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<th>Title</th>
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FIXED POINT SETS OF $S^1$-ACTIONS ON THE SPACES WHOSE
RATIONAL COHOMOLOGY RINGS ARE EVENLY GRADED

1. Introduction

Let $G = S^1$ be the circle group, and $X$ a connected
finite $G$-CW-complex whose rational cohomology ring is evenly
graded; that is

$$H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]/(\varphi_1, \ldots, \varphi_m),$$

where $\deg x_i = 2k_i \geq 2$ ($1 \leq i \leq n$) and $\varphi_i$ are homogeneous
elements. We establish a method of determining the possibili-
ties of the rational cohomology type of the fixed point set of
$G$ on $X$ (Theorem 3.7). The method is an application of that
originated and improved by K. Hokama in [2] and [4] respective-
ly. Combining Theorem 3.7 with a result of V. Puppe in [5]
(Theorem 3.8), the problem of existence for connected finite
$G$-CW-complex whose rational cohomology ring is evenly graded is
reduced to an algebraic one. We apply the result to three cases
(Theorems 4.1, 4.2 and 4.3). A result of G. E. Bredon [1]
applied to the case

$$X \sim \bigoplus_{i} S^{2m_i} \times S^{2n_i}$$

is improved slightly (Theorem 4.1). In the final section, we
construct $G$-CW-complexes which give examples in Theorems 4.2
and 4.3 except for the case (4).
2. Preliminaries

Let \( R \) (resp. \( a_R \)) be the polynomial ring \( \mathbb{Q}[t,x_1,\ldots,x_n] \) (resp. \( \mathbb{Q}[x_1,\ldots,x_n] \)), where \( \deg t = 2, \deg x_i = 2k_i \) (1 \( \leq i \) \( \leq n \)) and 1 \( \leq k_1 \leq \cdots \leq k_n \). Let \( a : R \to a_R \) denotes the ring homomorphism defined by

\[
a_F(x_1,\ldots,x_n) = F(1,x_1,\ldots,x_n)
\]

for every \( F \in R \), and \( h : a_R \setminus \{0\} \to R \) a map defined by

\[
h_f(t,x_1,\ldots,x_n) = t^{\delta(f)/2}f(x_1/t,\ldots,x_n/t^n)
\]

for every \( f \in a_R \setminus \{0\} \), where \( \delta(f) \) denotes the total degree of \( f \). For any ideal \( J \) in \( R \), we set \( a_J = (a_F|F \in J) \). Let \( \phi_i \in a_R \) and \( f_i \in R \) be homogeneous elements (1 \( \leq i \leq m \)), and suppose

1. \( f_i(0,x_1,\ldots,x_n) = \phi_i(x_1,\ldots,x_n) \) (1 \( \leq i \leq m \)).
2. \( \dim_{\mathbb{Q}} a_R/(\phi_1,\ldots,\phi_m) < \infty \).

Assume that \( a_R/(\phi_1,\ldots,\phi_m) \) has a basis \( M = \{y_i|1 \leq i \leq h\} \) over \( \mathbb{Q} \), where \( y_i \) is a homogeneous element (1 \( \leq i \leq h \)), \( y_1 = 1 \) and 0 = \( \deg y_1 \leq \cdots \leq \deg y_h = 2N \geq 2k_n \). Then we have the following lemma.

**Lemma 2.1.** (1) The \( \mathbb{Q}[t] \)-module \( R/(f_1,\ldots,f_m) \) is generated by \( M \).

(2) The \( \mathbb{Q} \)-module \( a_R/a(f_1,\ldots,f_m) \) is generated by \( M \).

(3) The following conditions are equivalent:

i) If \( tf \in (f_1,\ldots,f_m) \), then \( f \in (f_1,\ldots,f_m) \).

ii) \( R/(f_1,\ldots,f_m) \) is a free \( \mathbb{Q}[t] \)-module with a basis \( M \).

iii) \( \dim_{\mathbb{Q}} a_R/a(f_1,\ldots,f_m) = \dim_{\mathbb{Q}} a_R/(\phi_1,\ldots,\phi_m) = h \).
Proof. (1) Let $f(t, x) \in R$ be a homogeneous element of positive degree. Then, by the assumption

$$f(0, x) = \sum_{i=1}^{h} a_i y_i + \sum_{j=1}^{m} k_j(x) \varphi_j(x)$$

for some $a_i \in \mathbb{Q}$ $(1 \leq i \leq h)$ and $k_j(x) \in \mathbb{Q}[x]$ $(1 \leq j \leq m)$.

