数理解析研究所講究録
第 790 巻 1992 年 208-214

平衡弱合流性と項書換えシステムの正規戦略
Balanced Weakly Church-Rosser and Normalizing Strategy for Term Rewriting Systems
- Detailed Abstract -

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1 Introduction

The normalizing reduction strategies of reduction systems, such as leftmost-outermost evaluation of lambda calculus, combinatory logic, ordinal recursive program schemata and left-normal term rewriting systems, guarantee a *safe* evaluation which reduces a given expression to its normal form whenever possible [2, 5, 6, 8].

Strong sequentiality formalized by Huet and Lévy [3] is a well-known practical criterion guaranteeing an efficiently computable normalizing reduction strategy for orthogonal (i.e., left-linear and non-overlapping) term rewriting systems. They showed that for every strongly sequential orthogonal term rewriting system, index reduction is a normalizing strategy, that is, by rewriting a redex called an *index* at each step, every reduction starting with a term having a normal form eventually terminates at the normal from. Here, the index is defined as a *needed redex* concerning an approximation of R which is obtained by analyzing the left-hand sides alone of the rewriting rules of term rewriting systems. Moreover, Huet and Lévy [3] showed the decidability of strong sequentiality.

Extensions of strong sequentiality have been reported by [7, 9, 10]. Thatte [10] introduced left sequentiality and showed the equivalence of strong sequentiality and left sequentiality for the class of orthogonal constructor systems. Oyamaguchi [9] introduced sufficient sequentiality not only based on the analysis the left-hand sides of the rewriting rules of term rewriting systems but also on the non-variable parts of the right-hand sides. He showed that the notion of sufficient sequentiality properly extends the notion of strong sequentiality. Klop and Middeldorp [7] presented a simple proof of the decidability of strong sequentiality and compared these extensions of strong sequentiality. However, all these results are restricted to orthogonal term rewriting systems; hence, they cannot be applied to term rewriting systems with overlapping rules such as

$$\begin{cases} pred(succ(x)) \to x \\ succ(pred(x)) \to x \end{cases}$$

Concerning overlapping term rewriting systems, Kennway [5] proved the surprising fact that every weakly orthogonal term rewriting system has a computable normalizing strategy. How-

*Partially supported by grants from NWO, Vrije Universiteit Amsterdam, and Katholieke Universiteit Nijmegen

ever, the weak orthogonality in [5] is restricted to root overlapping rules such as *parallel-or*, i.e., $or(true, x) \to true$ and $or(x, true) \to true$. Thus, his result cannot be applied to the above non-root overlapping situation. Moreover, the algorithm proposed in [5] is too complicated to find a needed redex effectively.

In this paper, we extend the result by Huet and Lévy to *overlapping* term rewriting systems. The notions of index and strong sequentiality are naturally extended to the overlapping situation. Under these extensions, we show that index reduction is normalizing for the class of strongly sequential *balanced ambiguous* term rewriting systems. The balanced ambiguous term rewriting system is defined as a left-linear term rewriting system in which every critical pair can be joined with *the root balanced reductions*. We also show that this class includes all weakly orthogonal left-normal systems (not being restricted to root overlapping rules), for which leftmost-outermost reduction strategy is normalizing. For example, leftmost-outermost reduction strategy is normalizing for combinatory logic CL + $\{pred \cdot (succ \cdot x) \to x,\ succ \cdot (pred \cdot x) \to x\}$. Moreover, our result can be applied to term rewriting systems not having the Church-Rosser property too. For example, leftmost-outermost reduction strategy is again normalizing for CL +

$$\begin{cases} (K \cdot A) \cdot y \to (K \cdot B) \cdot y \\ (K \cdot B) \cdot y \to (K \cdot A) \cdot y \\ A \to A \\ B \to B, \end{cases}$$

though the system is not Church-Rosser since $(K \cdot A) \cdot y$ can be reduced into two terms A and B which cannot be joined.

