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Kyoto University
Boundary value problems and variational inequalities

1. Introduction and notations

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ ($n \geq 2$). We consider the obstacle problem for a quasilinear elliptic operator of second order in $\Omega$ as follows:

\[
L = -\text{div} A(x, u, \nabla u) + B(x, u, \nabla u).
\]

Here $A$ (resp. $B$) is a vector (resp. scalar) valued function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and the functions $A$ and $B$ are assumed to satisfy the following inequalities:

\[
\begin{align*}
|A(x, u, w)| &\leq a(|w|^{p-1} + |u|^{p-1} + 1), \\
|B(x, u, w)| &\leq b(|w|^{p-1} + |u|^{p-1} + 1), \\
w \cdot A(x, u, w) + uB(x, u, w) &\geq c_1|w|^p - c_2(|u|^p + 1), \\
(A(x, u, w_1) - A(x, u, w_2)) \cdot (w_1 - w_2) &> 0 \quad (w_1 \neq w_2)
\end{align*}
\]

for all $x \in \Omega$, $u \in \mathbb{R}$ and $w_1, w_2 \in \mathbb{R}^n$, where $a, b, c_1, p$ are positive real numbers satisfying $p > 1$ and $c_2$ is a nonnegative real number.

It has been known that for each continuous function $f$ on $\partial \Omega$ and a function $\phi$ on $\Omega$ satisfying

\[
\int_0^\infty t^{p-1}B_{1,p}(A(t))dt < +\infty
\]

there exists a solution $u \in W^{1,p}_{\text{loc}}(\Omega)$ to the obstacle problem with boundary data $f$, where

\[
A(t) = \{ y \in \Omega; \psi(y) > t \}.
\]

The obstacle problem is to find a function $u \in W^{1,p}_{\text{loc}}(\Omega)$ such that

\[
u \geq \psi \text{ on } \Omega \text{ except for a subset of } \Omega,
\]

\[
u = f \text{ weakly on } \partial \Omega.
\]
and
\[ \int_{\Omega} A(y, u(y), \nabla u(y)) \cdot \nabla \phi(y) dy + \int_{\Omega} B(y, u(y), \nabla u(y)) \phi(y) dy \geq 0 \]
for all \( \phi \in C^\infty_0(\Omega) \) satisfying \( \phi \geq \psi - u \) on \( \Omega \) except for a subset of \( \Omega \).

In this paper we will prove the existence of a weak solution to the obstacle problem for (possibly) non-bounded boundary function \( f \). To consider boundary functions which value \( \pm \infty \), we must distinguish functions not up to a set of \( n \)-dimensional Lebesgue measure zero, but up to a more fine set, for example, a set of \( B_{1,s} \)-capacity zero.

Recall that for \( s > 1 \) the Bessel capacity \( B_{1,s} \) with order 1 is defined by
\[ B_{1,s}(E) = \inf \{ \| g \| ; g \in L'(\mathbb{R}^n), \ g \geq 0, G_1 * g \geq 1 \text{ on } E \} \]
for a subset \( E \) of \( \mathbb{R}^n \). If a property holds on a subset \( X \) of \( \mathbb{R}^n \) except for a set of \( B_{1,s} \)-capacity zero, we say that it holds \( B_{1,s} \)-q.e. on \( X \). In the case \( s = p \) we use simply "q.e." instead of "\( B_{1,p} \)-q.e.”.

To distinguish functions up to a set of \( B_{1,s} \)-capacity zero, we construct a family of functions defined on \( \partial\Omega \), which contains all continuous functions and the restrictions of all Bessel potentials \( G_1 * g \ (g \in L'(\mathbb{R}^n)) \) to \( \partial\Omega \), where \( G_1 \) is the Bessel function with order 1, i.e.,
\[ G_1(x) = \frac{1}{(4\pi)^{1/2}} \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \exp(-\frac{\pi|x|^2}{t}) \exp(-\frac{t}{4\pi}) t^{(1-n)/2} \cdot \frac{1}{t} dt. \]

Recall that the Fourier function of \( G_1 \) is equal to
\[ \frac{1}{(1 + 4\pi|x|^2)^{1/2}}. \]

Let us define, for each extended real-valued function \( f \) on \( \partial\Omega \),
\[ \gamma_{1,s}(f) = \inf \{ \| g \| ; g \in L'(\mathbb{R}^n), g \geq 0, G_1 * g \geq |f| \text{ on } \partial\Omega \}. \]

Furthermore, denote by \( B(\gamma_{1,s}) \) the family of all Borel measurable functions on \( \partial\Omega \) such that \( \gamma_{1,s}(f) < +\infty \). We remark that \( B(\gamma_{1,s}) \supset C(\partial\Omega) \), where \( C(\partial\Omega) \) is the family of all continuous real-valued functions on \( \partial\Omega \).

