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On Protter-Weinberger's Algorithm for Obtaining Upper and Lower Bounds for the Initial Value Problem of O.D.E.

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Abstract. We are interested in a scheme due to M. H. Protter and H. F. Weinberger [1] for obtaining upper and lower bounds for the linear initial value problems of ordinary differential equations of the second order. An applicability to practical computation is tested by using interval arithmetic.

1. Introduction

It is one of important subjects in modern numerical analysis to find a numerical solution for differential equations with a prescribed accuracy, or to find upper and lower bounds for the exact solution.

The following theorem may be found in M. H. Protter and H. F. Weinberger [1]:

Theorem 1. *Consider the initial value problem*

$$(L+h)[u]\equiv u''+g(x)u'+h(x)u=f(x),\quad x\geq a, \tag{1}$$

$$u(a)=\gamma_1,\quad u'(a)=\gamma_2, \tag{2}$$

where

$$g(x),\ h(x) \text{ and } f(x) \text{ are bounded and } h(x)\leq 0 \text{ for } a\leq x\leq b. \tag{3}$$

Suppose that we can find the functions $\bar{z}(x)$ *and* $\tilde{z}(x)$ *with the properties*

$$(L+h)[\bar{z}]\leq f(x) \text{ for } a\leq x\leq b, \tag{4}$$

$$\bar{z}(a)\leq\gamma_1,\quad \bar{z}'(a)\leq\gamma_2, \tag{5}$$

and

$$(L+h)[\tilde{z}]\geq f(x) \text{ for } a\leq x\leq b,$$

$$\tilde{z}(a)\geq\gamma_1,\quad \tilde{z}'(a)\geq\gamma_2.$$

Then we have

$$\bar{z}(x)\leq u(x)\leq\tilde{z}(x),\quad \bar{z}'(x)\leq u'(x)\leq\tilde{z}'(x) \text{ for } a\leq x\leq b.$$

Futhermore, they have described an algorithm obtaining upper and lower bounds $\bar{z}(x)$ and $\tilde{z}(x)$. Hence a question arises: Is the sheme applicable to practical problems? In this paper, we test the applicability by a simple example. Results for the case of the sign of $h(x)$ being plus will be discussed in the forthcoming paper.

2. An Algorithm for Obtaining Upper and Lower Bounds

The argorithm due to Protter and Weinberger is stated as follows:

Algorithm 1. We divide the interval $[a, b]$ into subintervals, for instance

$$a=x_0<x_1<\cdots<x_{N-1}<x_N=b.$$

We shall select $\bar{z}(x)$ to be a quadratic polynomial in each subinterval

$$\bar{z}(x)=\bar{z}_i(x)\equiv\bar{c}_i(x-x_i)^2+\bar{d}_i(x-x_i)+\bar{e}_i, \text{ for } x_i\le x\le x_{i+1}, \quad i=0,1,2,\cdots,N-1,$$

where the constants $\bar{c}_i$, $\bar{d}_i$, $\bar{e}_i$ and the number N will be chosen so that all required conditions (4), (5) are satisfied. We first remark that the inequality

$$(L+h)[\bar{z}]\le f(x)$$

becomes

$$\bar{c}_i[2+2g(x)(x-x_i)+h(x)(x-x_i)^2]+g(x)\bar{d}_i+h(x)[\bar{d}_i(x-x_i)+\bar{e}_i]\le f(x) \qquad (6)$$

for $x_i\le x\le x_{i+1}$.