2 Reduction Systems

Assuming that the reader is familiar with the basic concepts and notations concerning reduction systems in [4, 6], we briefly explain notations and definitions.

A reduction system (or an abstract reduction system) is a structure $A = \langle D, \to \rangle$ consisting of some set D and some binary relation $\to$ on D (i.e., $\to \subseteq D \times D$), called a reduction relation. A reduction (starting with x_0) in A is a finite or infinite sequence $x_0 \to x_1 \to x_2 \to \cdots$. The identity of elements x, y of D is denoted by $x \equiv y$. $\overset{\equiv}{\to}$ is the reflexive closure of $\to$, $\leftrightarrow$ is the symmetric closure of $\to$, $\overset{+}{\to}$ is the transitive closure of $\to$, $\overset{*}{\to}$ is the transitive reflexive closure of $\to$, and $=$ is the equivalence relation generated by $\to$ (i.e., the transitive reflexive symmetric closure of $\to$). $\overset{m}{\to}$ denotes a reduction of m ($m \geq 0$) steps. If $x \in D$ is minimal with respect to $\to$, i.e., $\neg\exists y \in D[x \to y]$, then we say that x is a normal form; let NF be the set of normal forms. If $x \overset{*}{\to} y$ and $y \in NF$ then we say x has a normal form y and y is a normal form of x.

Definition 2.1 *$A = \langle D, \to \rangle$ is strongly normalizing (or terminating) iff every reduction in A terminates, i.e., there is no infinite sequence $x_0 \to x_1 \to x_2 \to \cdots$.*

Definition 2.2 *$A = \langle D, \to \rangle$ is Church-Rosser (or confluent) iff*
$\forall x, y, z \in D[x \overset{*}{\to} y \land x \overset{*}{\to} z \Rightarrow \exists w \in A, y \overset{*}{\to} w \land z \overset{*}{\to} w]$.

Definition 2.3 *$A = \langle D, \to \rangle$ is weakly Church-Rosser (or weakly confluent) iff*
$\forall x, y, z \in D[x \to y \land x \to z \Rightarrow \exists w \in A, y \overset{*}{\to} w \land z \overset{*}{\to} w]$.

The following propositions are well known [4, 6].

Proposition 2.4 *Let A be Church-Rosser, then,*

(1) $\forall x, y \in D[x = y \Rightarrow \exists w \in D, x \overset{*}{\to} w \land y \overset{*}{\to} w]$,

(2) $\forall x, y \in NF[x = y \Rightarrow x \equiv y]$,

(3) $\forall x \in D \forall y \in NF[x = y \Rightarrow x \xrightarrow{*} y]$.

Definition 2.5 (Reduction Strategy) *Let $A = \langle D, \rightarrow \rangle$ and let $\underset{s}{\rightarrow}$ be a subrelation of $\xrightarrow{+}$ (i.e., if $x \underset{s}{\rightarrow} y$ then $x \xrightarrow{+} y$) such that a normal form concerning $\underset{s}{\rightarrow}$ is also a normal form concerning $\rightarrow$ (i.e., two binary relations $\underset{s}{\rightarrow}$ and $\rightarrow$ have the same domain). Then, we say that $\underset{s}{\rightarrow}$ is a reduction strategy for A (or for $\rightarrow$). If $\underset{s}{\rightarrow}$ is a subrelation of $\rightarrow$ then we call it a one step reduction strategy; otherwise $\rightarrow$ is called a many step reduction strategy.*

Definition 2.6 (Normalizing Strategy) *A reduction strategy $\underset{s}{\rightarrow}$ is normalizing iff for each x having a normal form concerning $\rightarrow$, there exists no infinite sequence $x \equiv x_0 \underset{s}{\rightarrow} x_1 \underset{s}{\rightarrow} x_2 \underset{s}{\rightarrow} \cdots$ (i.e., every $\underset{s}{\rightarrow}$ reduction starting with x must eventually terminate at a normal form of x).*

3 Balanced Weak Church-Rosser Property

This section introduces the new concept of balanced weak Church-Rosser property. Though in the later sections this concept will play an important role for analyzing normalizing strategies of term rewriting systems, our results concerning the balanced weak Church-Rosser property can be presented in an abstract framework depending solely on the reduction relation.