We denote by \( L(\gamma_{1,s}) \) the family of all \( f \in B(\gamma_{1,s}) \) such that \( \gamma_{1,s}(f - f_j) \to 0 \ (j \to \infty) \) for some \( \{ f_j \} \subset C(\partial\Omega) \).

It is well-known that, if \( 1 < p < n \) and \( v \in W^{1,p}(\Omega) \), then
\[ \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} v(y) dy \]
exists (as a real number) for all \( x \in \Omega \) except for a set \( E \) with \( B_{1}(E) = 0 \), where \( B(x, r) \) is the ball with center \( x \) and radius \( r \) and \( |B(x, r)| \) stands for the volume of the ball \( B(x, r) \) (cf. [FZ]). For each \( v \in W^{1,p}(\Omega) \) we denote by \( v^{*} \) the function defined by (1.2) q.e. on \( \Omega \).

Under these notations we will prove the following theorem.

Theorem. Let \( 1 < p < n \), \( p < s \leq \overline{n}^{n} - \overline{p} \) \( \text{for} \) some \( g \in L^{n}(\mathbb{R}^{n})^{+} \).

for all \( x \in \partial\Omega \). Then there exists a function \( u \in W^{1,p}_{loc}(\Omega) \) having the following properties:

(i) \( u \geq \psi \) q.e. on \( \Omega \),

(ii) If \( \phi \in W^{1,p}(\Omega) \), supp \( \phi \subset \Omega \) and \( \phi^{*} \geq \psi - u \) q.e. on \( \Omega \), then

\[
\int_{\Omega} \{A(y, u(y), \nabla u(y)) \cdot \nabla \phi(y) + B(y, u(y), \nabla u(y)) \phi(y)\} \, dy \geq 0.
\]

(iii) \( u = f \) on \( \partial\Omega \) in the following sense:

Let \( \tau \) be a function on \( \tilde{\Omega} \) such that it is lower semi-continuous on \( \tilde{\Omega} \setminus K \) for some compact subset \( K \) of \( \Omega \) and

\[
\tau |_{\partial\Omega} = W^{1,s}(\Omega), \quad \tau \geq f + \delta \text{ on } \partial\Omega \text{ for some } \delta > 0.
\]

Then \( (u - \tau)^{+} \in W^{1,p}_{0}(\Omega) \). Further, let \( \lambda \) be a function on \( \tilde{\Omega} \) such that it is upper semicontinuous on \( \tilde{\Omega} \setminus K \) for some compact subset \( K \) of \( \Omega \) and

\[
\lambda |_{\partial\Omega} = W^{1,s}(\Omega), \quad \lambda \leq f - \delta
\]

for some \( \delta > 0 \) on \( \partial\Omega \). Then \( (u - \lambda)^{-} \in W^{1,p}_{0}(\Omega) \).

2. Properties of \( \gamma_{1,s} \)

In this section we study the properties of \( \gamma_{1,s} \). It is easy to see that the functional \( \gamma_{1,s} \) has the following properties similar to those of the upper integral.

Lemma 2.1. Let \( s > 1 \). Then the functional \( \gamma_{1,s} \) has the following properties:

\[
\begin{align*}
(c_{1}) \quad & \gamma_{1,s}(f) = \gamma_{1,s}(|f|), \\
(c_{2}) \quad & \gamma_{1,s}(bf) = b\gamma_{1,s}(f) \text{ for } b \in \mathbb{R}^{+},
\end{align*}
\]
$(c_3) f_j \geq 0 \Rightarrow \gamma_{1,\ast}(\sum_{j=1}^{\infty} f_j) \leq \sum_{j=1}^{\infty} \gamma_{1,\ast}(f_j),
(c_4) \gamma_{1,\ast}(\chi_E) = B_{1,\ast}(E)^{1/s} \text{ for } E \subset \mathbb{R}^n.$

Using Lemma 2.1, we can show the following lemma.

**Lemma 2.2.** (i) If $\gamma_{1,\ast}(f) < +\infty$, then the set $\{ x \in \partial \Omega; |f(x)| = +\infty \}$ is of $B_{1,\ast}$-capacity zero.
(ii) If $\gamma_{1,\ast}(f - g) = 0$, then $f = g$ $B_{1,\ast}$-q.e. on $\partial \Omega$.