This implies that

$$f(t, x) - \sum_{i=1}^{h} a_i y_i - \sum_{j=1}^{m} k_j(x)f_j(t, x) = tg(t, x)$$

for a homogeneous element $g(t, x) \in R$. It is shown by the induction with respect to $\deg f(t, x)$ that

$$g(t, x) = \sum_{i=1}^{h} h_i(t)y_i + \sum_{j=1}^{m} g_j(t, x)f_j(t, x)$$

for some $h_i(t) \in \mathbb{Q}[t]$ $(1 \leq i \leq h)$ and $g_j(t, x) \in R$ $(1 \leq j \leq m)$. Then we have

$$f(t, x) = \sum_{i=1}^{h} (a_i + th_i(t))y_i + \sum_{j=1}^{m} (k_j(x) + tg_j(t, x))f_j(t, x).$$

(2) Let $f(x) \in aR\setminus\{0\}$ be a polynomial. It follows from (1) that we have

$$h f(t, x) = \sum_{i=1}^{h} h_i(t)y_i + \sum_{j=1}^{m} k_j(t, x)f_j(t, x)$$

for some $h_i(t) \in \mathbb{Q}[t]$ $(1 \leq i \leq h)$ and $k_j(t, x) \in R$ $(1 \leq j \leq m)$. Then we have

$$f(x) = \sum_{i=1}^{h} h_i(1)y_i + \sum_{j=1}^{m} k_j(1, x)f_j(x).$$

(3) Suppose $i$ and $\sum_{i=1}^{h} h_i(t)y_i \in (f_1, \cdots, f_m)$. Then

$$\sum_{i=1}^{h} h_i(0)y_i \in (\varphi_1, \cdots, \varphi_m),$$

and hence $h_i(0) = 0$ $(1 \leq i \leq h)$. Suppose that $h_i(t) = tg_i(t)$ $(1 \leq i \leq h)$. Then

$$t\sum_{i=1}^{h} g_i(t)y_i = \sum_{i=1}^{h} h_i(t)y_i \in (f_1, \cdots, f_m),$$

and hence $\sum_{i=1}^{h} g_i(t)y_i \in (f_1, \cdots, f_m)$. It is shown by the induction with respect to the degree of $\sum_{i=1}^{h} h_i(t)y_i$ that

$g_i(t) = 0$ $(1 \leq i \leq h)$. Then $h_i(t) = 0$ $(1 \leq i \leq h)$. Thus i) implies ii).
Suppose that $f(t,x) \in R$ be a homogeneous element such that $tf(t,x) \in (f_1, \cdots, f_m)$ and $f(t,x)$ is not contained in $(f_1, \cdots, f_m)$. By the assumption,

$$f(0,x) = \sum_{i=1}^{h} a_i y_i + \sum_{j=1}^{m} k_j(x) \phi_j(x)$$

for some $a_i \in \mathbb{Q}$ $(1 \leq i \leq h)$ and $k_j(x) \in \mathbb{Q}[x]$ $(1 \leq j \leq m)$. If $a_i = 0$ $(1 \leq i \leq h)$, then

$$f(t,x) - \sum_{j=1}^{m} k_j(x)f_j(t,x) = tg(t,x)$$

for some $g(t,x) \in R$. Then $g(t,x)$ is not contained in $(f_1, \cdots, f_m)$ and $t^2g(t,x) \in (f_1, \cdots, f_m)$. Thus there exists $f(t,x) \in R$ and a positive integer $N_1$ such that $t^{N_1}f(t,x) \in (f_1, \cdots, f_m)$ and $[f(0,x)]$ has non-zero component with respect to the basis $M$. Suppose that

$$f(0,x) - \sum_{i=1}^{k} a_i y_i \in (\phi_1, \cdots, \phi_m)$$

for some integer $k$ with $1 \leq k \leq h$ and $a_i \in \mathbb{Q}$ $(1 \leq i \leq k)$ with $a_k \neq 0$. Then $R/(f_1, \cdots, f_m,f)$ is generated by $M \setminus \{y_k\}$ over $\mathbb{Q}[t]$, and $\mathcal{A}/\mathcal{A}(f_1, \cdots, f_m,f)$ is generated by $M \setminus \{y_k\}$ over $\mathbb{Q}$. Since

$$a(f_1, \cdots, f_m,f) = a(f_1, \cdots, f_m,t^N_1f) = a(f_1, \cdots, f_m),$$

this implies that $\dim_{\mathbb{Q}}^{\mathcal{A}/\mathcal{A}}(f_1, \cdots, f_m) \leq h-1$. Hence iii) does not hold. Thus iii) implies i).

Suppose that $\sum_{i=1}^{k} a_i y_i \in a(f_1, \cdots, f_m)$ for some $k$ with $1 \leq k \leq h$ and $a_i \in \mathbb{Q}$ $(1 \leq i \leq k)$ with $a_k \neq 0$. Then

$$\sum_{i=1}^{k} a_i t^{(\deg y_i)/2} y_i \in (f_1, \cdots, f_m)$$

for some integer $N_1 \geq \deg y_k$. Hence ii) does not holds. Thus ii) implies iii). This completes the proof of (3).

