<table>
<thead>
<tr>
<th>Title</th>
<th>Boundary value problems and variational inequalities (Nonlinear Analysis and Mathematical Economics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Watanabe, Hisako</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1992), 789: 124-137</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1992-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/82628">http://hdl.handle.net/2433/82628</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
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<td>Textversion</td>
<td>publisher</td>
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Boundary value problems and variational inequalities

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1. Introduction and notations

Let \(\Omega\) be a bounded open set in \(\mathbb{R}^n\) \((n \geq 2)\). We consider the obstacle problem for a quasilinear elliptic operator of second order in \(\Omega\) as follows:

\[
L = -\text{div} A(x, u, \nabla u) + B(x, u, \nabla u)
\]

Here \(A\) (resp. \(B\)) is a vector (resp. scalar) valued function defined on \(\Omega \times \mathbb{R} \times \mathbb{R}^n\) and the functions \(A\) and \(B\) are assumed to satisfy the following inequalities:

\[
|A(x, u, w)| \leq a(|w|^{p-1} + |u|^{p-1} + 1),
\]

\[
|B(x, u, w)| \leq b(|w|^{p-1} + |u|^{p-1} + 1),
\]

\[
w \cdot A(x, u, w) + uB(x, u, w) \geq c_1|w|^p - c_2(|u|^p + 1),
\]

\[
(A(x, u, w_1) - A(x, u, w_2)) \cdot (w_1 - w_2) > 0 \quad (w_1 \neq w_2)
\]

for all \(x \in \Omega\), \(u \in \mathbb{R}\) and \(w_1, w_2 \in \mathbb{R}^n\), where \(a, b, c_1, p\) are positive real numbers satisfying \(p > 1\) and \(c_2\) is a nonnegative real number.

It has been known that for each continuous function \(f\) on \(\partial\Omega\) and a function \(\phi\) on \(\Omega\) satisfying

\[
\int_0^{\infty} t^{p-1}B_{1,p}(A(t))dt < +\infty
\]

there exists a solution \(u \in W_{loc}^{1,p}(\Omega)\) to the obstacle problem with boundary data \(f\), where

\[
A(t) = \{y \in \Omega; \psi(y) > t\}.
\]

The obstacle problem is to find a function \(u \in W_{loc}^{1,p}(\Omega)\) such that

\[
\begin{align*}
u &\geq \psi \text{ on } \Omega \text{ except for a subset of } \Omega, \\
u &= f \text{ weakly on } \partial\Omega
\end{align*}
\]
and
\[ \int_{\Omega} A(y, u(y), \nabla u(y)) \cdot \nabla \phi(y) dy + \int_{\Omega} B(y, u(y), \nabla u(y)) \phi(y) dy \geq 0 \]
for all \( \phi \in C_c^\infty(\Omega) \) satisfying \( \phi \geq \psi - u \) on \( \Omega \) except for a subset of \( \Omega \).

In this paper we will prove the existence of a weak solution to the obstacle problem for (possibly) non-bounded boundary function \( f \). To consider boundary functions which value \( \pm \infty \), we must distinguish functions not up to a set of n-dimensional Lebesgue measure zero, but up to a more fine set, for example, a set of \( B_{1,s} \)-capacity zero.

Recall that for \( s > 1 \) the Bessel capacity \( B_{1,s} \) with order 1 is defined by
\[
B_{1,s}(E) = \inf\{\|g\|_{L'(\mathbb{R}^n)}; g \geq 0, G_1 * g \geq 1 \text{ on } E\}
\]
for a subset \( E \) of \( \mathbb{R}^n \). If a property holds on a subset \( X \) of \( \mathbb{R}^n \) except for a set of \( B_{1,s} \)-capacity zero, we say that it holds \( B_{1,s} \)-q.e. on \( X \). In the case \( s = p \) we use simply "q.e." instead of "\( B_{1,p} \)-q.e."

