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Boundary value problems and variational inequalities

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1. Introduction and notations

Let Ω be a bounded open set in $\mathbf{R}^n$ $(n \geq 2)$. We consider the obstacle problem for a quasilinear elliptic operator of second order in Ω as follows:

$$L = -\text{div } A(x, u, \nabla u) + B(x, u, \nabla u). \tag{1.1}$$

Here A (*resp.* B) is a vector (*resp.* scalar) valued function defined on $\Omega \times \mathbf{R} \times \mathbf{R}^n$ and the functions A and B are assumed to satisfy the following inequalities:

$$|A(x, u, w)| \leq a(|w|^{p-1} + |u|^{p-1} + 1),$$
$$|B(x, u, w)| \leq b(|w|^{p-1} + |u|^{p-1} + 1),$$
$$w \cdot A(x, u, w) + uB(x, u.w) \geq c_1|w|^p - c_2(|u|^p + 1),$$

$$(A(x, u, w_1) - A(x, u, w_2)) \cdot (w_1 - w_2) > 0 \quad (w_1 \neq w_2)$$

for all $x \in \Omega$, $u \in \mathbf{R}$ and w_1, $w_2 \in \mathbf{R}^n$, where a, b, c_1 p are positive real numbers satisfying $p > 1$ and c_2 is a nonnegative real number.

It has been known that for each continuous function f on $\partial\Omega$ and a function ϕ on Ω satisfying

$$\int_0^\infty t^{p-1} B_{1,p}(A(t))dt < +\infty$$

there exists a solution $u \in W^{1,p}_{loc}(\Omega)$ to the obstacle problem with boundary data f, where

$$A(t) = \{y \in \Omega;\ \psi(y) > t\}.$$

The obstacle problem is to find a function $u \in W^{1,p}_{loc}(\Omega)$ such that

$$u \geq \psi \text{ on } \Omega \text{ except for a subset of } \Omega,$$

$$u = f \text{ weakly on } \partial\Omega$$

and

$$\int_\Omega A(y,u(y),\nabla u(y))\cdot\nabla\phi(y)dy+\int_\Omega B(y,u(y),\nabla u(y))\phi(y)dy\geq 0$$

for all $\phi\in C_o^\infty(\Omega)$ satisfying $\phi\geq\psi-u$ on Ω except for a subset of Ω.

In this paper we will prove the existence of a weak solution to the obstacle problem for (possibly) non-bounded boundary function f. To consider boundary functions which value $\pm\infty$, we must distinglish functions not up to a set of n-dimensional Lebesgue measure zero, but up to a more fine set, for example, a set of $B_{1,s}$-capacity zero.

Recall that for $s>1$ the Bessel capacity $B_{1,s}$ with order 1 is defined by

$$B_{1,s}(E)=\inf\{\|g\|_s^s; g\in L^s(\mathbf{R}^n),\ g\geq 0, G_1*g\geq 1 \text{ on } E\}$$

for a subset E of $\mathbf{R}^n$. If a property holds on a subset X of $\mathbf{R}^n$ except for a set of $B_{1,s}$-capacity zero, we say that it holds $B_{1,s}$-q.e. on X. In the case $s=p$ we use simply "q.e." instead of "$B_{1,p}$-q.e.".

To distinglish functions up to a set of $B_{1,s}$-capacity zero, we construct a family of functions defined on $\partial\Omega$, which contains all continuous functions and the restrictions of all Bessel potentials G_1*g ($g\in L^s(\mathbf{R}^n)$) to $\partial\Omega$, where G_1 is the Bessel function with order 1, i.e.,

$$G_1(x)=\frac{1}{(4\pi)^{1/2}}\frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty \exp(-\frac{\pi|x|^2}{t})\exp(-\frac{t}{4\pi})t^{(1-n)/2}\cdot\frac{1}{t})dt.$$

Recall that the Fourier function of G_1 is equal to

$$\frac{1}{(1+4\pi|x|^2)^{1/2}}.$$

Let us define, for each extended real-valued function f on $\partial\Omega$,

$$\gamma_{1,s}(f)=\inf\{\|g\|_s;\ g\in L^s(\mathbf{R}^n), g\geq 0,\ G_1*g\geq|f| \text{ on } \partial\Omega\}.$$

Furthermore, denote by $\mathcal{B}(\gamma_{1,s})$ the family of all Borel measurable functions on $\partial\Omega$ such that $\gamma_{1,s}(f)<+\infty$. We remark that $\mathcal{B}(\gamma_{1,s})\supset C(\partial\Omega)$, where $C(\partial\Omega)$ is the family of all continuous real-valued functions on $\partial\Omega$.