Let $A = \langle D, \rightarrow \rangle$ be an abstract reduction system.

Definition 3.1 *$A = \langle D, \rightarrow \rangle$ is balanced weakly Church-Rosser (BWCR) iff*
$\forall x, y, z \in D[x \rightarrow y \wedge x \rightarrow z \Rightarrow \exists w \in D, \exists k \geq 0, y \xrightarrow{k} w \wedge z \xrightarrow{k} w]$.

We define the *local* Church-Rosser property and the *local* strong normalizing property for an element $x \in D$. x is strongly normalizing if every reduction starting with x terminates. x is Church-Rosser if
$\forall y, z \in D[x \xrightarrow{*} y \wedge x \xrightarrow{*} z \Rightarrow \exists w \in A, y \xrightarrow{*} w \wedge z \xrightarrow{*} w]$.
x is complete if x is Church-Rosser and strongly normalizing.

Lemma 3.2 (BWCR Lemma) *Let $A = \langle D, \rightarrow \rangle$ be BWCR. Let $x = y$ and $y \in NF$. Then,*

(1) *x is complete.*

(2) *All the reductions from x to y have the same length (i.e., the same number of reduction steps).*

The following lemma is essential to relate the balanced weak Church-Rosser property to a normalizing reduction strategy.

Lemma 3.3 *Let $\underset{\alpha}{\rightarrow}$ and $\underset{\beta}{\rightarrow}$ be two reduction relations on D such that:*

(1) *$\underset{\alpha}{\rightarrow}$ is balanced weakly Church-Rosser,*

(2) *If $x \underset{\beta}{\rightarrow} y$ then;*

  (i) *$x \underset{\alpha}{=} y$ or,*

  (ii) *$x \underset{\alpha}{=} \cdot \underset{\beta}{\rightarrow} \cdot \overset{+}{\underset{\alpha}{\leftarrow}} y$.*

If $x \overset{*}{\underset{\beta}{\rightarrow}} y$ and $y \in NF_\alpha$, then we have $x \overset{*}{\underset{\alpha}{\rightarrow}} y$. Here, NF_α is the set of the normal forms concerning $\underset{\alpha}{\rightarrow}$.

4 Term Rewriting Systems

In the following sections, we will explain how to apply the BWCR lemma to term rewriting systems. We briefly explain the basic notions and definitions concerning term rewriting systems [1, 4, 6].

Let $\mathcal{F}$ be an enumerable set of function symbols denoted by $f, g, h, \cdots$, and let $\mathcal{V}$ be an enumerable set of variable symbols denoted by $x, y, z, \cdots$ where $\mathcal{F} \cap \mathcal{V} = \phi$. By $T(\mathcal{F}, \mathcal{V})$, we denote the set of terms constructed from $\mathcal{F}$ and $\mathcal{V}$. The term set $T(\mathcal{F}, \mathcal{V})$ is sometimes denoted by T.

A substitution θ is a mapping from a term set $T(\mathcal{F}, \mathcal{V})$ to $T(\mathcal{F}, \mathcal{V})$ such that for a term t, $\theta(t)$ is completely determined by its values on the variable symbols occurring in t. Following common usage, we write this as $t\theta$ instead of $\theta(t)$.