q.e.d.
Let \((c_1^{(\alpha)}, \ldots, c_n^{(\alpha)}) | 1 \leq \alpha \leq k)\) be a set of rational zero points of the ideal \((f_1, \ldots, f_m) \subseteq {\mathbb R};\) that is, \(c_i^{(\alpha)} \in \mathbb{R}\) and 
\[ f_i(1, c_1^{(\alpha)}, \ldots, c_n^{(\alpha)}) = 0 \quad (1 \leq i \leq m, 1 \leq \alpha \leq k). \]
For \(1 \leq i \leq n\) and \(1 < j \leq k_i\), \(g_{i,j} \in \mathbb{R}\) denotes either 0 or a homogeneous element of degree \(2N_0\) with \(N_0 \geq N\) and 
\[ g_{i,j}(1, c_1^{(\alpha)}, \ldots, c_n^{(\alpha)}) = 0 \quad (1 \leq \alpha \leq k). \]
Let \(S\) (resp. \(aS\)) be the polynomial ring 
\[ \mathbb{R}[t] \cup \{x_{i,j} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \]
(resp. \( \mathbb{R}[\{(x_{i,j} | 1 \leq i \leq n, 1 \leq j \leq k_i\}])\), where \(\deg x_{i,j} = 2j\) \(1 \leq i \leq n\), \(1 \leq j \leq k_i\) and \(\deg t = 2\). Consider homomorphisms \(J_\alpha: R \to S\) (resp. \(aJ_\alpha: aR \to aS\)) defined by
\[ J_\alpha(f) = f(t, c_1^{(\alpha)} t^{k_1} + \sum_{j=1}^{k_1} x_{1,j} t^{j}, \ldots, c_n^{(\alpha)} t^{k_n} + \sum_{j=1}^{k_n} x_{n,j} t^{j}). \]
(resp. \(aJ_\alpha(f) = f(c_1^{(\alpha)} + \sum_{j=1}^{k_1} x_{1,j} t^{j}, \ldots, c_n^{(\alpha)} + \sum_{j=1}^{k_n} x_{n,j} t^{j})\)). Consider the induced homomorphisms 
\[ J_\alpha: R/(f_1, \ldots, f_m) \to S/I_\alpha S \]
and \(aJ_\alpha: aR/a(f_1, \ldots, f_m) \to aS/I_\alpha aS\).

**Lemma 2.2.** Let \(\alpha\) be an integer with \(1 \leq \alpha \leq k\).

1. The graded group \(aS/I_\alpha aS\) equals to zero in the degrees \(> 2N_0\).
2. The ideal \(a_\alpha\) is primary with the radical
\[ \sqrt{a_\alpha} = (x_1^{c_1^{(\alpha)}} t^{k_1}, \ldots, x_n^{c_n^{(\alpha)}} t^{k_n}). \]
3. The ideal \(a_\alpha\) coincides with \(aJ_\alpha^{-1}(I_\alpha)\), and \(aJ_\alpha\) induces an isomorphism \(aJ_\alpha: aR/a_\alpha aS \to aS/I_\alpha aS.\)
Proof. (1) Let \( g(x_{i,j}) \in \mathfrak{a}S \) be a homogeneous element. Let \( f = g(\xi_{i,j}^t) \in R \), where

\[
g_{i,1} = x_i^{N_0 - k_i} - c_i^{(\alpha)} t^{N_0} - \sum_{j=2}^{k_i} g_{i,j} (1 \leq i \leq n).
\]

It follows from the definition that \( J_\alpha(f) \) is congruent to \( (N_0 - 1)(\deg g)/2 \) \( g_t \mod I_\alpha S \). By Lemma 2.1 (1) there exist \( h_i(t) \in \mathfrak{G}[t] (1 \leq i \leq h) \) and \( k_i(t,x) \in R \ (1 \leq i \leq m) \) with

\[
f = \sum_{i=1}^{h} h_i(t)y_i + \sum_{i=1}^{m} k_i(t,x)f_i(t,x).
\]

Then \( g_t \mod I_\alpha S \), the degree with respect to \( x_{i,j} \), is at most \( 2N \). If \( \deg g > 2N \), then \( g \in I_\alpha \).

(2) It follows from (1) that \( \sqrt{I_\alpha} = (x_{i,j}) \). This implies that \( \sqrt{I_\alpha S} = (x_{i,j}) \), and \( I_\alpha S \) is a primary ideal. Since \( q_\alpha \) is the inverse image of \( I_\alpha S \) by a ring homomorphism \( J_\alpha \), we obtain (2).

(3) Suppose that \( f \in \mathfrak{q}_\alpha \). Choose \( F \in q_\alpha \) with \( \mathfrak{a}F = f \). Then, the coefficients of \( J_\alpha(F) \) with respect to \( t \) are contained in \( I_\alpha \), and the sum of which is equal to \( \mathfrak{a}J_\alpha(f) = \mathfrak{a}(J_\alpha(F)) \). This implies that \( f \in \mathfrak{a}J_\alpha^{-1}(I_\alpha) \). Conversely, suppose that \( f \in \mathfrak{a}J_\alpha^{-1}(I_\alpha) \) and \( f \neq 0 \). Then \( \mathfrak{a}(J_\alpha(hf)) = \mathfrak{a}J_\alpha(f) \in I_\alpha \). Since \( I_\alpha \) is a homogeneous ideal and the set of the coefficients of \( J_\alpha(hf) \) with respect to \( t \) coincides to that of the homogeneous components of \( \mathfrak{a}J_\alpha(f) \), we have \( J_\alpha(hf) \in I_\alpha S \). This implies that \( hf \in q_\alpha \), and \( f = \mathfrak{a}(hf) \in \mathfrak{a}q_\alpha \). Thus we obtain the first part of (3) and a monomorphism \( \mathfrak{a}_\alpha : \mathfrak{a}R/\mathfrak{a}q_\alpha \rightarrow \mathfrak{a}S/I_\alpha \). It follows from the proof of (1) that \( \mathfrak{a}_\alpha \) is also an epimorphism. This completes the proof of (3).
In order to state the next lemma, we set
\[ j = \Phi_\alpha^k, j : \mathbb{R}/(f_1, \ldots, f_m) \to \Phi_{\alpha=1}^k S/\alpha S \]
and
\[ a_j = \Phi_\alpha^k, a_j : aR/\alpha(f_1, \ldots, f_m) \to \Phi_{\alpha=1}^k aS/\alpha S. \]