**Lemma 2.3.** Let $g$ be a nonnegative function in $L^s(\mathbb{R}^n)$. Then the Bessel potential $G_{1} \ast g$ belongs to $\mathcal{L}(\gamma_{1,\ast})$.

**Proof.** We can assume that $g$ is nonnegative. Set $g_j = \min\{g, j\}$ and $h_j = g - g_j$.

Noting $G_1 \in L^1(\mathbb{R}^n)$, we see that $G_1 \ast g_j$ is continuous on $\partial \Omega$. Since $|G_1 \ast g - G_1 \ast g_j| \leq G_1 \ast h_j$ and $\|h_j\| \to 0$ as $j \to \infty$, we have the conclusion. Q.E.D

**Lemma 2.4.** The set $\mathcal{L}$ of the restrictions of all Lipschitz functions on $\Omega$ to $\partial \Omega$ is dense in $\mathcal{L}(\gamma_{1,\ast})$.

**Proof.** We can choose a nonnegative function $h = G_1 \ast g$ $(g \in L^s(\mathbb{R}^n)^{+})$ such that $h \geq 1$ on $\partial \Omega$. Since $\mathcal{L}$ is uniformly dense in $C(\partial \Omega)$, it is dense in $\mathcal{L}(\gamma_{1,\ast})$. Q.E.D.

Noting that

$$G_1(y) = O(e^{-c|y|}) \text{ for some } c > 0 \text{ as } |y| \to \infty,$$

we can easily show the following lemma.

**Lemma 2.5.** Let $p, s$ be positive real numbers satisfying $1 < p \leq s$ and $E$ be a relatively compact subset of $\mathbb{R}^n$. If $B_{1,\ast}(E) = 0$, then $B_{1,p}(E) = 0$.

For a function $f$ defined on $\mathbb{R}^n$ we define

$$\gamma_{1,\ast}(f) = \inf\{\|f\|_s; g \in L^s(\mathbb{R}^n)^{+}; G_1 \ast g \geq |f| \text{ on } \mathbb{R}^n\}$$

It is easy to see that this functional $\gamma_{1,\ast}$ also has the properties in Lemmas 2.1 and 2.2 in which $\partial \Omega$ is replaced by $\mathbb{R}^n$.

**Lemma 2.6.** Let $\{f_j\}$ be a sequence of functions on $\mathbb{R}^n$ such that $\gamma_{1,\ast}(f_j) \to 0$ $(j \to \infty)$. Then there exists a subsequence $\{g_k\}$ of $\{f_j\}$ such that $g_k \to 0$ pointwisely $B_{1,\ast}$-q.e. on $\mathbb{R}^n$. 

Proof. Choose a subsequence \( \{g_k\} \) of \( \{f_i\} \) satisfying

\[
\sum_{k=1}^{\infty} 2^k \gamma_{1,*}(g_{k+1} - g_k) < +\infty.
\] (2.1)

To show that \( \{g_k\} \) is the desired subsequence, set

\[ E = \bigcup_{k=1}^{\infty} \{x \in \mathbb{R}^n; |g_k(x)| = +\infty\}. \]

Then we have \( B_{1,*}(E) = 0 \) by Lemmas 2.1 and 2.2. Further set

\[ O'_k = \{x \in \mathbb{R}^n \setminus E; |g_{k+1}(x) - g_k(x)| \geq 2^{-k}\} \]

and

\[ O_k = \bigcup_{i=k}^{\infty} O'_i \quad \text{and} \quad F_k = \mathbb{R}^n \setminus (O_k \cup E). \]

Setting \( g_0 = 0 \) and noting that

\[ g_k = \sum_{i=0}^{k-1} (g_{i+1} - g_i) \]

on \( \mathbb{R}^n \setminus E \), we see that \( \{g_k\} \) converges to 0 on \( \bigcup_{i=1}^{\infty} F_i \). We put

\[ E_0 = \mathbb{R}^n \setminus (\bigcup_{k=1}^{\infty} F_k \cup E) = \bigcap_{k=1}^{\infty} O_k. \]

From

\[ \chi_{O_k} \leq \sum_{i=k}^{\infty} \chi_{O'_i} \leq \sum_{i=k}^{\infty} 2^i |g_{i+1} - g_i| \]

and Lemma 2.1 we deduce

\[ \gamma_{1,*}(\chi_{O_k}) \leq \sum_{i=k}^{\infty} 2^i \gamma_{1,*}(g_{i+1} - g_i). \]

On account of (2.1) we have

\[ \gamma_{1,*}(\chi_{O_k}) \to 0 \quad (k \to \infty) \]

and hence \( \gamma_{1,*}(\chi_{E_0}) = 0 \). Therefore we see by Lemma 2.1 that

\[ B_{1,*}(E \cup E_0)^{1/s} = \gamma_{1,*}(\chi_{E \cup E_0}) = 0 \quad \text{and} \quad \{g_k\} \text{ converges to } 0 \text{ on } \mathbb{R}^n \setminus (E \cup E_0). \]

Q.E.D.