To distinguish functions up to a set of \( B_{1,s} \)-capacity zero, we construct a family of functions defined on \( \partial \Omega \), which contains all continuous functions and the restrictions of all Bessel potentials \( G_1 * g \) (\( g \in L'(\mathbb{R}^n) \)) to \( \partial \Omega \), where \( G_1 \) is the Bessel function with order 1, i.e.,
\[
G_1(x) = \frac{1}{(4\pi)^{1/2}} \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \exp\left(-\frac{\pi|x|^2}{t}\right) \exp\left(-\frac{t}{4\pi}\right) t^{(1-n)/2} \cdot \frac{1}{t} dt.
\]
Recall that the Fourier function of \( G_1 \) is equal to
\[
\frac{1}{(1 + 4\pi|x|^2)^{1/2}}.
\]
Let us define, for each extended real-valued function \( f \) on \( \partial \Omega \),
\[
\gamma_{1,s}(f) = \inf\{\|g\|_{L'(\mathbb{R}^n)}; g \geq 0, G_1 * g \geq |f| \text{ on } \partial \Omega\}.
\]
Furthermore, denote by \( B(\gamma_{1,s}) \) the family of all Borel measurable functions on \( \partial \Omega \) such that \( \gamma_{1,s}(f) < +\infty \). We remark that \( B(\gamma_{1,s}) \supset C(\partial \Omega) \), where \( C(\partial \Omega) \) is the family of all continuous real-valued functions on \( \partial \Omega \).

We denote by \( \mathcal{L}(\gamma_{1,s}) \) the family of all \( f \in B(\gamma_{1,s}) \) such that \( \gamma_{1,s}(f-f_j) \rightarrow 0 \) (\( j \rightarrow \infty \)) for some \( \{f_j\} \subset C(\partial \Omega) \).

It is well-known that, if \( 1 < p < n \) and \( v \in W^{1,p}(\Omega) \), then
\[
\lim_{r \rightarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} v(y) dy
\]
exists (as a real number) for all $x \in \Omega$ except for a set $E$ with $B_{1,p}(E) = 0$, where $B(x, r)$ is the ball with center $x$ and radius $r$ and $|B(x, r)|$ stands for the volume of the ball $B(x, r)$ (cf. [FZ]). For each $v \in W^{1,p}(\Omega)$ we denote by $v^*$ the function defined by (1.2) q.e. on $\Omega$.

Under these notations we will prove the following theorem.

**Theorem.** Let $1 < p < n$, $p < s \leq \frac{n}{n-p}$, $f \in L^{1}(\gamma_{1,\sigma})$ and $\psi$ be a real-valued function on $\Omega$ such that $|\psi| \leq G_{1} \ast g$ on $\Omega$ for some $g \in L^{1}(\mathbb{R}^{n})^{+}$. If

$$\limsup_{y \to x} \psi(y) < +\infty \text{ and } \limsup_{y \to x} \psi(y) \leq f(y)$$

for all $x \in \partial \Omega$. Then there exists a function $u \in W^{1,p}(\Omega)$ having the following properties:

(i) $u \geq \psi$ q.e. on $\Omega$,

(ii) If $\phi \in W^{1,p}(\Omega)$, supp $\phi \subset \Omega$ and $\phi^{*} \geq \psi - u$ q.e. on $\Omega$, then

$$\int_{\Omega} \{A(y, u(y), \nabla u(y)) \cdot \nabla \phi(y) + B(y, u(y), \nabla u(y)) \phi(y)\} dy \geq 0.$$

(iii) $u = f$ on $\partial \Omega$ in the following sense:

Let $\tau$ be a function on $\bar{\Omega}$ such that it is lower semi-continuous on $\bar{\Omega} \setminus K$ for some compact subset $K$ of $\Omega$ and

$$\tau|_{\Omega} \in W^{1,s}(\Omega), \quad \tau \geq f + \delta \text{ on } \partial \Omega \text{ for some } \delta > 0.$$

Then $(u - \tau)^{+} \in W^{1,p}_{0}(\Omega)$. Further, let $\lambda$ be a function on $\bar{\Omega}$ such that it is upper semicontinuous on $\bar{\Omega} \setminus K$ for some compact subset $K$ of $\Omega$ and

$$\lambda|_{\Omega} \in W^{1,s}(\Omega), \quad \lambda \leq f - \delta$$

for some $\delta > 0$ on $\partial \Omega$. Then $(u - \lambda)^{-} \in W^{1,p}_{0}(\Omega)$.

2. Properties of $\gamma_{1,\sigma}$

In this section we study the properties of $\gamma_{1,\sigma}$. It is easy to see that the functional $\gamma_{1,\sigma}$ has the following properties similar to those of the upper integral.

**Lemma 2.1.** Let $s > 1$. Then the functional $\gamma_{1,\sigma}$ has the following properties:

(c1) $\gamma_{1,\sigma}(f) = \gamma_{1,\sigma}(|f|)$,

(c2) $\gamma_{1,\sigma}(bf) = b\gamma_{1,\sigma}(f)$ for $b \in \mathbb{R}^{+}$;
\[(c_3) \ f_j \geq 0 \Rightarrow \gamma_{1,*}(\sum_{j=1}^{\infty} f_j) \leq \sum_{j=1}^{\infty} \gamma_{1,*}(f_j),
(c_4) \ \gamma_{1,*}(\chi_E) = B_{1,*}(E)^{1/s} \text{ for } E \subset \mathbb{R}^n.\]

Using Lemma 2.1, we can show the following lemma.