**Lemma 2.3.** (1) \( j \) is surjective in the degrees \( \geq 2N \).

(2) \( a_j \) is an epimorphism.

(3) \( \sum_{\alpha=1}^k \dim q_{\alpha} aS/I_{\alpha} \leq h. \)

(4) \( \sum_{\alpha=1}^k \dim q_{\alpha} aR/\alpha q_{\alpha} \leq h. \)

(5) \( (f_1, \ldots, f_m) \subset \bigcap_{\alpha=1}^k q_{\alpha}. \)

(6) \( a(f_1, \ldots, f_m) \subset \bigcap_{\alpha=1}^k aq_{\alpha}. \)

(7) The following conditions are equivalent:

i) \( j \) is a monomorphism,

ii) \( a_j \) is an isomorphism,

iii) \( \sum_{\alpha=1}^k \dim q_{\alpha} aS/I_{\alpha} = h, \)

iv) \( \sum_{\alpha=1}^k \dim q_{\alpha} aR/\alpha q_{\alpha} = h, \)

v) \( (f_1, \ldots, f_m) = \bigcap_{\alpha=1}^k q_{\alpha}. \)

vi) \( a(f_1, \ldots, f_m) = \bigcap_{\alpha=1}^k aq_{\alpha}. \)

If this is the case, the set of zero points of the ideal \( a(f_1, \ldots, f_m) \subset aR \) coincides with \( \{ (c_1^{(\alpha)}, \ldots, c_n^{(\alpha)}) | 1 \leq \alpha \leq k \}. \)

**Proof.** (1) We show that for each \( \alpha (1 \leq \alpha \leq k) \) and homogeneous element \( g \in aS \), there exist a homogeneous element \( f \in R \) and an integer \( N_1 \) with \( 0 \leq N_1 \leq \max \{0, N-(\deg g)/2\} \), \( J_\beta(f) \in I_\beta S (\beta \neq \alpha) \) and \( J_\alpha(f)-gt \in I_\alpha S \). If \( \deg g > 2N \), then \( f = 0 \) and \( N_1 = 0 \) satisfies the above condition. Let \( \deg g = 2l \leq 2N \), and set
\[ F_1 = g(g_1, j \cdot \pi_{\beta \neq \alpha} (x_1^N - c_i^N) \cdot \sum_{j=2}^{k_1} g_1, j) \quad (1 \leq i \leq n) \] and \( g_i, j = x_j^N - c_i^N \cdot \sum_{j=2}^{k_1} g_1, j \). \( i(\beta) \) is an integer with \( c_i(\beta) = c_i(\alpha) \quad (\beta \neq \alpha) \). Then we have \( J_{\beta}(F_1) \in l_1 \) and \[ \ell(N_0 - 1) + N(\alpha) \\ N(\alpha) - 1 \quad \ell(N_0 - 1) + i \] and \( \deg g_i > 2\ell \) if \( g_i \neq 0 \). Set \( N_2 = \max(N, \ell N_0 + N(\alpha)) \). It follows from the inductive hypothesis that there exists a homogeneous element \( F \in R \) of degree \( 2N_2 \) with \( J_{\beta}(F) \in l_1 \) \( (\beta \neq \alpha) \) and \( J_{\alpha}(F) - gt^{N - \ell} \in l_1 \). By Lemma 2.1 (1) there exist \( h_i(t) \in \mathbb{C}[t] \quad (1 \leq i \leq h) \) and \( k_i(t, x) \in R \quad (1 \leq i \leq m) \) with \[ F = \sum_{i=1}^{h} h_i(t)y_i + \sum_{i=1}^{m} k_i(t, x)f_i(t, x). \] Set \( \sum_{i=1}^{h} h_i(t)y_i = ft^{\ell} \), where \( f \in R \) is a homogeneous element of degree \( 2N \). Then, we have \( J_{\beta}(f) \in l_1 \) \( (\beta \neq \alpha) \) and \( J_{\alpha}(f) - gt^{N - \ell} \in l_1 \). Thus (1) is proved by the induction with respect to \( (\deg g) / 2 \).

(2) In the proof above, \( a_j([a_f]) = [g] \). This implies (2).

(3) It follows from (2) and Lemma 2.1 (2) that we have \[ \sum_{\alpha = 1}^{K} \dim_{\alpha}^A S / l_\alpha \leq \dim_{\alpha}^R [a_{(f_1, \ldots, f_m)}] \leq h. \] (4) is a direct consequence of (3) and Lemma 2.2 (3).