3. Boundedness of solutions
For an open subset $\Omega_0 \neq \emptyset$ of $\Omega$ we denote by $A(\Omega_0, \cdot)$ the mapping
$$W^{1,p}(\Omega_0) \rightarrow W^{1,p}(\Omega_0)'$$
defined by
$$\langle A(\Omega_0, v), w \rangle = \int_{\Omega_0} \{A(y, v(y), \nabla v(y)) \cdot \nabla w(y) + B(y, v(y), \nabla v(y))w(y)\}$$

The following theorem is fundamental.

**Theorem A** ([MZ, Theorem 3.1]). Let $p < s$ and $\Omega_0$ be a nonempty open subset of $\Omega$, $\epsilon = +$ or $-$ and $v, \eta$ be functions in $W^{1,p}(\Omega_0)$ such that $(v - \eta)^\epsilon \in W^{1,p}_0(\Omega_0)$ and
$$\langle A(\Omega_0, v), -\epsilon(v - \eta)^\epsilon \rangle \geq 0.$$ 

Then
$$\|(v - \eta)^\epsilon\|_{W^{1,p}(\Omega_0)} \leq c + c \left(1 + \int_{\Omega_0} (|\eta|^s + \sum_{i=1}^n |\partial \eta/\partial y_i|^p)dy \right)^{1/p},$$
where $c$ is a constant independent of $v, \eta$.

It is well-known that for each $s > 1$
$$W^{1,s}(\mathbb{R}^n) = \{G_1 * g; g \in L^s(\mathbb{R}^n)\}$$
and
$$\frac{1}{M} \|g\|_s \leq \|G_1 * g\|_{W^{1,s}(\mathbb{R}^n)} \leq M \|g\|_s,$$
where $M$ is a constant independent of $g$ (cf. [S, Theorem 3 on p.135]).

**Lemma 3.1.** Let $f$ be a Lipschitz function on $\Omega$ such that $|f| \leq G_1 * g_1$ for some $g_1 \in L^p(\mathbb{R}^n)^+$. Furthermore, let $\psi$ be a real-valued function on $\Omega$ such that
$$\limsup_{y \rightarrow x, y \in \Omega} \psi(y) < f(x) - \delta$$
for all $x \in \partial \Omega$ and for some $\delta > 0$, and
$$|\psi| \leq G_1 * g_\delta \text{ for some } g_\delta \in L^p(\mathbb{R}^n)^+.$$
Then there exists a function $u \in W^{1,p}(\Omega)$ such that $u$ has the properties:

(i) $u \geq \psi$ q.e. on $\Omega$,

(ii) If $\phi \in W^{1,p}_{0}(\Omega)$ and $\phi^{*} \geq \psi - u$ q.e. on $\Omega$, then

$$\int_{\Omega} \{ A(y, u(y), \nabla u(y)) \cdot \nabla \phi(y) + B(y, u(y), \nabla u(y)) \phi(y) \} dy \geq 0.$$ 

(iii) $u - f \in W^{1,p}_{0}(\Omega)$,

(iv) If $\eta$ (resp. $\lambda$) is a function on $\overline{\Omega}$, such that it is lower (resp. upper) semicontinuous on $\overline{\Omega} \setminus K$ for some compact subset $K$ of $\Omega$ and $\eta \in W^{1,p}(\Omega)$ (resp. $\lambda \in W^{1,p}(\Omega)$), $\eta(y) > \phi(y)$ (resp. $\lambda(y) < \phi(y)$) for all $y \in \partial \Omega$, then $(u - \eta)^{+} \in W_{0}^{1,p}(\Omega)$ (resp. $(u - \lambda)^{-} \in W_{0}^{1,p}(\Omega)$).