Consider an extra constant $\square$ called a hole and the set $T(\mathcal{F} \cup \{\square\}, \mathcal{V})$. Then $C \in T(\mathcal{F} \cup \{\square\}, \mathcal{V})$ is called a context on $\mathcal{F}$. We use the notation $C[\ ,\ldots,\ ]$ for the context containing n holes ($n \geq 0$), and if $t_1, \ldots, t_n \in T(\mathcal{F}, \mathcal{V})$, then $C[t_1, \ldots, t_n]$ denotes the result of placing $t_1, \ldots, t_n$ in the holes of $C[\ ,\ldots,\ ]$ from left to right. In particular, $C[\ ]$ denotes a context containing precisely one hole. s is called a subterm of $t \equiv C[s]$. If s is a subterm occurrence of t, then we write $s \subseteq t$. If a term t has an occurrence of some (function or variable) symbol e, we write $e \in t$. The variable occurrences $z_1, \cdots, z_n$ of $C[z_1, \cdots, z_n]$ are fresh if $z_1, \cdots, z_n \notin C[\ ,\cdots,\ ]$ and $z_i \not\equiv z_j$ $(i \neq j)$.

A rewriting rule is a pair $\langle l, r \rangle$ of terms such that $l \notin \mathcal{V}$ and any variable in r also occurs in l. We write $l \to r$ for $\langle l, r \rangle$. A redex is a term $l\theta$, where $l \to r$. In this case $r\theta$ is called a contractum of $l\theta$. The set of rewriting rules defines a reduction relation $\to$ on T as follows:

$t \to s$ iff $t \equiv C[l\theta]$, $s \equiv C[r\theta]$
for some rule $l \to r$, and some $C[\ ]$, θ.

When we want to specify the redex occurrence $\Delta \equiv l\theta$ of t in this reduction, we write $t \xrightarrow{\Delta} s$.

Definition 4.1 *A term rewriting system R is a reduction system $R = \langle T(\mathcal{F}, \mathcal{V}), \to \rangle$ such that the reduction relation $\to$ on $T(\mathcal{F}, \mathcal{V})$ is defined by a set of rewriting rules. If R has $l \to r$ as a rewriting rule, we write $l \to r \in R$.*

We say that R is left-linear if for any $l \to r \in R$, l is linear (i.e., every variable in l occurs only once).

Let $l \to r$ and $l' \to r'$ be two rules in R. Assume that we have renamed the variables appropriately, so that l and l' share no variables. Assume that $s \notin V$ is a subterm occurrence in t, i.e., $t \equiv C[s]$, such that s and l' are unifiable, i.e., $s\theta \equiv l'\theta$, with a minimal unifier θ [4, 6]. Then we say that $l \to r$ and $l' \to r'$ are overlapping, and that the pair $\langle C[r']\theta, r\theta \rangle$ of terms is critical in R [4] . We may choose $l \to r$ and $l' \to r'$ to be the same rule, but in this case we shall not consider the case $s \equiv l$, which gives the trivial pair $\langle r, r \rangle$. If R has no critical pair, then we say that R is non-overlapping. If every critical pair $\langle s, t \rangle$ is trivial, i.e., $s \equiv t$, then R is weakly overlapping [4, 6].

R is orthogonal if R is left-linear and non-overlapping. R is weakly orthogonal if R is left-linear and weakly overlapping. The following result is well known [4, 6].

Proposition 4.2 *Let R be orthogonal (or weakly orthogonal). Then R is Church-Rosser.*

5 Strong Sequentiality

The fundamental concept of strong sequentiality for orthogonal term rewriting systems was introduced by Huet and Lévy [3]. In this section we explain the basic notions and properties related to strong sequentiality, according to Huet and Lévy [3], and Klop and Middeldorp [7]. Instead of the orthogonality, we assume only left-linearity for term rewriting systems. Thus, strong sequentiality defined here is an extension of the original one in [3].

Note. From here on we assume that R is a left-linear term rewriting system which may have *overlapping* rules.

Consider an extra constant Ω and the set $T(\mathcal{F} \cup \{\Omega\}, \mathcal{V})$, denoted by T_Ω. The element of T_Ω is called a Ω-term.