(5) and (6) are immediate from the definition.

(7) It is evident that i) is equivalent to v), ii) is equivalent to vi), iii) is equivalent to iv) and v) is equivalent to vi).

Suppose i). Then \( R/(f_1, \ldots, f_m) \) is a torsionfree
\( \mathbb{Q}[t] \)-module \( (\phi^k_{\alpha=1} S/\alpha S \text{ is a free } \mathbb{Q}[t] \text{-module}). \) It follows from Lemma 2.1 (3) and (ii) that
\[
h = \dim_{\mathbb{Q}} aR/a(f_1, \ldots, f_m) = \sum_{\alpha=1}^k \dim_{\mathbb{Q}} aS/\alpha S.
\]
Thus (i) implies (iii).

Suppose (iii). It follows from (2) and Lemma 2.1 (2) that
\[
h \geq \dim_{\mathbb{Q}} aR/a(f_1, \ldots, f_m) \geq \sum_{\alpha=1}^k \dim_{\mathbb{Q}} aS/\alpha S = h.
\]
This implies that \( h = \dim_{\mathbb{Q}} aR/a(f_1, \ldots, f_m) \) and \( a_j \) is an isomorphism. Thus (iii) implies (ii). This completes the proof of (7).

q.e.d.

3. Equivariant cohomology rings

Let \( X \) be a connected finite \( G \)-CW-complex whose rational cohomology ring is evenly graded; that is, there exists an isomorphism
\[(3.1) \quad i(X): \mathbb{Q}[x_1, \ldots, x_n]/(\phi_1, \ldots, \phi_m) \longrightarrow H^*(X; \mathbb{Q}), \]
where \( \deg x_i = 2k_i \geq 2 \) (\( 1 \leq i \leq n \)) and \( \phi_i \) is a homogeneous element (\( 1 \leq i \leq m \)). Let \( \tau: EG \times_G \mathbb{C} \longrightarrow BG \) be the complex line bundle associated to a universal \( G \)-bundle \( EG \longrightarrow BG \). Then we have

\[(3.2) \quad H^*(BG; \mathbb{Q}) \cong \mathbb{Q}[t], \text{ where } t \in H^2(BG; \mathbb{Q}) \text{ is the Euler class of } \tau. \]

Let \( \pi: X_G = EG \times_G X \longrightarrow BG \) be the associated bundle with fiber \( X \), and \( i: X \longrightarrow X_G \) the inclusion of a fiber. The equivariant cohomology ring of \( X \) is defined by \( H^*_G(X; \mathbb{Q}) \).
\( = H^*(X_G; \mathbb{Q}) \). The induced homomorphism \( \pi^*: H^*(BG; \mathbb{Q}) \rightarrow H^*_G(X; \mathbb{Q}) \) gives a \( \mathbb{Q}[t] \)-algebra structure to \( H^*_G(X; \mathbb{Q}) \).

Then \( H^{\text{odd}}_G(X; \mathbb{Q}) \cong 0 \) and the sequence

\[
(3.3) \quad 0 \rightarrow H^{2i}_G(X; \mathbb{Q}) \xrightarrow{I} H^{2i+2}_G(X; \mathbb{Q}) \xrightarrow{i^*} H^{2i+2}_G(X; \mathbb{Q}) \rightarrow 0
\]

is exact. Choose \( a_i \in H^i_G(X; \mathbb{Q}) \) with \( i^*(a_i) = i(X)([X_i]) \) \( (1 \leq i \leq n) \). Then \( H^*_G(X; \mathbb{Q}) \) is generated by \( (a_i | 1 \leq i \leq n) \) as a \( \mathbb{Q}[t] \)-algebra. Let \( I(X_G): R \rightarrow H^*_G(X; \mathbb{Q}) \) be the \( \mathbb{Q}[t] \)-algebra homomorphism defined by setting \( I(X_G)(x_i) = a_i \) \( (1 \leq i \leq n) \). Choose a homogeneous element \( f_i \in R \) with \( I(X_G)(f_i) = 0 \) and \( f_i(t, x_1, \ldots, x_n) = \phi_i(x_1, \ldots, x_n) \) \( (1 \leq i \leq m) \). The proof of following lemma is similar to that of \([3, \text{Lemma 2.2}]\).

**Lemma 3.4.** The kernel of \( I(X_G) \) coincides with the ideal \( (f_1, \ldots, f_m) \) of \( R \) and \( I(X_G) \) induces an isomorphism

\[
i(X_G): R/(f_1, \ldots, f_m) \longrightarrow H^*_G(X; \mathbb{Q}).
\]