Proof. Set

$$K = \{ v \in W^{1,p}(\Omega); v - f \in W^{1,p}_{0}(\Omega), \ v^{*} \geq \psi \text{ q.e. on } \Omega \}.$$ 

We claim that $K$ is not empty. Indeed, noting that $G_{1} \ast g_{1}$ is lower semicontinuous, we can choose, by the aid of (3.2), an open set $\Omega_{0}$ such that $\Omega_{0} \subset \Omega$ and

$$\psi(y) < f(y) \text{ for all } y \in \Omega \setminus \Omega_{0}.$$ 

Choose a Lipschitz function $h$ such that

$$\text{supp } h \subset \Omega, \ h = 1 \text{ on } \overline{\Omega}_{o}, \ 0 \leq h \leq 1,$$

and define

$$\phi(y) = h(y)w(y) + (1 - h(y))f(y),$$

where $w = G_{1} \ast g_{1}$. We note that supp $h$ stands for the closure of the set $\{ y; h(y) \neq 0 \}$. Then $\phi^{*} \geq \psi$ q.e. on $\Omega$ and $\phi - f = h(w - f) \in W^{1,p}_{0}(\Omega)$. Therefore we see that $\phi \in K$.

The family $K$ is a convex closed subset of $W^{1,p}(\Omega)$ and hence weakly closed. The mapping $A(\Omega, \cdot)$ from $W^{1,p}(\Omega)$ to $W^{1,p}(\Omega)'$ is pseudomonotone by Theorem 3.9 in [MZ]. Furthermore we see that

$$\frac{\langle A(\Omega, v), v - v_{o} \rangle}{\| v \|_{W^{1,p}(\Omega)}} \rightarrow \infty$$

as $\| v \|_{W^{1,p}(\Omega)} \rightarrow \infty$ ($v \in K$). It follows from Theorem 8.2 on p.247 in [L] that there exists $u_{o} \in K$ such that

$$\langle A(\Omega, u_{o}), v - u_{o} \rangle \geq 0 \text{ for all } v \in K.$$
Setting $u = u^*_o$, we will show that $u$ is the desired function. It is obvious that (i) and (iv) hold. To show (ii), let $\phi$ be a function in $W^{1,p}_o(\Omega)$ such that

$$\phi^* \geq \psi - u \text{ q.e. on } \Omega.$$ 

From $u - f \in W^{1,p}_o(\Omega)$ and $\phi + u - f \in W^{1,p}_o(\Omega)$, it follows that

$$\langle A(\Omega, u), \phi \rangle \geq 0.$$ 

Finally, to show (v), let $\eta$ be a lower semicontinuous function on $\overline{\Omega}$ in $W^{1,p}_o(\Omega)$ such that $\eta > f$ on $\partial \Omega$. Since $f - \eta < 0$ outside a compact subset of $\Omega$ and $u - f \in W^{1,p}_o(\Omega)$, we have

$$(u - \eta)^+ \in W^{1,p}_o(\Omega).$$ 

Similarly we can show that $(u - \lambda)^- \in W^{1,p}_o(\Omega).$ Q.E.D.

4. Proof of Theorem

Let us prove Theorem. Suppose that $f \in \mathcal{L}(\gamma_{1,\sigma})$. On account of Lemma 2.4 we can choose a sequence $\{f_j\}$ of Lipschitz functions on $\overline{\Omega}$ and a sequence $\{g_j\}$ of functions in $L^+(\mathbb{R}^n)$ such that

$$|f - f_j| \leq G_1 * g_j \text{ on } \partial \Omega \quad ||g_j||_\ast < 2^{-j}.$$ 

Since $\gamma_{1,\sigma}(G_1 * g_j) \to 0$, we can choose, by Lemma 2.6, a subsequence $\{G_1 * h_k\}$ converges pointwisely to $0 B_{1,\sigma}$-q.e. on $\mathbb{R}^n$. Therefore, by Lemma 2.5, it converges to $0$ q.e. on $\overline{\Omega}$. Noting that

$$\lim_{y \to x} \sup_{\Omega} \psi(y) \leq f(x) \leq G_1 * h_k(x) + f_{j_k}(x)$$

for all $x \in \partial \Omega$, we define

$$\psi_k(y) = \psi(y) - G_1 * h_k(y) - 2^{-k} \quad \text{if } G_1 * h_k(y) < +\infty$$

and

$$\psi_k(y) = -\sup_{x \in \partial \Omega} |f_{j_k}(x)| - 1 \quad \text{otherwise.}$$

Then we have

$$\lim_{y \to x} \sup_{\Omega} \psi_k(y) \leq f_{j_k}(x) - 2^{-k}.$$
Pick $h_o, h' \in L'(\mathbb{R}^n)^+$ such that

$$G_1 * h_o \geq |f| \text{ on } \partial \Omega, \quad G_1 * h' \geq 1 \text{ on } \partial \Omega.$$ 

If we set

$$h = h' + \sum_{k=0}^{\infty} h_k, \quad (4.1)$$

then

$$|f_j| + 1 \leq |f| + G_1 * h_k + G_1 * h' \leq G_1 * h \quad (4.2)$$
on $\partial \Omega$.