We denote by $\mathcal{L}(\gamma_{1,s})$ the family of all $f\in\mathcal{B}(\gamma_{1,s})$ such that $\gamma_{1,s}(f-f_j)\to 0$ ($j\to\infty$) for some $\{f_j\}\subset C(\partial\Omega)$.

It is well-known that, if $1<p<n$ and $v\in W^{1,p}(\Omega)$, then

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)}v(y)dy \tag{1.2}$$

exists (as a real number) for all $x \in \Omega$ except for a set E with $B_{1,p}(E) = 0$, where $B(x,r)$ is the ball with center x and radius r and $|B(x,r)|$ stands for the volume of the ball $B(x,r)$ (cf.[FZ]). For each $v \in W^{1,p}(\Omega)$ we denote by v^* the function defined by (1.2) q.e. on Ω.

Under these notations we will prove the following theorem.

Theorem. *Let $1 < p < n$, $p < s \leq \frac{np}{n-p}$, $f \in \mathcal{L}(\gamma_{1,s})$ and ψ be a real-valued function on Ω such that $|\psi| \leq G_1 * g$ on Ω for some $g \in L^s(\mathbf{R}^n)^+$. If*

$$\limsup_{y \to x} \psi(y) < +\infty \text{ and } \limsup_{y \to x} \psi(y) \leq f(y)$$

for all $x \in \partial\Omega$. Then there exists a function $u \in W^{1,p}_{loc}(\Omega)$ having the following properties:

(i) *$u \geq \psi$ q.e. on Ω,*

(ii) *If $\phi \in W^{1,p}(\Omega)$, supp $\phi \subset \Omega$ and $\phi^* \geq \psi - u$ q.e. on Ω, then*

$$\int_\Omega \{A(y, u(y), \nabla u(y)) \cdot \nabla\phi(y) + B(y, u(y), \nabla u(y))\phi(y)\} dy \geq 0.$$

(iii) *$u = f$ on $\partial\Omega$ in the following sense:*

Let τ be a function on $\bar{\Omega}$ such that it is lower semi-continuous on $\bar{\Omega} \setminus K$ for some compact subset K of Ω and

$$\tau\,|_\Omega \in W^{1,s}(\Omega), \quad \tau \geq f + \delta \text{ on } \partial\Omega \text{ for some } \delta > 0.$$

Then $(u - \tau)^+ \in W^{1,p}_0(\Omega)$. Further, let λ be a function on $\bar{\Omega}$ such that it is upper semicontinuous on $\bar{\Omega} \setminus K$ for some compact subset K of Ω and

$$\lambda\,|_\Omega \in W^{1,s}(\Omega), \quad \lambda \leq f - \delta$$

for some $\delta > 0$ on $\partial\Omega$. Then $(u - \lambda)^- \in W^{1,p}_0(\Omega)$.

2. Properties of $\gamma_{1,s}$

In this section we study the properties of $\gamma_{1,s}$. It is easy to see that the functional $\gamma_{1,s}$ has the following properties similar to those of the upper integral.

Lemma 2.1. *Let $s > 1$. Then the functional $\gamma_{1,s}$ has the following properties:*

(c_1) $\gamma_{1,s}(f) = \gamma_{1,s}(|f|)$,

(c_2) $\gamma_{1,s}(bf) = b\gamma_{1,s}(f)$ *for* $b \in \mathbf{R}^+$,

(c_3) $f_j \geq 0 \Rightarrow \gamma_{1,s}(\sum_{j=1}^{\infty} f_j) \leq \sum_{j=1}^{\infty} \gamma_{1,s}(f_j)$,
(c_4) $\gamma_{1,s}(\chi_E) = B_{1,s}(E)^{1/s}$ *for* $E \subset \mathbf{R}^n$.

Using Lemma 2.1, we can show the follwing lemma.