If x_{i+1} is so close to x_i that the coefficient of $\bar{c}_i$ in (6) is positive, then we can take $\bar{c}_i$ so small that (6) holds, since $g(x)$, $h(x)$ and $f(x)$ are bounded on $[a, b]$. Accordingly, we can chose $\bar{c}_i$, $\bar{d}_i$, $\bar{e}_i$ as follows:

From (5), we set

$$\bar{e}_0=\bar{z}_0(x_0)=\bar{z}(a)=\gamma_1,$$

$$\bar{d}_0=\bar{z}'_0(x_0)=\bar{z}'(a)=\gamma_2.$$

To insure the continuity of $\bar{z}$ and $\bar{z}'$, we choose

$$\bar{e}_{i+1}=\bar{c}_i(x_{i+1}-x_i)^2+\bar{d}_i(x_{i+1}-x_i)+\bar{e}_i,$$

$$\bar{d}_{i+1}=2\bar{c}_i(x_{i+1}-x_i)+\bar{d}_i, \quad i=0,1,2,\cdots,N-1,$$

where $\bar{c}_i$ will be chosen so that (6) holds at each step.

3. Programming

In numerical computation, we use interval arithmetic to avoid that rounding-off errors violate the property of the lower bound and obtain a useful value of $\bar{c}_i$. That is, we set $\bar{e}_{i+1}$, $\bar{d}_{i+1}$, and $\bar{c}_{i+1}$ to the lower bounds of the intervals

$$\bar{c}_i(x_{i+1}-x_i)^2+\bar{d}_i(x_{i+1}-x_i)+\bar{e}_i,$$

$$2\bar{c}_i(x_{i+1}-x_i)+\bar{d}_i,$$

and

$$\{f([x_i, x_{i+1}])-g([x_i, x_{i+1}])\bar{d}_i-h([x_i, x_{i+1}])(\bar{d}_i[0, x_{i+1}-x_i]+\bar{e}_i)\}$$

$$/\{2+2g([x_i, x_{i+1}])[0, x_{i+1}-x_i]+h([x_i, x_{i+1}])[0, x_{i+1}-x_i]^2\},$$

respectively.

We then realize machine interval arithmetic on Macintosh SE/30, whose numerical environment is so-called Standard Apple Numerical Environment (SANE) which is the implementation of IEEE Standard 754 (cf. [2]).

4. Numerical Result

We now show the computational result of the Algorithm 1 applied to the problem (1), (2), and (3), with

$$g(x) = x\left(x - \frac{1}{2}\right)(x - 1) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$

$$h(x) = -\left(x - \frac{1}{2}\right)^2 = -x^2 + x - \frac{1}{4},$$

$$f(x) = \frac{1287}{8}x^9 - \frac{8151}{32}x^8 - \frac{5511}{64}x^7 + \frac{9009}{32}x^6 + \frac{68355}{64}x^5 - \frac{2205}{32}x^4 - \frac{53865}{64}x^3 + \frac{35}{32}x^2 + \frac{7525}{64}x,$$

$$u(x) = \frac{429}{16}x^7 - \frac{693}{16}x^5 + \frac{315}{16}x^3 - \frac{35}{16}x,$$

and

$$[a, b] = [0, 1].$$

Figures 1-4 are the graphs of u and $\bar{z}$, in which the interval [0, 1] is divided into 2^5, 2^6, 2^7, and 2^8 equally spaced subintervals, respectively.