We denote by $u_k$ the solution $u$ in Lemma 3.1 corresponding to $f = f_j$ and $\psi = \psi_k$. Let $\Omega_o$ be an arbitrary subset of $\Omega$ such that $\overline{\Omega}_o \subset \Omega$.

To show that

$$\{|u_k|_{W^{1,s}(\Omega_o)}\}$$
is uniformly bounded, let us take a Lipschitz function $\eta$ such that

$$\text{supp } \eta \subset \Omega, \quad \eta = 1 \text{ on } \overline{\Omega}_o, \quad 0 \leq \eta \leq 1.$$ 

Further, take $g \in L^p(\mathbb{R}^n)^+$ satisfying $|\psi| \leq G_1 * g$ on $\Omega$ and define

$$\beta(y) = \eta(y)G_1 * g(y) + (1 - \eta(y))G_1 * h(y),$$
$$\phi(y) = \eta(y)\beta(y) - (1 - \eta(y))G_1 * h(y)$$

for $y \in \Omega$. Then $\phi \in W^{1,p}(\Omega)$. Applying the Sobolev inequality and (3.1), and noting that $s \leq \frac{np}{n-p}$, we obtain

$$\int_{\partial \Omega} |\beta(y)|^s + \sum_{j=1}^{n} \int_{\partial \Omega} |\frac{\partial \beta}{\partial y_j}(y)|^p dy < +\infty$$

and

$$\int_{\partial \Omega} |\phi(y)|^s + \sum_{j=1}^{n} \int_{\partial \Omega} |\frac{\partial \phi}{\partial y_j}(y)|^p dy < +\infty.$$ 

Note that

$$u_k - \phi \geq 0 \geq \psi_k - u_k \quad \text{q.e. on } \Omega.$$ 

\( (4.3) \)
By the aid of (4.2) we have

$$f_{j_{k}} - 1 \geq -G_{1} \ast h = \phi \quad \text{on } \partial \Omega.$$  

From Lemma 3.1, (iv) we deduce \((u_{k} - \phi)^{-} \in W_{1,p}^{1}(\Omega)\), which and (4.3) lead to

$$\langle A(\Omega, u_{k}), (u_{k} - \phi)^{-} \rangle \geq 0.$$  

Therefore we obtain, by Theorem A,

$$\| (u_{k} - \phi)^{-} \|_{W_{1,p}^{1}(\Omega)} \leq c + c \left( \int_{\Omega} |\phi(y)|^p + \sum_{j=1}^{n} \int_{\Omega} |\frac{\partial \phi}{\partial y_j}(y)|^p dy \right)^{1/p}$$  

and hence

$$\| (u_{k} - \beta)^{-} \|_{W_{1,p}^{1}(\partial \Omega)} \leq c + c \left( \int_{\Omega} |\phi(y)|^p + \sum_{j=1}^{n} \int_{\Omega} |\frac{\partial \phi}{\partial y_j}(y)|^p dy \right)^{1/p} \quad (4.4)$$  

On the other hand, since

$$f_{j_{k}} + 1 < G_{1} \ast h \leq \beta \quad \text{on } \partial \Omega,$$

we have \((u_{k} - \beta)^{+} \in W_{1,p}^{1}(\Omega)\). Moreover, if \(u_{k}(y) \leq \beta(y)\), then