Definition 5.1 *The preordering $\geq$ on T_Ω is defined as follows:*

$t \geq \Omega$ *for all* $t \in T_\Omega$,

$f(t_1, \cdots, t_n) \geq f(s_1, \cdots, s_n) \quad (n \geq 0) \quad$ *if* $t_i \geq s_i$ *for* $i = 1, \cdots, n$.

We write $t > s$ if $t \geq s$ and $t \not\equiv s$.

Definition 5.2 (Compatibility) *Two Ω-terms t and s are compatible, denoted by $t \uparrow s$, if there exists some Ω-term r such that $r \geq t$ and $r \geq s$; otherwise, t and s are incompatible, denoted by $t \# s$. Let S be a set of Ω-term. Then $t \uparrow S$ if there exists some $s \in S$ such that $t \uparrow s$; otherwise, $t \# S$.*

Let t_Ω denote the Ω-term obtained from a term t by replacing each variable in t with Ω. The set of redex schemata of R is $Red = \{\, l_\Omega \mid l \to r \in R\}$. The Ω-reduction $\to_\Omega$ is defined on T_Ω as $C[s] \to_\Omega C[\Omega]$ where $s \uparrow Red$ and $s \not\equiv \Omega$. $\omega(t)$ denotes the normal form of t concerning $\to_\Omega$. Note that $\omega(t)$ is well-defined according to the completeness of $\to_\Omega$ [7].

Definition 5.3 (Index) *Let Δ be a redex occurrence in $C[\Delta]$ such that $z \in \omega(C[z])$ where z is a fresh variable. Then the redex occurrence Δ is called an index of t. If Δ is an index of $C[\Delta]$ then we write $C[\Delta_I]$; otherwise $C[\Delta_{NI}]$.*

The original definition of index in Huet and Lévy [3] is restricted to orthogonal term rewriting systems; hence, any two indexes occurring in a term must be disjoint. On the other hand we assume only left-linearity for term rewriting systems. Hence, if a term rewriting system is overlapping then two indexes may be overlapping as follows.

Example 5.4 *Let $Red = \{p(s(\Omega)), s(p(\Omega))\}$. Then we have the overlapping indexes $f(s(p(s(x))_I)_I)$ since $\omega(f(z)) \equiv f(z)$ and $\omega(f(s(z))) \equiv f(s(z))$.*

One might think that overlapping redex occurrences always make overlapping indexes, but this is not the case from the following example.

Example 5.5 *Let $Red = \{0, f(0), g(f(\Omega), 1)\}$. Then we have $g(f(0_{NI})_I, 0_I)$. Note that two redex occurrences $f(0)$ and 0 are overlapping but 0 occurring in $f(0)$ is not an index.*

$t \xrightarrow{\Delta} s$ is an index reduction if Δ is an index of t. We indicate the index reduction with $t \underset{I}{\to} s$; otherwise $t \underset{NI}{\to} s$.

We say that R is strongly sequential if for each term $t \notin NF$, t has an index [3, 7, 6]. Note that index reduction of a strongly sequential system R is a reduction strategy because we can always apply an index reduction to a term being not a normal form.

The decidability of strong sequentiality for orthogonal term rewriting systems was first proven by Huet and Lévy [3], through a complicated decision procedure. A simple proof by Klop and Middeldorp can be found in [7]. This result can be immediately generalized to left-linear term rewriting systems.

Theorem 5.6 *Strong sequentiality of left-linear term rewriting systems (which may have overlapping rules) is decidable.*

6 Index Reduction of Overlapping Systems

We will now explain how to prove the normalizing property of index reduction for *balanced ambiguous* term rewriting systems by using the BWCR lemma. We first define balanced ambiguous term rewriting systems.

Let R be a term rewriting system. The root reduction $t \underset{r}{\to} s$ is defined as $t \overset{\Delta}{\to} s$ and $\Delta \equiv t$.