Lemma 2.2. (i) *If* $\gamma_{1,s}(f) < +\infty$, *then the set* $\{x \in \partial\Omega;\ |f(x)| = +\infty\}$ *is of* $B_{1,s}$*-capacity zero.*
(ii) *If* $\gamma_{1,s}(f - g) = 0$, *then* $f = g$ $B_{1,s}$*- q.e. on* $\partial\Omega$.

Lemma 2.3. Let g be a nonnegative function in $L^s(\mathbf{R}^n)$. Then the Bessel potential $G_1 * g$ belongs to $\mathcal{L}(\gamma_{1,s})$.

Proof. We can assume that g is nonnegative. Set

$$g_j = \min\{g,\ j\} \text{ and } h_j = g - g_j.$$

Noting $G_1 \in L^1(\mathbf{R}^n)$, we see that $G_1 * g_j$ is continuous on $\partial\Omega$. Since $|G_1 * g - G_1 * g_j| \leq G_1 * h_j$ and $\|h_j\|_s \to 0$ as $j \to \infty$, we have the conclusion. Q.E.D

Lemma 2.4. *The set* $\mathcal{L}$ *of the restrictions of all Lipschitz functions on* $\bar{\Omega}$ *to* $\partial\Omega$ *is dense in* $\mathcal{L}(\gamma_{1,s})$.

Proof. We can choose a nonnegative function $h = G_1 * g$ $(g \in L^s(\mathbf{R}^n)^+)$ such that $h \geq 1$ on $\partial\Omega$. Since $\mathcal{L}$ is uniformly dense in $C(\partial\Omega)$, it is dense in $\mathcal{L}(\gamma_{1,s})$. Q.E.D.

Noting that

$$G_1(y) = O(e^{-c|y|}) \text{ for some } c > 0 \quad \text{as } |y| \to \infty,$$

we can easily show the following lemma.

Lemma 2.5. *Let* p, s *be positive real numbers satisfying* $1 < p \leq s$ *and* E *be a relatively compact subset of* $\mathbf{R}^n$. *If* $B_{1,s}(E) = 0$, *then* $B_{1,p}(E) = 0$.

For a function f defined on $\mathbf{R}^n$ we define

$$\gamma_{1,s}(f) = \inf\{\|f\|_s;\ g \in L^s(\mathbf{R}^n)^+;\ G_1 * g \geq |f| \text{ on } \mathbf{R}^n\}$$

It is easy to see that this functional $\gamma_{1,s}$ also has the properties in Lemmas 2.1 and 2.2 in which $\partial\Omega$ is replaced by $\mathbf{R}^n$.

Lemma 2.6. *Let* $\{f_j\}$ *be a sequence of functions on* $\mathbf{R}^n$ *such that* $\gamma_{1,s}(f_j) \to 0$ $(j \to \infty)$. *Then there exists a subsequence* $\{g_k\}$ *of* $\{f_j\}$ *such that* $g_k \to 0$ *pointwisely* $B_{1,s}$*-q.e. on* $\mathbf{R}^n$.

Proof. Choose a subsequence $\{g_k\}$ of $\{f_j\}$ satisfying

$$\sum_{k=1}^{\infty} 2^k \gamma_{1,s}(g_{k+1} - g_k) < +\infty. \tag{2.1}$$

To show that $\{g_k\}$ is the desired subsequence, set

$$E = \cup_{k=1}^{\infty}\{x \in \mathbf{R}^n; |g_k(x)| = +\infty\}.$$

Then we have $B_{1,s}(E) = 0$ by Lemmas 2.1 and 2.2. Further set

$$O'_k = \{x \in \mathbf{R}^n \setminus E;\ |g_{k+1}(x) - g_k(x)| \geq 2^{-k}\}$$

and

$$O_k = \cup_{i=k}^{\infty} O'_i \quad \text{and } F_k = \mathbf{R}^n \setminus (O_k \cup E).$$

Setting $g_0 = 0$ and noting that

$$g_k = \sum_{i=0}^{k-1}(g_{i+1} - g_i)$$

on $\mathbf{R}^n \setminus E$, we see that $\{g_k\}$ converges to 0 on $\cup_{i=1}^{\infty} F_i$. We put

$$E_o = \mathbf{R}^n \setminus (\cup_{k=1}^{\infty} F_k \cup E) = \cap_{k=1}^{\infty} O_k.$$