Let \( F = X^G \) be the fixed point set of the \( G \)-action on \( X \) with connected components \( F_1, \ldots, F_k \). Let

\[
j: F_G = BG \times F \rightarrow X_G
\]

be the inclusion map. Then the sequence

\[
(3.5) \quad 0 \rightarrow H^*_G(X; \mathbb{Q}) \xrightarrow{j^*} H^*_G(F; \mathbb{Q}) \rightarrow H^{*+1}(X/G, F; \mathbb{Q}) \rightarrow 0
\]

is exact. For \( 1 \leq \alpha \leq k \), let \( p^\alpha_\alpha: (F^\alpha_G) \rightarrow F^\alpha_\alpha \) be the projection to the second factor. Then we have an isomorphism

\[
t_\alpha: \mathbb{Q}[t] \otimes H^*(F^\alpha_\alpha; \mathbb{Q}) \longrightarrow H^*_G(F^\alpha_\alpha; \mathbb{Q})
\]

defined by setting \( t_\alpha(h \otimes a) = h p^\alpha_\alpha(a) \) for every \( h \in \mathbb{Q}[t] \).
and \( a \in H^*(F_\alpha; \mathbb{Q}) \). For \( 1 \leq i \leq n \), set
\[
j^*(a_{i,j}) = \sum_{\alpha=1}^k (c_i^{(\alpha)} t_i^k + \sum_{j=1}^{k_i} t_i^{k_i-j} p_\alpha^*(a_{i,j})),
\]
where \( a_{i,j}^{(\alpha)} \in H^j_{\alpha}(F_\alpha; \mathbb{Q}) \) and \( c_i^{(\alpha)} \in \mathbb{Q} \) (\( 1 \leq \alpha \leq k \)).

**Lemma 3.6.** (1) For \( 1 \leq \alpha \leq k \), \( H^*(F_\alpha; \mathbb{Q}) \) is generated by \( (a_{i,j}^{(\alpha)})_{1 \leq i \leq n, 1 \leq j \leq k_i} \).

(2) \( (c_1^{(\alpha)}, \ldots, c_n^{(\alpha)}) \neq (c_1^{(\beta)}, \ldots, c_n^{(\beta)}) \) if \( \alpha \neq \beta \).

**Proof.** Suppose \( a \in H^*(F_\alpha; \mathbb{Q}) \). It follows from the exactness of the sequence (3.5) that there exist an integer \( N_1 \geq 0 \) and an element \( f \in R \) such that \( j^*(l(X_\alpha)(f)) = t_1^{N_1} p_\alpha^*(a) \).

By the isomorphism \( l_\alpha \), we see that there exist a polynomial \( g \in \mathfrak{a} \mathfrak{S} \) such that \( g(a_{i,j}^{(\alpha)}) = a \). This completes the proof of (1).

In the proof above, set \( a = 1 \). Then we have
\[
f(1, c_1^{(\alpha)}, \ldots, c_n^{(\alpha)}) = 1
\]
and \( f(1, c_1^{(\beta)}, \ldots, c_n^{(\beta)}) = 0 \) if \( \beta \neq \alpha \). This completes the proof of (2).

q.e.d.

For \( 1 \leq \alpha \leq k \), let \( l(F_\alpha): \mathfrak{a} \mathfrak{S} \to H^*(F_\alpha; \mathbb{Q}) \) be the ring homomorphism defined by setting \( l(F_\alpha)(x_i) = a_{i,j}^{(\alpha)} \) (\( 1 \leq i \leq n, 1 \leq j \leq k_i \)), and \( l((F_\alpha)_G): \mathfrak{S} \to H^*_G(F_\alpha; \mathbb{Q}) \) a \( \mathbb{Q}[t] \)-algebra homomorphism defined by \( l(F_\alpha) \) and \( l_\alpha \). Choose \( N_0 \geq N \) and \( g_{i,j} \in \mathfrak{a} \mathfrak{R} \) (\( 1 \leq i \leq n, 1 < j \leq k_i \)) such that
\[
l((F_\alpha)_G)(J_{\alpha}(g_{i,j})) = t_1^{N_0-j} p_\alpha^*(a_{i,j}^{(\alpha)}) \) (\( 1 \leq \alpha \leq k \)).
\]

For \( 1 \leq \alpha \leq k \), let \( \mathfrak{I}_\alpha \) be the ideal generated in \( \mathfrak{a} \mathfrak{S} \) by the coefficients of \( J_{\alpha}(f_i) \) (\( 1 \leq i \leq m \)) and \( J_{\alpha}(g_{i,j}) - x_i t_j^{N_0-j} \).
(1 ≤ i ≤ n, 1 ≤ j ≤ k_1) with respect to t, and set \( q_\alpha = J_\alpha^{-1}(I_\alpha S) \).

**Theorem 3.7.** Let \( \alpha \) be an integer with \( 1 ≤ \alpha ≤ k \). Then we have

(1) The kernel of the homomorphism \( I(F_\alpha) \) coincides with \( I_\alpha \), and \( I(F_\alpha) \) induces the isomorphism \( i(F_\alpha): aS/I_\alpha → H^*(F_\alpha; \mathbb{Q}) \).

(2) \((f_1, \ldots, f_m) = \cap_{\alpha=1}^k q_\alpha \) is the reduced primary decomposition, where \( \sqrt{q_\alpha} = (x_1 - c_1^{(\alpha)})^{k_1}, \ldots, x_n - c_n^{(\alpha)})^{k_n} \).

(3) \( g_{i,j}(1, c_1^{(\alpha)}, \ldots, c_n^{(\alpha)}) = 0 \) \((1 ≤ i ≤ n, 1 ≤ j ≤ k_1) \).