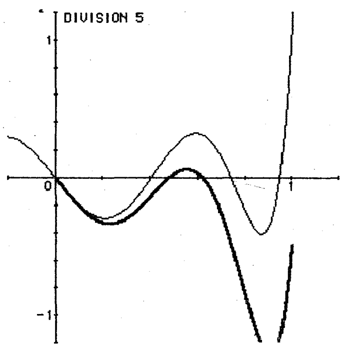

Fig. 1

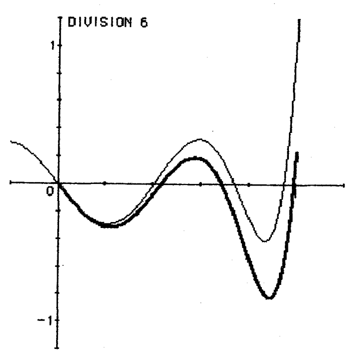

Fig. 2

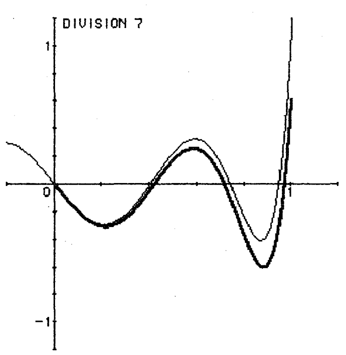

Fig. 3

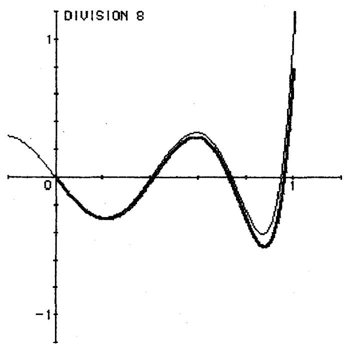

Fig. 4

List 1 shows the values of x_i, $\bar{e}_i$ (which is equal to $\bar{z}(x_i)$), and the error $\bar{z}(x_i)$-$u(x_i)$ and computation time, when the number of divisions N=2^6, 2^8, $\cdots$, 2^{16}.

```
? DIVISION 6
Xi= 2.5000000e-1    Ei=-3.0313528e-1    Error: -0.023216618843
Xi= 5.0000000e-1    Ei= 1.3493995e-1    Error: -0.088204581242
Xi= 7.5000000e-1    Ei=-2.7657491e-1    Error: -0.242391409028
Xi= 1.0000000e+0    Ei= 2.7396954e-1    Error: -0.726030450813
00h00m01s31t

? DIVISION 8
Xi= 2.5000000e-1    Ei=-2.8568750e-1    Error: -0.005768838216
Xi= 5.0000000e-1    Ei= 2.0205694e-1    Error: -0.021087586439
Xi= 7.5000000e-1    Ei=-8.9114330e-2    Error: -0.054930828025
Xi= 1.0000000e+0    Ei= 8.3138204e-1    Error: -0.168617951255
00h00m06s55t

? DIVISION 10
Xi= 2.5000000e-1    Ei=-2.8134926e-1    Error: -0.001430594403
Xi= 5.0000000e-1    Ei= 2.1803968e-1    Error: -0.005104850571
Xi= 7.5000000e-1    Ei=-4.7318260e-2    Error: -0.013134758516
Xi= 1.0000000e+0    Ei= 9.5908239e-1    Error: -0.040917602221
00h00m27s05t

? DIVISION 12
Xi= 2.5000000e-1    Ei=-2.8027615e-1    Error: -0.000357485603
Xi= 5.0000000e-1    Ei= 2.2187228e-1    Error: -0.001272248811
Xi= 7.5000000e-1    Ei=-3.7455251e-2    Error: -0.003271749575
Xi= 1.0000000e+0    Ei= 9.8978600e-1    Error: -0.010213995585
00h01m48s33t

? DIVISION 14
Xi= 2.5000000e-1    Ei=-2.8000799e-1    Error: -0.000089325465
Xi= 5.0000000e-1    Ei= 2.2282713e-1    Error: -0.000317398818
Xi= 7.5000000e-1    Ei=-3.4999081e-2    Error: -0.000815579497
Xi= 1.0000000e+0    Ei= 9.9745133e-1    Error: -0.002548662773
00h07m14s09t

? DIVISION 16
Xi= 2.5000000e-1    Ei=-2.7994100e-1    Error: -0.000022330258
Xi= 5.0000000e-1    Ei= 2.2306520e-1    Error: -0.000079328962
Xi= 7.5000000e-1    Ei=-3.4387330e-2    Error: -0.000203828038
Xi= 1.0000000e+0    Ei= 9.9936294e-1    Error: -0.000637056665
00h29m03s07t
```

List 1

Acknowledgment. The author wishes to thank Professor T. Yamamoto of Ehime University for bringing the reference [1] to his attention and for giving the opportunity of this report.

References

[1] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations* (Springer-Verlag, 1984).

[2] The Official Publications from Apple Computer, Inc., *Apple Numerics Manual, Second Edition* (Addison-Wesley, 1988).