**Lemma 2.2.** (i) If \(\gamma_{1,*}(f) < +\infty\), then the set \(\{x \in \partial \Omega; |f(x)| = +\infty\}\) is of \(B_{1,*}\)-capacity zero.

(ii) If \(\gamma_{1,*}(f - g) = 0\), then \(f = g\) \(B_{1,*}\)-q.e. on \(\partial \Omega\).

**Lemma 2.3.** Let \(g\) be a nonnegative function in \(L^s(\mathbb{R}^n)\). Then the Bessel potential \(G_1 * g\) belongs to \(L(\gamma_{1,*})\).

**Proof.** We can assume that \(g\) is nonnegative. Set
\[g_j = \min\{g, j\} \text{ and } h_j = g - g_j.\]

Noting \(G_1 \in L^1(\mathbb{R}^n)\), we see that \(G_1 * g_j\) is continuous on \(\partial \Omega\). Since \(|G_1 * g - G_1 * g_j| \leq G_1 * h_j\) and \(\|h_j\|_s \to 0\) as \(j \to \infty\), we have the conclusion.

**Lemma 2.4.** The set \(\mathcal{L}\) of the restrictions of all Lipschitz functions on \(\Omega\) to \(\partial \Omega\) is dense in \(L(\gamma_{1,*})\).

**Proof.** We can choose a nonnegative function \(h = G_1 * g\) \((g \in L^s(\mathbb{R}^n)^+)\) such that \(h \geq 1\) on \(\partial \Omega\). Since \(\mathcal{L}\) is uniformly dense in \(C(\partial \Omega)\), it is dense in \(L(\gamma_{1,*})\).

**Lemma 2.5.** Let \(p, s\) be positive real numbers satisfying \(1 < p \leq s\) and \(E\) be a relatively compact subset of \(\mathbb{R}^n\). If \(B_{1,*}(E) = 0\), then \(B_{1,p}(E) = 0\).

**Proof.** For a function \(f\) defined on \(\mathbb{R}^n\) we define
\[\gamma_{1,*}(f) = \inf\{\|f\|_s; \ g \in L^s(\mathbb{R}^n)^+; \ G_1 * g \geq |f| \text{ on } \mathbb{R}^n\}\]

It is easy to see that this functional \(\gamma_{1,*}\) also has the properties in Lemmas 2.1 and 2.2 in which \(\partial \Omega\) is replaced by \(\mathbb{R}^n\).

**Lemma 2.6.** Let \(\{f_j\}\) be a sequence of functions on \(\mathbb{R}^n\) such that \(\gamma_{1,*}(f_j) \to 0\) \((j \to \infty)\). Then there exists a subsequence \(\{g_k\}\) of \(\{f_j\}\) such that \(g_k \to 0\) pointwisely \(B_{1,*}\)-q.e. on \(\mathbb{R}^n\).
Proof. Choose a subsequence \( \{g_k\} \) of \( \{f_j\} \) satisfying

\[
\sum_{k=1}^{\infty} 2^k \gamma_{1,\tau}(g_{k+1} - g_k) < +\infty.
\]

To show that \( \{g_k\} \) is the desired subsequence, set

\[
E = \bigcup_{k=1}^{\infty} \{x \in \mathbb{R}^n; |g_k(x)| = +\infty\}.
\]

Then we have \( B_{1,\tau}(E) = 0 \) by Lemmas 2.1 and 2.2. Further set

\[
O'_k = \{x \in \mathbb{R}^n \setminus E; |g_{k+1}(x) - g_k(x)| \geq 2^{-k}\}
\]

and

\[
O_k = \bigcup_{i=k}^{\infty} O'_i \quad \text{and} \quad F_k = \mathbb{R}^n \setminus (O_k \cup E).
\]

Setting \( g_0 = 0 \) and noting that

\[
g_k = \sum_{i=0}^{k-1} (g_{i+1} - g_i)
\]

on \( \mathbb{R}^n \setminus E \), we see that \( \{g_k\} \) converges to 0 on \( \bigcup_{i=1}^{\infty} F_i \). We put

\[
E_\circ = \mathbb{R}^n \setminus (\bigcup_{i=1}^{\infty} F_i \cup E) = \bigcap_{k=1}^{\infty} O_k.
\]