From

$$\chi_{O_k} \leq \sum_{i=k}^{\infty} \chi_{O'_i} \leq \sum_{i=k}^{\infty} 2^i |g_{i+1} - g_i|$$

and Lemma 2.1 we deduce

$$\gamma_{1,s}(\chi_{O_k}) \leq \sum_{i=k}^{\infty} 2^i \gamma_{1,s}(g_{i+1} - g_i).$$

On account of (2.1) we have

$$\gamma_{1,s}(\chi_{O_k}) \to 0 \quad (k \to \infty)$$

and hence $\gamma_{1,s}(\chi_{E_o}) = 0$. Therefore we see by Lemma 2.1 that $B_{1,s}(E \cup E_o)^{1/s} = \gamma_{1,s}(\chi_{E \cup E_o}) = 0$ and $\{g_k\}$ converges to 0 on $\mathbf{R}^n \setminus (E \cup E_o)$. Q.E.D.

3. Boundedness of solutions

For an open subset $\Omega_o \neq \emptyset$ of Ω we denote by $A(\Omega_o, \cdot)$ the mapping

$$W^{1,p}(\Omega_o) \to W^{1,p}(\Omega_o)'$$

defined by

$$\langle A(\Omega_o, v), w\rangle = \int_{\Omega_o} \{A(y, v(y), \nabla v(y)) \cdot \nabla w(y) + B(y, v(y), \nabla v(y))w(y)\}$$

The following theorem is fundamental.

Theorem A ([MZ, Theorem 3.1]). *Let $p < s$ and Ω_o be a nonempty open subset of Ω, $\varepsilon = +$ or $-$ and v, η be functions in $W^{1,p}(\Omega_o)$ such that $(v-\eta)^\varepsilon \in W_o^{1,p}(\Omega_o)$ and*

$$\langle A(\Omega_o, v), -\varepsilon(v-\eta)^\varepsilon\rangle \geq 0.$$

Then

$$\|(v-\eta)^\varepsilon\|_{W^{1,p}(\Omega_o)} \leq c + c\left(1 + \int_{\Omega_o} (|\eta|^s + \sum_{j=1}^n |\frac{\partial \eta}{\partial y_j}|^p) dy\right)^{1/p},$$

where c is a constant independent of v, η.

It is well-known that for each $s > 1$

$$W^{1,s}(\mathbf{R}^n) = \{G_1 * g;\ g \in L^s(\mathbf{R}^n)\}$$

and

$$\frac{1}{M}\|g\|_s \leq \|G_1 * g\|_{W^{1,s}(\mathbf{R}^n)} \leq M\|g\|_s, \tag{3.1}$$

where M is a constant independent of g (cf. [S, Theorem 3 on p.135]).

Lemma 3.1. *Let f be a Lipschitz function on $\bar{\Omega}$ such that $|f| \leq G_1 * g_1$ for some $g_1 \in L^p(\mathbf{R}^n)^+$. Furthermore, let ψ be a real-valued function on Ω such that*

$$\limsup_{y \to x, y \in \Omega} \psi(y) < f(x) - \delta \tag{3.2}$$

for all $x \in \partial\Omega$ and for some $\delta > 0$, and

$$|\psi| \leq G_1 * g_o \text{ for some } g_o \in L^p(\mathbf{R}^n)^+.$$

Then there exists a function $u \in W^{1,p}(\Omega)$ such that u has the properties:

(i) $u \geq \psi$ *q.e. on* Ω,

(ii) *If* $\phi \in W_o^{1,p}(\Omega)$ *and* $\phi^* \geq \psi - u$ *q.e. on* Ω, *then*

$$\int_\Omega \{A(y,u(y),\nabla u(y)) \cdot \nabla\phi(y) + B(y,u(y),\nabla u(y))\phi(y)\}dy \geq 0.$$

(iii) $u - f \in W_o^{1,p}(\Omega)$,

(iv) *If* η *(resp.* λ*) is a function on* $\bar{\Omega}$, *such that it is lower (resp. upper) semicontinuous on* $\bar{\Omega} \setminus K$ *for some compact subset* K *of* Ω *and* $\eta \in W^{1,p}(\Omega)$ *(resp.* $\lambda \in W^{1,p}(\Omega)$*),* $\eta(y) > f(y)$ *(resp.* $\lambda(y) < f(y)$*) for all* $y \in \partial\Omega$, *then* $(u-\eta)^+ \in W_o^{1,p}(\Omega)$ *(resp.* $(u-\lambda)^- \in W_o^{1,p}(\Omega)$*).*