$$-(u_{k} - \beta)^{+}(y) = 0 \geq \psi_{k}(y) - u_{k}(y)$$

for q.e. \(y\). If \(u_{k}(y) > \beta(y)\), then

$$-(u_{k} - \beta)^{+}(y) = \beta(y) - u_{k}(y) \geq \psi_{k}(y) - u_{k}(y)$$

for q.e. \(y\). Therefore we have

$$\langle A(\Omega, u_{k}), -(u_{k} - \beta)^{+} \rangle \geq 0$$

by Lemma 3.1, (ii). This and Theorem A lead to

$$\| (u_{k} - \beta)^{+} \|_{W_{1,p}^{1}(\Omega)} \leq c + c \left( \int_{\Omega} |\beta(y)|^p + \sum_{j=1}^{n} \int_{\Omega} |\frac{\partial \beta}{\partial y_j}(y)|^p dy \right)^{1/p} \quad (4.5)$$
Thus we see by (4.4) and (4.5) that
\[ \{ \| u_k - \beta \|_{W^{1,p}(\Omega)} \} \]
and hence
\[ \{ \| u_k \|_{W^{1,p}(\Omega)} \} \]
is uniformly bounded for every open set \( \Omega \) satisfying \( \bar{\Omega} \subset \Omega \). Therefore we can choose a subsequence \( \{ u_{k_i} \} \) and \( w \in W^{1,p}_{loc}(\Omega) \) such that for every open set \( \Omega \) satisfying \( \bar{\Omega} \subset \Omega \) the sequence \( \{ u_{k_i} \} \) converges weakly to \( w \) in \( W^{1,p}(\Omega) \). Since identity mapping is a compact operator from \( W^{1,p}(\Omega) \) to \( L^p(\Omega) \), we may suppose that \( \{ u_{k_i} \} \) converges strongly to \( w \) in \( L^p(\Omega) \).

We note that
\[ u_k = u_k^{*} \geq \psi_k \text{ q.e. on } \Omega, \quad \lim_{k \to \infty} \psi_k = \psi \text{ q.e. on } \Omega. \]

Since there exists a subsequence of convex combinations of the functions \( u_{k_i} \), which converges strongly to \( w \) in \( W^{1,p}(\Omega) \), we conclude that
\[ (4.6) \]
\[ w^{*} \geq \psi \text{ q.e. on } \Omega. \]

Moreover, by the same method as in the proof of Lemma 4.5 in [MZ] we can show that
\[ u_k \to w \text{ strongly in } W^{1,p}(\Omega). \]

Putting \( u = w^{*} \), we will show that \( u \) is the desired function. The assertion (i) follows from (4.6). To show (ii), suppose that \( \phi \in W^{1,p}(\Omega) \), \( \text{supp } \phi \subset K \) and
\[ \phi^{*} \geq \psi - u \text{ q.e. on } \Omega. \]
Take an open set \( \Omega \) such that \( \text{supp } \phi \subset \Omega \subset \bar{\Omega} \subset \Omega \) and choose a Lipschitz function \( \tau \) on \( \Omega \) such that
\[ \text{supp } \tau \subset \Omega, \quad \tau = 1 \text{ on } \bar{\Omega}, \quad 0 \leq \tau \leq 1. \]
We define
\[ \phi_i = \phi + \tau(u - u_{k_i})^{+}. \]

We note that
\[ \phi_i^{*} = \phi^{*} + u - u^{*}_{k_i} \geq \psi_{k_i} - u_{k_i} \quad \text{q.e. on } \Omega \cap \{ u \geq u_{k_i} \} \]
and
\[ \phi_i^{*} = \phi^{*} \geq \psi - u \geq \psi_{k_i} - u^{*}_{k_i} \quad \text{q.e. on } \Omega \cap \{ u < u_{k_i} \}. \]
On account of Lemma 3.1, (ii) we have

$$\langle A(\Omega, u), \phi_i \rangle \geq 0$$

for each $i$.

Since

$$u_{k_i} \rightharpoonup u \text{ strongly in } W^{1,p}(\Omega)$$

and

$$\phi_i \rightharpoonup \phi \text{ strongly in } W^{1,p}(\Omega),$$

we see that

$$\langle A(\Omega, u), \phi \rangle \geq 0$$

and hence

$$\langle A(\Omega, u), \phi \rangle \geq 0.$$

Next, to show (iii), denote by $E$ the set

$$\{x \in \partial \Omega; |f(x)| = +\infty\} \cup \{x \in \partial \Omega; \lim_{k \to \infty} G_1 * h_k(x) \neq 0\}.$$

Then $\gamma_1(\chi_E) = 0$ and

(4.7) $\lim_{k \to \infty} f_{j_k}(x) = f(x)$ for $x \in \partial \Omega \setminus E$.