From

\[
\chi_{O_k} \leq \sum_{i=k}^{\infty} \chi_{O'_i} \leq \sum_{i=k}^{\infty} 2^i |g_{i+1} - g_i|
\]

and Lemma 2.1 we deduce

\[
\gamma_{1,\tau}(\chi_{O_k}) \leq \sum_{i=k}^{\infty} 2^i \gamma_{1,\tau}(g_{i+1} - g_i).
\]

On account of (2.1) we have

\[
\gamma_{1,\tau}(\chi_{O_k}) \to 0 \quad (k \to \infty)
\]

and hence \( \gamma_{1,\tau}(\chi_{E_\circ}) = 0 \). Therefore we see by Lemma 2.1 that \( B_{1,\tau}(E \cup E_\circ)^{1/s} = \gamma_{1,\tau}(\chi_{E \cup E_\circ}) = 0 \) and \( \{g_k\} \) converges to 0 on \( \mathbb{R}^n \setminus (E \cup E_\circ) \). Q.E.D.

3. Boundedness of solutions
For an open subset $\Omega_0 \neq \emptyset$ of $\Omega$ we denote by $A(\Omega_0, \cdot)$ the mapping

$$W^{1,p}(\Omega_0) \to W^{1,p}(\Omega_0)'$$

defined by

$$\langle A(\Omega_0, v), w \rangle = \int_{\Omega_0} \{A(y, v(y), \nabla v(y)) \cdot \nabla w(y) + B(y, v(y), \nabla v(y))w(y)\}$$

The following theorem is fundamental.

**Theorem A** ([MZ, Theorem 3.1]). Let $p < s$ and $\Omega_0$ be a nonempty open subset of $\Omega$, $\epsilon = +$ or $-$ and $v, \eta$ be functions in $W^{1,p}(\Omega_0)$ such that $(v - \eta)^\epsilon \in W^{1,p}_0(\Omega_0)$ and

$$\langle A(\Omega_0, v), -\epsilon(v - \eta)^\epsilon \rangle \geq 0.$$

Then

$$\| (v - \eta)^\epsilon \|_{W^{1,p}(\Omega_0)} \leq c + c \left( 1 + \int_{\Omega_0} (|\eta|^s + \sum_{i=1}^n |\partial y_i|^s) dy \right)^{1/p},$$

where $c$ is a constant independent of $v, \eta$.

It is well-known that for each $s > 1$

$$W^{1,s}(\mathbb{R}^n) = \{G_1 * g; g \in L^s(\mathbb{R}^n)\}$$

and

$$\frac{1}{M} \| g \|_s \leq \| G_1 * g \|_{W^{1,s}(\mathbb{R}^n)} \leq M \| g \|_s,$$

where $M$ is a constant independent of $g$ (cf. [S, Theorem 3 on p.135]).

**Lemma 3.1.** Let $f$ be a Lipschitz function on $\Omega$ such that $|f| \leq G_1 * g_1$ for some $g_1 \in L^p(\mathbb{R}^n)^+$. Furthermore, let $\psi$ be a real-valued function on $\Omega$ such that

$$\lim_{y \to x, y \in \Omega} \sup_{y \to x, y \in \Omega} \psi(y) < f(x) - \delta$$

for all $x \in \partial \Omega$ and for some $\delta > 0$, and

$$|\psi| \leq G_1 * g_o \text{ for some } g_o \in L^p(\mathbb{R}^n)^+.$$
Then there exists a function $u \in W^{1,p}(\Omega)$ such that $u$ has the properties:

(i) $u \geq \psi$ q.e. on $\Omega$,

(ii) If $\phi \in W^{1,p}_o(\Omega)$ and $\phi^* \geq \psi - u$ q.e. on $\Omega$, then

$$
\int_{\Omega} \{A(y, u(y), \nabla u(y)) \cdot \nabla \phi(y) + B(y, u(y), \nabla u(y))\phi(y)\}dy \geq 0.
$$

(iii) $u - f \in W^{1,p}_o(\Omega)$,

(iv) If $\eta$ (resp. $\lambda$) is a function on $\overline{\Omega}$ such that it is lower (resp. upper) semicontinuous on $\overline{\Omega}\setminus K$ for some compact subset $K$ of $\Omega$ and $\eta \in W^{1,p}(\Omega)$ (resp. $\lambda \in W^{1,p}(\Omega)$), $\eta(y) > f(y)$ (resp. $\lambda(y) < f(y)$) for all $y \in \partial\Omega$, then $(u - \eta)^+ \in W^{1,p}_o(\Omega)$ (resp. $(u - \lambda)^- \in W^{1,p}_o(\Omega)$).