Definition 6.1 *A critical pair $\langle s, t\rangle$ is root balanced joinable if $s \overset{k}{\underset{r}{\to}} t'$ and $t \overset{k}{\underset{r}{\to}} t'$ for some t' and $k \geq 0$. A term rewriting system R is root balanced joinable if every critical pair is root balanced joinable.*

Definition 6.2 *A term rewriting system R is balanced ambiguous if R is left-linear and root balanced joinable.*

Note that every weakly orthogonal term rewriting system is trivially balanced ambiguous since every critical pair is root balanced joinable with $k = 0$.

Lemma 6.3 *Let R be balanced ambiguous. Let $t \underset{I}{\to} t'$, $t \underset{I}{\to} t''$. Then, we have $t' \overset{k}{\underset{I}{\to}} s$ and $t'' \overset{k}{\underset{I}{\to}} s$ for some s and $k \geq 0$ (i.e., $\underset{I}{\to}$ is BWCR).*

The parallel reduction $t \mathrel{+\!\!\!\mapsto} s$ is defined with $t \equiv C[\Delta_1, \cdots, \Delta_n] \overset{\Delta_1}{\to} \cdots \overset{\Delta_n}{\to} s$ $(n \geq 0)$.

Lemma 6.4 *Let R l · strongly sequential and balanced ambiguous, and let $t \mathrel{+\!\!\!\mapsto} s$. Then $t \underset{I}{=} s$ or $t \underset{I}{=} \cdot \mathrel{+\!\!\!\mapsto} \cdot \overset{+}{\underset{I}{\leftarrow}} s$.*

Theorem 6.5 *Let R be strongly sequential and balanced ambiguous. Then index reduction $\underset{I}{\to}$ is normalizing.*

7 Balanced Ambiguous Left-Normal Systems

In this section we discuss syntactical characterization of strongly sequential overlapping term rewriting systems. An answer concerning orthogonal term rewriting systems was found by O'Donnell [8]. He proved that if an orthogonal term rewriting system R is left-normal then R is strongly sequential and leftmost-outermost reduction is normalizing. We show that his result can be naturally extended to balanced ambiguous term rewriting systems.

Definition 7.1 *The set T_L of the left-normal terms is inductively defined as follows:*

1. *$x \in T_L$ if x is a variable,*
2. *$f(t_1, \cdots, t_{p-1}, t_p, t_{p+1} \cdots, t_n)$ $(0 \leq p \leq n)$ if $t_1, \cdots, t_{p-1}$ are ground terms (i.e., no variable occurs in $t_1, \cdots, t_{p-1}$), $t_p \in T_L$, and $t_{p+1}, \cdots, t_n$ are variables.*

The set of the left-normal schemata is $T_{L\Omega} = \{t_\Omega \mid t \in T_L\}$. We say that R is left-normal [8, 3, 6] iff for any rule $l \to r$ in R, l is a left-normal term, i.e., $Red \subseteq T_{L\Omega}$.

Lemma 7.2 *Let R be left-linear and left-normal (note that R may be overlapping). If a term t is not a normal form, then the leftmost-outermost redex of t is an index.*

Theorem 7.3 *Let R be balanced ambiguous left-normal. Then, leftmost-outermost reduction is normalizing.*

Note that every weakly orthogonal left-normal term rewriting system is balanced ambiguous left-normal. Thus the following corollary holds.

Corollary 7.4 *Let R be weakly orthogonal left-normal. Then, leftmost-outermost reduction is normalizing.*

Acknowledgments

The author would like to thank H. P. Barendregt and J. W. Klop for offering a stimulating environment for this research and for valuable discussions. The author would also like to thank M. C. J. D. van Eekelen, J. R. Kennaway, M. J. Plasmeijer, M. R. Sleep, J. E. W. Smetsers, R. C. de Vrijer for valuable discussions. Specially thanks to A. Middeldorp and F. J. de Vries for carefully reading a previous version of this paper.

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