**Proof.** By the definition, we have \( I(F_\alpha)(I_\alpha) = 0 \). Let \( f \in \ker I(F_\alpha) \) be a homogeneous element. By Lemma 3.6 (2), there is an integer \( i(\beta) \) with \( c_i(\beta) ≠ c_i^{(\alpha)} \) for each \( \beta ≠ \alpha \). Set

\[
g = f(g_{i,j}(t^{N_0(j-1)}) \cdot \prod_{\beta ≠ \alpha} (x_i - c_i^{(\beta)})^{k_i(\beta)})^{N_1},
\]

where \( N_1 \) is an integer such that \( H^i(F; \mathbb{Q}) = 0 \) for \( i ≥ 2N_1 \) and \( g_{i,1} = x_i t^{N_0 - k_i} - c_i^{(\alpha)} t^{N_0} - \sum_{j=2}^{k_i} g_{i,j} \) \((1 ≤ i ≤ n) \). Then we have \( j^*(I(X_\alpha)(g)) = 0 \), and hence \( g \in (f_1, \ldots, f_m) \). The coefficient of the highest degree with respect to \( t \) in the polynomial \( J_\alpha(g) \) is congruent to a multiple of \( f \) by some non-zero constant \((\mod I_\alpha) \). This implies that \( f \in I_\alpha \). This completes the proof of (1).

Since \( \dim_{\mathbb{Q}} aS/I_\alpha = \dim_{\mathbb{Q}} H^*(F_\alpha; \mathbb{Q}) < \infty \), we have \( \sqrt{I_\alpha} = (x_i, j) \), and hence \( I_\alpha \) is a primary ideal. It follows that \( I_\alpha S \) is also a primary ideal and \( \sqrt{I_\alpha S} = (x_i, j) \). By the definition, \( q_\alpha \) is a primary ideal with the radical
\( f_1^* \) is a monomorphism, we have \( (f_1, \ldots, f_m) = \bigcap_{\alpha=1}^{k} q_\alpha \). This completes the proof of (2).

Since \( g_{i,j}(1,c_1^{(\alpha)}, \ldots, c_n^{(\alpha)}) \in \mathbb{Q} \) is equal to the coefficient of \( t \) of the polynomial \( J_\alpha(g_{i,j}) - x_i,j^t \), and \( I_\alpha \neq \mathbb{R} \), we have \( g_{i,j}(1,c_1^{(\alpha)}, \ldots, c_n^{(\alpha)}) = 0 \). q.e.d.

By Lemma 2.3, it is easy to see that we can assume \( N_0 = N \). According to [5] and Lemma 2.3, we have the following theorem.

**Theorem 3.8 (V. Puppe [5]).** Let \( f_i \in R \) and \( \phi_i \in \mathbb{R} \)

\( 1 \leq i \leq m \) be homogeneous elements that satisfy (1) and (11). Let \( \{c_1^{(\alpha)}, \ldots, c_n^{(\alpha)}\} 1 \leq \alpha \leq k \) be a set of rational zero points of the ideal \( \mathbb{R}(f_1, \ldots, f_m) \subset \mathbb{R} \).

For \( 1 \leq i \leq n \) and \( 1 < j \leq k \), \( g_{i,j} \in \mathbb{R} \) denotes either 0 or a homogeneous element of degree \( 2N \) and \( g_{i,j}(1,c_1^{(\alpha)}, \ldots, c_n^{(\alpha)}) = 0 \) \( (1 \leq \alpha \leq k) \). Set \( I_\alpha \) as in the section 2 \( (1 \leq \alpha \leq k) \). If one of the properties i)-vi) of (7) of Lemma 2.3 is satisfied, then there is a finite \( G \)-CW-pair \( (X, F) \) and \( \mathbb{Q}[t] \)-algebra isomorphisms

\[
\begin{align*}
i(X_G): R/(f_1, \ldots, f_m) &\longrightarrow H^*_G(X; \mathbb{Q}) \\
i((F_\alpha)_G): S/I_\alpha S &\longrightarrow H^*_G(F_\alpha; \mathbb{Q}) \quad (1 \leq \alpha \leq k),
\end{align*}
\]

such that \( F = X^G \) with connected components \( F_1, \ldots, F_k \) and the following diagram commutes:

\[
\begin{array}{ccc}
R/(f_1, \ldots, f_m) & \xrightarrow{j} & S/I_\alpha S & \xrightarrow{i(X_G)} & H^*_G(X; \mathbb{Q}) \\
\downarrow & & \downarrow & & \downarrow \\
H^*_G(X; \mathbb{Q}) & \xrightarrow{j^*} & \mathbb{Q} \quad \quad (1 \leq \alpha \leq k)
\end{array}
\]
4. Applications

Applying Theorems 3.7 and 3.8 to corresponding cases, we obtain the following theorems.

**Theorem 4.1.** Let $X \simeq G S^{2m} \times S^{2n}$ be a finite $G$-CW-complex, $1 \leq m \leq n$. Then one of the following possibilities must occur:

1. $F \simeq G S^{2q} \times S^{2r}$, $m \geq q$, $n \geq r$.
2. $F \simeq G P^2(2q)$, $1 \leq q \leq n/2 < m$ or $1 \leq q \leq m/2 \leq n/4$.
3. $F \simeq G (\text{point} + P^2(2q))$, $1 \leq q \leq n/2 < m$.
4. $F \simeq G S^{2q} + S^{2r}$, $n \geq q$, $n \geq r$.