**Proof.** Set

$$K = \{v \in W^{1,p}(\Omega); \; v - f \in W^{1,p}_o(\Omega), \; v^* \geq \psi \text{ q.e. on } \Omega\}.$$

We claim that $K$ is not empty. Indeed, noting that $G_1 \ast g_1$ is lower semicontinuous, we can choose, by the aid of (3.2), an open set $\Omega_0$ such that $\overline{\Omega_0} \subset \Omega$ and

$$\psi(y) < f(y) \quad \text{for all } y \in \Omega \setminus \Omega_0.$$ 

Choose a Lipschitz function $h$ such that

$$\operatorname{supp} h \subset \Omega, \quad h = 1 \text{ on } \overline{\Omega_0}, \quad 0 \leq h \leq 1,$$

and define

$$\phi(y) = h(y)w(y) + (1 - h(y))f(y),$$

where $w = G_1 \ast g_o$. We note that $\operatorname{supp} h$ stands for the closure of the set $\{y; \; h(y) \neq 0\}$. Then $\phi^* \geq \psi$ q.e. on $\Omega$ and $\phi - f = h(w - f) \in W^{1,p}_o(\Omega)$. Therefore we see that $\phi \in K$.

The family $K$ is a convex closed subset of $W^{1,p}(\Omega)$ and hence weakly closed. The mapping $A(\Omega, \cdot)$ from $W^{1,p}(\Omega)$ to $W^{1,p}(\Omega)'$ is pseudomonotone by Theorem 3.9 in [MZ]. Furthermore we see that

$$\frac{\langle A(\Omega, v), v - v_o \rangle}{\|v\|_{W^{1,p}(\Omega)}} \to \infty$$

as $\|v\|_{W^{1,p}(\Omega)} \to \infty$ ($v \in K$). It follows from Theorem 8.2 on p.247 in [L] that there exists $u_o \in K$ such that

$$\langle A(\Omega, u_o), v - u_o \rangle \geq 0 \text{ for all } v \in K.$$
Setting \( u = u_{o}^{*} \), we will show that \( u \) is the desired function. It is obvious that (i) and (iv) hold. To show (ii), let \( \phi \) be a function in \( W_{o}^{1,p}(\Omega) \) such that
\[
\phi^{*} \geq \psi - u \text{ q.e. on } \Omega.
\]
From \( u - f \in W_{o}^{1,p}(\Omega) \) and \( \phi + u - f \in W_{o}^{1,p}(\Omega) \), it follows that
\[
(A(\Omega, u), \phi) \geq 0.
\]
Finally, to show (v), let \( \eta \) be a lower semicontinuous function on \( \Omega \) in \( W^{1,p}(\Omega) \) such that \( \eta > f \) on \( \partial \Omega \). Since \( f - \eta < 0 \) outside a compact subset of \( \Omega \) and \( u - f \in W_{o}^{1,p}(\Omega) \), we have
\[
(u - \eta)^{+} \in W_{o}^{1,p}(\Omega).
\]
Similarly we can show that \( (u - \lambda)^{-} \in W_{o}^{1,p}(\Omega) \). Q.E.D.

4. Proof of Theorem

Let us prove Theorem. Suppose that \( f \in \mathcal{L}(\gamma_{1,s}) \). On account of Lemma 2.4 we can choose a sequence \( \{f_{j}\} \) of Lipschitz functions on \( \Omega \) and a sequence \( \{g_{j}\} \) of functions in \( L^{1}(\mathbb{R}^{n})^{+} \) such that
\[
|f - f_{j}| \leq G_{1} * g_{j} \text{ on } \partial \Omega \quad ||g_{j}||_{s} < 2^{-j}.
\]
Since \( \gamma_{1,s}(G_{1} * g_{j}) \rightarrow 0 \), we can choose, by Lemma 2.6, a subsequence \( \{G_{1} * h_{k}\} \) converges pointwisely to \( 0 \) \( B_{1,s} \)-q.e. on \( \mathbb{R}^{n} \). Therefore, by Lemma 2.5, it converges to 0 q.e. on \( \Omega \). Noting that
\[
\lim_{y \rightarrow x} \sup_{y \rightarrow x} \psi(y) \leq f(x) \leq G_{1} * h_{k}(x) + f_{j_{k}}(x)
\]
for all \( x \in \partial \Omega \), we define
\[
\psi_{k}(y) = \psi(y) - G_{1} * h_{k}(y) - 2^{-k} \quad \text{if } G_{1} * h_{k}(y) < +\infty
\]
and
\[
\psi_{k}(y) = -\sup_{x \in \partial \Omega} |f_{j_{k}}(x)| - 1 \quad \text{otherwise}.
\]
Then we have
\[
\lim_{y \rightarrow x} \sup_{y \rightarrow x} \psi_{k}(y) \leq f_{j_{k}}(x) - 2^{-k}.
\]
Pick $h_0, h' \in L'(\mathbb{R}^n)^+$ such that
\[ G_1 * h_0 \geq |f| \text{ on } \partial \Omega, \quad G_1 * h' \geq 1 \text{ on } \partial \Omega. \]