Proof. Set

$$K = \{v \in W^{1,p}(\Omega);\ v - f \in W_o^{1,p}(\Omega),\ v^* \geq \psi \text{ q.e. on } \Omega\}.$$

We claim that K is not empty. Indeed, noting that $G_1 * g_1$ is lower semicontinuous, we can choose, by the aid of (3.2), an open set Ω_o such that $\bar{\Omega}_o \subset \Omega$ and

$$\psi(y) < f(y) \quad \text{for all } y \in \Omega \setminus \Omega_o.$$

Choose a Lipschitz function h such that

$$\text{supp } h \subset \Omega, \quad h = 1 \text{ on } \bar{\Omega}_o, \quad 0 \leq h \leq 1,$$

and define

$$\phi(y) = h(y)w(y) + (1 - h(y))f(y),$$

where $w = G_1 * g_o$. We note that supp h stands for the closure of the set $\{y;\ h(y) \neq 0\}$. Then $\phi^* \geq \psi$ q.e. on Ω and $\phi - f = h(w - f) \in W_o^{1,p}(\Omega)$. Therefore we see that $\phi \in K$.

The family K is a convex closed subset of $W^{1,p}(\Omega)$ and hence weakly closed. The mapping $A(\Omega,\cdot)$ from $W^{1,p}(\Omega)$ to $W^{1,p}(\Omega)'$ is pseudomonotone by Theorem 3.9 in [MZ]. Furthermore we see that

$$\frac{\langle A(\Omega,v), v - v_o\rangle}{\|v\|_{W^{1,p}(\Omega)}} \longrightarrow \infty$$

as $\|v\|_{W^{1,p}(\Omega)} \to \infty$ $(v \in K)$. It follows from Theorem 8.2 on p.247 in [L] that there exists $u_o \in K$ such that

$$\langle A(\Omega,u_o), v - u_o\rangle \geq 0 \text{ for all } v \in K.$$

Setting $u = u_o^*$, we will show that u is the desired function. It is obvious that (i) and (iv) hold. To show (ii), let ϕ be a function in $W_o^{1,p}(\Omega)$ such that

$$\phi^* \geq \psi - u \text{ q.e. on } \Omega.$$

From $u - f \in W_o^{1,p}(\Omega)$ and $\phi + u - f \in W_o^{1,p}(\Omega)$, it follows that

$$\langle A(\Omega, u), \phi \rangle \geq 0.$$

Finally, to show (v), let η be a lower semicontinuous function on $\bar{\Omega}$ in $W^{1,p}(\Omega)$ such that $\eta > f$ on $\partial\Omega$. Since $f - \eta < 0$ outside a compact subset of Ω and $u - f \in W_o^{1,p}(\Omega)$, we have

$$(u - \eta)^+ \in W_o^{1,p}(\Omega).$$

Similarly we can show that $(u - \lambda)^- \in W_o^{1,p}(\Omega)$. Q.E.D.

4. Proof of Theorem

Let us prove Theorem. Suppose that $f \in \mathcal{L}(\gamma_{1,s})$. On account of Lemma 2.4 we can choose a sequence $\{f_j\}$ of Lipschitz functions on $\bar{\Omega}$ and a sequence $\{g_j\}$ of functions in $L^s(\mathbf{R}^n)^+$ such that

$$|f - f_j| \leq G_1 * g_j \text{ on } \partial\Omega \quad \|g_j\|_s < 2^{-j}.$$

Since $\gamma_{1,s}(G_1 * g_j) \to 0$, we can choose, by Lemma 2.6, a subsequence $\{G_1 * h_k\}$ converges pointwisely to 0 $B_{1,s}$-q.e. on $\mathbf{R}^n$. Therefore, by Lemma 2.5, it converges to 0 q.e. on $\bar{\Omega}$. Noting that

$$\limsup_{y \to x} \psi(y) \leq f(x) \leq G_1 * h_k(x) + f_{j_k}(x)$$

for all $x \in \partial\Omega$, we define

$$\psi_k(y) = \psi(y) - G_1 * h_k(y) - 2^{-k} \quad \text{if } G_1 * h_k(y) < +\infty$$

and

$$\psi_k(y) = -\sup_{x \in \partial\Omega} |f_{j_k}(x)| - 1 \quad \text{otherwise.}$$

