smallest root of the equation $t=q\frac{K}{2}t^2+\eta$, as $k\to\infty$, with any m, we can conclude from (6) that $\{x^{k,i}\}$ converges to a solution x^* of (10), that is, the ball $B(x^*,r)$ is a convergence ball for the multistep simplified Newton method (9) with any m. Furthermore, we note that both convergence balls $B(x^*,2/(5K))$ (with any m) and $B(x^*,2/(3K))$ (with m=1) contain Rall's ball.

Remark 3. The above discussion implies the semilocal convergence for the multistep simplified Newton method under Kantorovich's conditions: If there is an x^0 such that $f'(x^0)^{-1}$ exists, $\|f'(x^0)^{-1}(f'(x)-f'(y))\|\leqq K\|x-y\|$ and $h=K\eta\leqq\frac{1}{2}$, where $\eta=\|f'(x^0)^{-1}f(x^0)\|$, then the multistep simplified Newton method starting from x^0 converges to a solution x^* of (10) and $x^*\in B(x^0,t^*)$, $t^*=\frac{1-\sqrt{1-2h}}{K}$. Furthermore, we have error estimates $\|x^*-x^{k,i}\|\leq t^*-t_{k,i}$, which include the usual error estimates $\|x^*-x^{k,1}\|\leq t^*-t_{k,1}$ for Newton's method (m=1), where the sequence $\{t_{k,i}\}$ is defined by

$$t_{0,0}=0,\quad t_{0,1}=\eta,\quad t_{k,i}=t_{k,i}-\frac{\phi(t_{k,i})}{\phi'(t_k)},\quad i=2,\dots,m+1$$

$$t_{k+1,0}=t_{k,m},\quad t_{k+1,1}=t_{k,m+1},\quad k=0,1,\dots$$

and $\phi(t)=\frac{K}{2}t^2-t+\eta$.

Example 1. We use the example in [2] to illustrate that the ball $B(x^*,r),r=2/(3K)$ is not a convergence ball for the multistep simplified Newton method with m>1.

We consider the equation $F(x)=f(x)=\frac{1}{2}(x^2-x^{*2})$ in R^1, where $x^*\neq 0$. Then we have $\|f'(x^*)^{-1}(f'(x)-f'(y))\|\leq K\|x-y\|$ with $K=\frac{1}{x^*}$. Take $x^0=\frac{1}{3}x^*$, then $x^*-x^0=\frac{2}{3}x^*=\frac{2}{3K}$ and

$$x^{0,1}-x^*=x^0-\frac{1}{2x^0}(x^0+x^*)(x^0-x^*)-x^*=x^*-x^0=\frac{2}{3K}.$$

If m=1, we have

$$x^1=x^{0,1}$$

$$x^{1,1}-x^*=x^1-\frac{1}{2x^1}(x^1+x^*)(x^1-x^*)-x^*=\frac{1}{5}(x^*-x^1).$$

Hence $\{x^k\}$ converges to x^* because of the convexity of f.

If m=2, then

$$x^{0,2}-x^*=x^{0,1}-\frac{1}{2x^0}(x^{0,1}+x^*)(x^{0,1}-x^*)-x^*$$

$$=-3(x^{0,1}-x^*)=-2x^*,$$

and

$$x^1=x^{0,2}=-x^*.$$

That is, $\{x^k\}$ converges to $-x^*$. Since the behaviour of the sequence $\{x^k\}$ depends continuously on x^0, the above result implies that if m>1, $x^0\in B(x^*,2/(3K))$ and x^0 is sufficiently close to $x^*/3$, then we have $x^1=x^{0,m}\in B(-x^*,2/(5K))$. Hence $\{x^k\}$ converges to $-x^*$ by Corollary 2 and $B(x^*,r)$, $r=2/(3K)$ is not a convergence ball for the iteration (9).

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