Suppose that $\tau$ is a function on $\tilde{\Omega}$ such that it is lower semicontinuous on $\tilde{\Omega} \setminus K$ for some compact subset $K$ of $\Omega$ and $\tau \in W^{1,*}(\Omega)$, $\tau \geq f + \delta$ on $\partial \Omega$ for some $\delta > 0$. Since

$$\limsup_{y \to x} \psi(y) < +\infty \quad \text{and} \quad \limsup_{y \to x} \psi(y) \leq f(x)$$

for all $x \in \partial \Omega$, there is an open set $\Omega_1$ satisfying $\tilde{\Omega}_1 \subset \Omega$ and

$$\psi(y) < \tau(y) \quad \text{for all } y \in \Omega \setminus \Omega_1.$$

Take a Lipschitz function $\eta$ such that

$$\text{supp } \eta \subset \Omega, \quad \eta = 1 \text{ on } \tilde{\Omega}_1, \quad 0 \leq \eta \leq 1$$

and define

$$v_o = (G_1 * h)\eta,$$

where $h$ is the function defined in (4.1).

Let us show that $(u - \tau)^+ \in W^{1,p}(\Omega)$. Since $\text{supp } v_o \subset \Omega$ and $v_o \in W^{1,p}(\Omega)$, it suffices to show that $(u - \tau - v_o)^+ \in W^{1,p}(\Omega)$. From the inequality

$$f_{j_k} < \tau + v_o + G_1 * h_k \quad \text{on } \partial \Omega$$
and lemma 3.1 we deduce
\[(u_k - (\tau + v_o + G_1 \ast h_k))^+ \in W^{1,p}_0(\Omega)\].

We will show that
\[
\{(u_k - (\tau + v_o + G_1 \ast h_k))^+\} \subseteq W^{1,p}(\Omega)
\]
is uniformly bounded. We claim that
\[-(u_k - (\tau + v_o + G_1 \ast h_k))^+ \geq \psi_k - u_k \quad \text{q.e. on } \Omega.
\]
Indeed we have
\[-(u_k - (\tau + v_o + G_1 \ast h_k))^+ \geq -(u_k - (\tau + v_o + G_1 \ast h_k)) \geq \psi_k - u_k \quad \text{q.e. on } \Omega \cap \{u_k \geq \tau + v_o + G_1 \ast h_k\}
\]
and
\[-(u_k - (\tau + v_o + G_1 \ast h_k))^+ = 0 \geq \psi_k - u_k \quad \text{q.e. on } \Omega \cap \{u_k < \tau + v_o + G_1 \ast h_k\}.
\]
Therefore, from Lemma 3.1, (ii) it follows that
\[(A(\Omega, u_k), -(u_k - (\tau + v_o + G_1 \ast h_k))^+) \geq 0.
\]
Using Theorem A and (3.1), we have
\[
||u_k - (\tau + v_o + G_1 \ast h_k)||_{W^{1,p}(\Omega)} \leq c_1
\]
\[
+ c_1 \left(\int_\Omega |(\tau + v_o + G_1 \ast h_k)|^r \sum_{j=1}^n \int_\Omega \left|\frac{\partial}{\partial y_j}(\tau + v_o + G_1 \ast h_k)\right|^p dy\right)^{1/p}
\]
\[
\leq c_2 + c_2 M \left(\int_\Omega |(\tau + v_o)|^r \sum_{j=1}^n \int_\Omega \left|\frac{\partial}{\partial y_j}(\tau + v_o)\right|^p dy\right)^{1/p}
\]
Thus we see that
\[
\{(u_k - (\tau + v_o + G_1 \ast h_k))^+\} \subseteq W^{1,p}(\Omega)
\]
is uniformly bounded. We note that for every open set \(\Omega_o\) satisfying \(\overline{\Omega} \subseteq \Omega\), \(\{u_{k_i}\}\) converges to \(u\) strongly in \(W^{1,p}(\Omega_o)\) and
\[
||G_1 \ast h_k||_{W^{1,r}(\Omega_o)} \leq M||h_k||_r \to 0
\]
as $i \to \infty$. Using Lemma 4.6 in [MZ], we conclude that $(u - (\tau + v_0))^+ \in W^{1,p}_0(\Omega)$ and hence $(u - \tau)^+ \in W^{1,p}_0(\Omega)$.

Finally, suppose that $\lambda$ is a function on $\Omega$ such that it is upper semi-continuous on $\Omega \setminus K$ for some compact subset $K$ of $\Omega$ and $\lambda \in W^{1,p}(\Omega)$, $\lambda \leq f - \delta$ on $\partial \Omega$. In this case we can also show directly, without the aid of $v_0$, that $(u - \lambda)^- \in W^{1,p}_0(\Omega)$. Thus we see that (iii) also holds. This completes the proof.

Q.E.D.

References