Then we have

$$\limsup_{y \to x} \psi_k(y) \leq f_{j_k}(x) - 2^{-k}.$$

Pick $h_o,\ h' \in L^s(\mathbf{R}^n)^+$ such that

$$G_1 * h_o \geq |f| \text{ on } \partial\Omega, \quad G_1 * h' \geq 1 \text{ on } \partial\Omega.$$

If we set

$$h = h' + \sum_{k=0}^{\infty} h_k, \tag{4.1}$$

then

$$|f_{j_k}| + 1 \leq |f| + G_1 * h_k + G_1 * h' \leq G_1 * h \tag{4.2}$$

on $\partial\Omega$.

We denote by u_k the solution u in Lemma 3.1 corresponding to $f = f_{j_k}$ and $\psi = \psi_k$. Let Ω_o be an arbitrary subset of Ω such that $\bar{\Omega}_o \subset \Omega$.

To show that

$$\{\|u_k\|_{W^{1,p}(\Omega_o)}\}$$

is uniformly bounded, let us take a Lipschitz function η such that

$$\operatorname{supp} \eta \subset \Omega, \quad \eta = 1 \text{ on} \bar{\Omega}_o \quad 0 \leq \eta \leq 1.$$

Further, take $g \in L^p(\mathbf{R}^n)^+$ satisfying $|\psi| \leq G_1 * g$ on Ω and define

$$\beta(y) = \eta(y) G_1 * g(y) + (1 - \eta(y)) G_1 * h(y),$$

$$\phi(y) = \eta(y)\beta(y) - (1 - \eta(y)) G_1 * h(y)$$

for $y \in \bar{\Omega}$. Then $\phi \in W^{1,p}(\Omega)$. Applying the Sobolev inequality and (3.1), and noting that $s \leq \frac{np}{n-p}$, we obtain

$$\int_\Omega |\beta(y)|^s + \sum_{j=1}^{n} \int_\Omega |\frac{\partial \beta}{\partial y_j}(y)|^p dy < +\infty$$

and

$$\int_\Omega |\phi(y)|^s + \sum_{j=1}^{n} \int_\Omega |\frac{\partial \phi}{\partial y_j}(y)|^p dy < +\infty.$$

Note that

$$(u_k - \phi)^- \geq 0 \geq \psi_k - u_k \quad \text{q.e. on } \Omega. \tag{4.3}$$

By the aid of (4.2) we have

$$f_{j_k} - 1 \geq -G_1 * h = \phi \quad \text{on } \partial\Omega.$$

From Lemma 3.1, (iv) we deduce $(u_k - \phi)^- \in W_o^{1,p}(\Omega)$, which and (4.3) lead to

$$\langle A(\Omega, u_k), (u_k - \phi)^- \rangle \geq 0.$$

Therefore we obtain, by Theorem **A**,

$$\|(u_k - \phi)^-\|_{W^{1,p}(\Omega)} \leq c + c\left(\int_\Omega |\phi(y)|^s + \sum_{j=1}^n \int_\Omega |\frac{\partial \phi}{\partial y_j}(y)|^p dy\right)^{1/p}$$

and hence

$$\|(u_k - \beta)^-\|_{W^{1,p}(\Omega_o)} \leq c + c\left(\int_\Omega |\phi(y)|^s + \sum_{j=1}^n \int_\Omega |\frac{\partial \phi}{\partial y_j}(y)|^p dy\right)^{1/p}. \tag{4.4}$$

On the other hand, since

$$f_{j_k} + 1 < G_1 * h \leq \beta \quad \text{on } \partial\Omega,$$

we have $(u_k - \beta)^+ \in W_o^{1,p}(\Omega)$. Moreover, if $u_k(y) \leq \beta(y)$, then

$$-(u_k - \beta)^+(y) = 0 \geq \psi_k(y) - u_k(y)$$

for q.e. y. If $u_k(y) > \beta(y)$, then

$$-(u_k - \beta)^+(y) = \beta(y) - u_k(y) \geq \psi_k(y) - u_k(y)$$

for q.e. y. Therefore we have

$$\langle A(\Omega, u_k), -(u_k - \beta)^+ \rangle \geq 0$$

by Lemma 3.1, (ii). This and Theorem A lead to

$$\|(u_k - \beta)^+\|_{W^{1,p}(\Omega)} \leq c + c\left(\int_\Omega |\beta(y)|^s + \sum_{j=1}^n \int_\Omega |\frac{\partial \beta}{\partial y_j}(y)|^p dy\right)^{1/p}. \tag{4.5}$$