If we set
\[
(4.1) \quad h = h' + \sum_{k=0}^{\infty} h_k,
\]
then
\[
(4.2) \quad |f_{j_k}| + 1 \leq |f| + G_1 * h_k + G_1 * h' \leq G_1 * h
\]
on $\partial \Omega$.

We denote by $u_k$ the solution $u$ in Lemma 3.1 corresponding to $f = f_{j_k}$ and $\psi = \psi_k$. Let $\Omega_o$ be an arbitrary subset of $\Omega$ such that $\Omega_o \subset \Omega$.

To show that
\[ \{||u_k||_{W^{1,p}(\Omega_o)}\} \]
is uniformly bounded, let us take a Lipschitz function $\eta$ such that
\[ \supp \eta \subset \Omega, \quad \eta = 1 \text{ on } \overline{\Omega_o}, \quad 0 \leq \eta \leq 1. \]

Further, take $g \in L^p(\mathbb{R}^n)^+$ satisfying $|\psi| \leq G_1 * g$ on $\Omega$ and define
\[ \beta(y) = \eta(y)G_1 * g(y) + (1 - \eta(y))G_1 * h(y), \]
\[ \phi(y) = \eta(y)\beta(y) - (1 - \eta(y))G_1 * h(y) \]
for $y \in \Omega$. Then $\phi \in W^{1,p}(\Omega)$. Applying the Sobolev inequality and (3.1), and noting that $s \leq \frac{np}{n-p}$, we obtain
\[ \int_{\Omega} |\beta(y)|^p + \sum_{j=1}^{n} \int_{\Omega} |\frac{\partial \beta}{\partial y_j}(y)|^p dy < +\infty \]
and
\[ \int_{\Omega} |\phi(y)|^p + \sum_{j=1}^{n} \int_{\Omega} |\frac{\partial \phi}{\partial y_j}(y)|^p dy < +\infty. \]

Note that
\[
(4.3) \quad (u_k - \phi)^- \geq 0 \geq \psi_k - u_k \quad \text{q.e. on } \Omega.
\]
By the aid of (4.2) we have

$$f_{j_{k}} - 1 \geq -G_{1} \ast h = \phi \quad \text{on } \partial \Omega.$$ 

From Lemma 3.1, (iv) we deduce \((u_{k} - \phi)^{-} \in W_{o}^{1,p}(\Omega)\), which and (4.3) lead to

$$\langle A(\Omega, u_{k}), (u_{k} - \phi)^{-} \rangle \geq 0.$$ 

Therefore we obtain, by Theorem A,

$$\| (u_{k} - \phi)^{-} \|_{W^{1,p}(\Omega)} \leq c + c \left( \int_{\Omega} |\phi(y)|^{r} + \sum_{j=1}^{n} \int_{\Omega} \left| \frac{\partial \phi}{\partial y_{j}}(y) \right|^{p} dy \right)^{1/p}$$

and hence

$$\| (u_{k} - \beta)^{-} \|_{W^{1,p}(\Omega_{o})} \leq c + c \left( \int_{\Omega} |\beta(y)|^{r} + \sum_{j=1}^{n} \int_{\Omega} \left| \frac{\partial \beta}{\partial y_{j}}(y) \right|^{p} dy \right)^{1/p} \quad (4.4)$$

On the other hand, since

$$f_{j_{k}} + 1 < G_{1} \ast h \leq \beta \quad \text{on } \partial \Omega,$$

we have \((u_{k} - \beta)^{+} \in W_{o}^{1,p}(\Omega)\). Moreover, if \(u_{k}(y) \leq \beta(y)\), then