Thus we see by (4.4) and (4.5) that

$$\{\|u_k - \beta\|_{W^{1,p}(\Omega_o)}\}$$

and hence

$$\{\|u_k\|_{W^{1,p}(\Omega_o)}\}$$

is uniformly bounded for every open set Ω_o satisfying $\bar{\Omega}_o \subset \Omega$. Therefore we can choose a subsequence $\{u_{k_i}\}$ and $w \in W^{1,p}_{loc}(\Omega)$ such that for every open set Ω_o satisfying $\bar{\Omega}_o \subset \Omega$ the sequence $\{u_{k_i}\}$ converges weakly to w in $W^{1,p}(\Omega_o)$. Since identity mapping is a compact operator from $W^{1,p}(\Omega_o)$ to $L^p(\Omega_o)$, we may suppose that $\{u_{k_i}\}$ converges strongly to w in $L^p(\Omega_o)$. We note that

$$u_k = u_k^* \geq \psi_k \text{ q.e. on } \Omega, \quad \lim_{k\to\infty} \psi_k = \psi \text{ q.e. on } \Omega.$$

Since there exists a subsequence of convex combinations of the functions u_{k_i} which converges strongly to w in $W^{1,p}(\Omega_o)$, we conclude that

$$w^* \geq \psi \quad \text{q.e. on } \Omega. \tag{4.6}$$

Moreover, by the same method as in the proof of Lemma 4.5 in [MZ] we can show that

$$u_k \to w \quad \text{strongly in } W^{1,p}(\Omega_o).$$

Putting $u = w^*$, we will show that u is the desired function. The assertion (i) follows from (4.6). To show (ii), suppose that $\phi \in W^{1,p}(\Omega)$, supp $\phi \subset K$ and

$$\phi^* \geq \psi - u \quad \text{q.e. on } \Omega.$$

Take an open set Ω_o such that supp $\phi \subset \Omega_o \subset \bar{\Omega}_o \subset \Omega$ and choose a Lipschitz function τ on Ω such that

$$\operatorname{supp} \tau \subset \Omega, \quad \tau = 1 \text{ on } \bar{\Omega}_o, \quad 0 \leq \tau \leq 1.$$

We define

$$\phi_i = \phi + \tau(u - u_{k_i})^+.$$

We note that

$$\phi_i^* = \phi^* + u - u_{k_i}^* \geq \psi_{k_i} - u_{k_i} \text{ q.e. on } \Omega_o \cap \{u \geq u_{k_i}\}$$

and

$$\phi_i^* = \phi^* \geq \psi - u \geq \psi_{k_i} - u_{k_i}^* \text{ q.e. on } \Omega_o \cap \{u < u_{k_i}\}.$$

On account of Lemma 3.1, (ii) we have

$$\langle A(\Omega_o, u_{k_i}), \phi_i \rangle \geq 0 \quad \text{for each } i.$$

Since

$$u_{k_i} \to u \text{ strongly in } W^{1,p}(\Omega_o)$$

and

$$\phi_i \to \phi \text{ strongly in } W^{1,p}(\Omega_o),$$

we see that

$$\langle A(\Omega_o, u), \phi \rangle \geq 0$$

and hence

$$\langle A(\Omega, u), \phi \rangle \geq 0.$$

Next, to show (iii), denote by E the set

$$\{x \in \partial\Omega;\ |f(x)| = +\infty\} \cup \{x \in \partial\Omega; \lim_{k\to\infty} G_1 * h_k(x) \neq 0\}.$$

Then $\gamma_{1,s}(\chi_E) = 0$ and

$$\lim_{k\to\infty} f_{j_k}(x) = f(x) \quad \text{for } x \in \partial\Omega \setminus E. \tag{4.7}$$