$$-(u_{k} - \beta)^{+}(y) = 0 \geq \psi_{k}(y) - u_{k}(y)$$

for q.e. \(y\). If \(u_{k}(y) > \beta(y)\), then

$$-(u_{k} - \beta)^{+}(y) = \beta(y) - u_{k}(y) \geq \psi_{k}(y) - u_{k}(y)$$

for q.e. \(y\). Therefore we have

$$\langle A(\Omega, u_{k}), -(u_{k} - \beta)^{+} \rangle \geq 0$$

by Lemma 3.1, (ii). This and Theorem A lead to

$$\| (u_{k} - \beta)^{+} \|_{W^{1,p}(\Omega)} \leq c + c \left( \int_{\Omega} |\beta(y)|^{r} + \sum_{j=1}^{n} \int_{\Omega} \left| \frac{\partial \beta}{\partial y_{j}}(y) \right|^{p} dy \right)^{1/p} \quad (4.5)$$
Thus we see by (4.4) and (4.5) that
\[ \{ \| u_k - \beta \|_{W^{1,p}(\Omega_o)} \} \]
and hence
\[ \{ \| u_k \|_{W^{1,p}(\Omega_o)} \} \]
is uniformly bounded for every open set \( \Omega_o \) satisfying \( \bar{\Omega}_o \subset \Omega \). Therefore we can choose a subsequence \( \{ u_{k_i} \} \) and \( w \in W^{1,p}_{\text{loc}}(\Omega) \) such that for every open set \( \Omega_o \) satisfying \( \bar{\Omega}_o \subset \Omega \) the sequence \( \{ u_{k_i} \} \) converges weakly to \( w \) in \( W^{1,p}(\Omega_o) \). Since identity mapping is a compact operator from \( W^{1,p}(\Omega_o) \) to \( L^p(\Omega_o) \), we may suppose that \( \{ u_{k_i} \} \) converges strongly to \( w \) in \( L^p(\Omega_o) \).

We note that
\[ u_k = u_k^* \geq \psi_k \text{ q.e. on } \Omega, \quad \lim_{k \to \infty} \psi_k = \psi \text{ q.e. on } \Omega. \]

Since there exists a subsequence of convex combinations of the functions \( u_k \), which converges strongly to \( w \) in \( W^{1,p}(\Omega_o) \), we conclude that
\[ w^* \geq \psi \text{ q.e. on } \Omega. \]

Moreover, by the same method as in the proof of Lemma 4.5 in [MZ] we can show that
\[ u_k \to w \text{ strongly in } W^{1,p}(\Omega_o). \]

Putting \( u = w^* \), we will show that \( u \) is the desired function. The assertion (i) follows from (4.6). To show (ii), suppose that \( \phi \in W^{1,p}(\Omega) \), supp \( \phi \subset K \) and
\[ \phi^* \geq \psi - u \text{ q.e. on } \Omega. \]

Take an open set \( \Omega_o \) such that supp \( \phi \subset \Omega_o \subset \bar{\Omega}_o \subset \Omega \) and choose a Lipschitz function \( \tau \) on \( \Omega \) such that
\[ \text{supp } \tau \subset \Omega, \quad \tau = 1 \text{ on } \bar{\Omega}_o, \quad 0 \leq \tau \leq 1. \]

We define
\[ \phi_i = \phi + \tau(u - u_{k_i})^+. \]

We note that
\[ \phi_i^* = \phi^* + u - u_{k_i}^* \geq \psi_k - u_{k_i} \text{ q.e. on } \Omega_o \cap \{ u \geq u_{k_i} \} \]
and
\[ \phi_i^* = \phi^* \geq \psi - u \geq \psi_k - u_{k_i}^* \text{ q.e. on } \Omega_o \cap \{ u < u_{k_i} \}. \]
On account of Lemma 3.1, (ii) we have

\[ \langle A(\Omega_o, u_k), \phi_i \rangle \geq 0 \quad \text{for each} \quad i. \]

Since

\[ u_k \rightharpoonup u \text{ strongly in } W^{1,p}(\Omega_o) \]

and

\[ \phi_i \rightharpoonup \phi \text{ strongly in } W^{1,p}(\Omega_o), \]

we see that

\[ \langle A(\Omega_o, u), \phi \rangle \geq 0 \]

and hence

\[ \langle A(\Omega, u), \phi \rangle \geq 0. \]

Next, to show (iii), denote by \( E \) the set

\[ \{ x \in \partial\Omega; |f(x)| = +\infty \} \cup \{ x \in \partial\Omega; \lim_{k \to \infty} G_1 * h_k(x) \neq 0 \}. \]

Then \( \gamma_{1,*}(\chi_E) = 0 \) and

\[ \lim_{k \to \infty} f_{j_k}(x) = f(x) \quad \text{for } x \in \partial\Omega \setminus E. \]

Suppose that \( \tau \) is a function on \( \bar{\Omega} \) such that it is lower semicontinuous on \( \bar{\Omega} \setminus K \) for some compact subset \( K \) of \( \Omega \) and \( \tau \in W^{1,*}(\Omega) \), \( \tau \geq f + \delta \) on \( \partial\Omega \) for some \( \delta > 0 \). Since

\[ \limsup_{y \to x} \psi(y) < +\infty \quad \text{and} \quad \limsup_{y \to x} \psi(y) \leq f(x) \]

for all \( x \in \partial\Omega \), there is an open set \( \Omega_1 \) satisfying \( \bar{\Omega}_1 \subset \Omega \) and

\[ \psi(y) < \tau(y) \quad \text{for all } y \in \Omega \setminus \Omega_1. \]

Take a Lipschitz function \( \eta \) such that

\[ \text{supp} \ \eta \subset \Omega, \quad \eta = 1 \text{ on } \bar{\Omega}_1, \quad 0 \leq \eta \leq 1 \]

and define

\[ v_o = (G_1 * h) \eta, \]

where \( h \) is the function defined in (4.1).

Let us show that \( (u - \tau)^+ \in W^{1,p}_0(\Omega) \). Since \( \text{supp} \ \eta \subset \Omega \) and \( v_o \in W^{1,p}(\Omega) \), it suffices to show that \( (u - \tau - v_o)^+ \in W^{1,p}_0(\Omega) \). From the inequality

\[ f_{j_k} < \tau + v_o + G_1 * h_k \quad \text{on } \partial\Omega \]
and lemma 3.1 we deduce
\[(u_k - (\tau + v_o + G_1 * h_k))^+ \in W^{1,p}_0(\Omega).\]

We will show that
\[\{(u_k - (\tau + v_o + G_1 * h_k))^+\} \in W^{1,p}(\Omega)\]
is uniformly bounded. We claim that
\[-(u_k - (\tau + v_o + G_1 * h_k))^+ \geq \psi_k - u_k \quad \text{q.e. on } \Omega.
\]
Indeed we have
\[-(u_k - (\tau + v_o + G_1 * h_k))^+ \geq -(u_k - (\tau + v_o + G_1 * h_k)) \geq \psi_k - u_k \quad \text{q.e. on } \Omega \cap \{u_k \geq \tau + v_o + G_1 * h_k\}\]
and
\[-(u_k - (\tau + v_o + G_1 * h_k))^+ = 0 \geq \psi_k - u_k \quad \text{q.e. on } \Omega \cap \{u_k < \tau + v_o + G_1 * h_k\}.
\]

Therefore, from Lemma 3.1, (ii) it follows that
\[\{A(\Omega, u_k), -(u_k - (\tau + v_o + G_1 * h_k))^+\} \geq 0.
\]

Using Theorem A and (3.1), we have
\[\|(u_k - (\tau + v_o + G_1 * h_k))^+\|_{W^{1,p}(\Omega)} \leq c_1
\]
\[+ c_1 \left( \int_\Omega |(\tau + v_o + G_1 * h_k)|^p \sum_{j=1}^n \int_\Omega \frac{\partial}{\partial y_j} (\tau + v_o + G_1 * h_k)^p \right)^{1/p}
\]
\[\leq c_2 + c_2 M \left( \int_\Omega |(\tau + v_o)|^p + \sum_{j=1}^n \int_\Omega \frac{\partial}{\partial y_j} (\tau + v_o)^p \right)^{1/p}
\]

Thus we see that
\[\{(u_k - (\tau + v_o + G_1 * h_k))^+\} \in W^{1,p}(\Omega)\]
is uniformly bounded. We note that for every open set \(\Omega_o\) satisfying \(\Omega \subset \Omega_o\), \(\{u_k\}\) converges to \(u\) strongly in \(W^{1,p}(\Omega_o)\) and
\[\|G_1 * h_k\|_{W^{1,\infty}(\Omega_o)} \leq M\|h_k\|_* \to 0\]
as $i \to \infty$. Using Lemma 4.6 in [MZ], we conclude that $(u - (\tau + v_o))^+ \in W^{1,p}_0(\Omega)$ and hence $(u - \tau)^+ \in W^{1,p}_0(\Omega)$.

Finally, suppose that $\lambda$ is a function on $\overline{\Omega}$ such that it is upper semi-continuous on $\overline{\Omega} \setminus K$ for some compact subset $K$ of $\Omega$ and $\lambda \in W^{1,p}(\Omega)$, $\lambda \leq f - \delta$ on $\partial \Omega$. In this case we can also show directly, without the aid of $v_o$, that $(u - \lambda)^- \in W^{1,p}_0(\Omega)$. Thus we see that (iii) also holds. This completes the proof. Q.E.D.

References