Suppose that τ is a function on $\bar{\Omega}$ such that it is lower semicontinuous on $\bar{\Omega} \setminus K$ for some compact subset K of Ω and $\tau \in W^{1,s}(\Omega)$, $\tau \geq f + \delta$ on $\partial\Omega$ for some $\delta > 0$. Since

$$\limsup_{y\to x} \psi(y) < +\infty \quad \text{and} \ \limsup_{y\to x} \psi(y) \leq f(x)$$

for all $x \in \partial\Omega$, there is an open set Ω_1 satisfying $\bar{\Omega}_1 \subset \Omega$ and

$$\psi(y) < \tau(y) \quad \text{for all } y \in \Omega \setminus \Omega_1.$$

Take a Lipschitz function η such that

$$\operatorname{supp} \eta \subset \Omega, \quad \eta = 1 \text{ on } \bar{\Omega}_1, \quad 0 \leq \eta \leq 1$$

and define

$$v_o = (G_1 * h)\eta,$$

where h is the function defined in (4.1).

Let us show that $(u - \tau)^+ \in W_o^{1,p}(\Omega)$. Since $\operatorname{supp} v_o \subset \Omega$ and $v_o \in W^{1,p}(\Omega)$, it suffices to show that $(u - \tau - v_o)^+ \in W_o^{1,p}\Omega)$. From the inequality

$$f_{j_k} < \tau + v_o + G_1 * h_k \quad \text{on } \partial\Omega$$

and lemma 3.1 we deduce

$$(u_k - (\tau + v_o + G_1 * h_k))^+ \in W_o^{1,p}(\Omega).$$

We will show that

$$\{\|(u_k - (\tau + v_o + G_1 * h_k))^+\|_{W^{1,p}(\Omega)}\}$$

is uniformly bounded. We claim that

$$-(u_k - (\tau + v_o + G_1 * h_k))^+ \geq \psi_k - u_k \quad \text{q.e. on } \Omega.$$

Indeed we have

$$-(u_k - (\tau + v_o + G_1 * h_k))^+ \geq -(u_k - (\tau + v_o + G_1 * h_k)) \geq \psi_k - u_k$$

q.e. on $\Omega \cap \{u_k \geq \tau + v_o + G_1 * h_k\}$ and

$$-(u_k - (\tau + v_o + G_1 * h_k))^+ = 0 \geq \psi_k - u_k$$

q.e. on $\Omega \cap \{u_k < \tau + v_o + G_1 * h_k\}$. Therefore, from Lemma 3.1, (ii) it follows that

$$\langle A(\Omega, u_k), -(u_k - (\tau + v_o + G_1 * h_k))^+\rangle \geq 0.$$

Using Theorem **A** and (3.1), we have

$$\|(u_k - (\tau + v_o + G_1 * h_k))^+\|_{W^{1,p}(\Omega)} \leq c_1$$
$$+ c_1 \left(\int_\Omega |(\tau + v_o + G_1 * h_k)|^s + \sum_{j=1}^n \int_\Omega |\frac{\partial}{\partial y_j}(\tau + v_o + G_1 * h_k)|^p dy \right)^{1/p}$$
$$\leq c_2 + c_2 M \left(\int_\Omega |(\tau + v_o)|^s + \sum_{j=1}^n \int_\Omega |\frac{\partial}{\partial y_j}(\tau + v_o)|^p dy \right)^{1/p}.$$

Thus we see that

$$\{\|(u_k - (\tau + v_o + G_1 * h_k))^+\|_{W^{1,p}(\Omega)}\}$$

is uniformly bounded. We note that for every open set Ω_o satisfying $\bar{\Omega} \subset \Omega$, $\{u_{k_i}\}$ converges to u strongly in $W^{1,p}(\Omega_o)$ and

$$\|G_1 * h_{k_i}\|_{W^{1,s}(\Omega_o)} \leq M\|h_{k_i}\|_s \to 0$$

as $i \to \infty$. Using Lemma 4.6 in [MZ], we conclude that $(u-(\tau+v_o))^+ \in W_o^{1,p}(\Omega)$ and hence $(u-\tau)^+ \in W_o^{1,p}(\Omega)$.

Finally, suppose that λ is a function on $\bar{\Omega}$ such that it is upper semi-continuous on $\bar{\Omega} \setminus K$ for some compact subset K of Ω and $\lambda \in W^{1,p}(\Omega)$, $\lambda \leq f-\delta$ on $\partial\Omega$. In this case we can also show directly, without the aid of v_o, that $(u-\lambda)^- \in W_o^{1,p}(\Omega)$. Thus we see that (iii) also holds. This completes the proof. Q.E.D.

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