Conversely, each type of (1)-(4) can be realized by the fixed point set of a $G$-CW-complex.

**Theorem 4.2.** Let $X \simeq G \text{HP}(2) \# \text{CP}(4)$ be a finite $G$-CW-complex. Then one of the following possibilities must occur:

1. $F \simeq G X$.
2. $F \simeq G \text{CP}(2) \# \text{CP}(2) + S^2$.
3. $F \simeq G F_1 + 2$ points,

where $H^*(F; \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2]/(x_1^2 - \alpha x_1^2)$, $0 \neq \alpha \in \mathbb{Q}$ and $\deg x_1 = \deg x_2 = 2$.
4. $F \simeq G \text{CP}(3) + 2$ points.
5. $F \simeq G \text{CP}(2) + \text{CP}(2)$.
6. $F \simeq G S^{2k} + S^{2m} + 2$ points $(k \leq 2, m \leq 1)$.

Conversely, each type of (1)-(6) can be realized by the fixed point set of a $G$-CW-complex.
Theorem 4.3. Let $X \sim_{\mathbb{Q}} \text{HP}(2) \# (-\text{CP}(4))$ be a finite $G$-CW-complex. Then one of the following possibilities must occur:

1. $F \sim_{\mathbb{Q}} X$.
2. $F \sim_{\mathbb{Q}} S^2 \times S^2 + S^2$.
3. $F \sim_{\mathbb{Q}} F_1 + 2$ points, where $H^*(F; \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2]/(x_1 x_2, x_2^2 + \alpha x_1^2)$, $0 \neq \alpha \in \mathbb{Q}$ and $\deg x_1 = \deg x_2 = 2$.
4. $F \sim_{\mathbb{Q}} \text{CP}(3) + S^{2k}$ ($k \leq 2$).
5. $F \sim_{\mathbb{Q}} \text{CP}(2) + \text{CP}(2)$.
6. $F \sim_{\mathbb{Q}} \text{CP}(2) + S^{2k} + \text{point}$ ($k \leq 2$).
7. $F \sim_{\mathbb{Q}} S^{2k} + S^{2m} + S^{2n}$ ($k \leq 2$, $m \leq 1$, $n \leq 1$).

Conversely, each type of (1)-(6) can be realized by the fixed point set of a $G$-CW-complex.

5. Construction of $S^1$-CW-complexes

Finally we construct some $G$-CW-complexes which give examples in the Theorems 4.2 and 4.3. Set $S^{11} = \{(u_1 + jv_1, u_2 + jv_2, u_3 + jv_3) \in (\mathbb{C} \oplus \mathbb{C})^3 | \sum_{i=1}^3 (|u_i|^2 + |v_i|^2) = 1\}$ and $S^3 = \{(x+jy) \in \mathbb{C} \oplus \mathbb{C} | |x|^2 + |y|^2 = 1\}$. Then $\text{HP}(2)$ is defined as the orbit space $S^{11}/S^3$, where the $S^3$-action on $S^{11}$ is defined by

$$(u_1 + jv_1, u_2 + jv_2, u_3 + jv_3) \cdot (x+jy) = (u_1 \overline{x} - v_1 \overline{y} + j(v_1 x + \overline{u}_1 y), u_2 x - \overline{v}_2 y + j(v_2 x + \overline{u}_2 y), u_3 x - \overline{v}_3 y + j(v_3 x + \overline{u}_3 y))$$

for every $(u_1 + jv_1, u_2 + jv_2, u_3 + jv_3, x+jy) \in S^{11} \times S^3$. For each $(c_1, c_2, c_3) \in \mathbb{Z}^3$, $\Psi(c_1, c_2, c_3) : G \times \text{HP}(2) \rightarrow \text{HP}(2)$ denotes the $G$-action on $\text{HP}(2)$ defined by
\[ \Psi(c_1, c_2, c_3)(z, [u_1 + jv_1, u_2 + jv_2, u_3 + jv_3]) \]
\[ = [z^{c_1}u_1 + jz^{c_1}v_1, z^{c_2}u_2 + jz^{c_2}v_2, z^{c_3}u_3 + jz^{c_3}v_3] \]
for every \( (z, [u_1 + jv_1, u_2 + jv_2, u_3 + jv_3]) \in G \times \text{HP}(2) \). Set
\[ D_1^8 = ([u_1 + jv_1, u_2 + jv_2, u_3 + jv_3] \in \text{HP}(2) | |u_1|^2 + |v_1|^2 > 1/2), \]
\[ S_1^7 = ([u_1 + jv_1, u_2 + jv_2, u_3 + jv_3] \in \text{HP}(2) | |u_1|^2 + |v_1|^2 = 1/2) \]
and \( S_7^q = ([u_1 + jv_1, u_2 + jv_2] \in (\mathbb{C} \oplus j\mathbb{C})^2 | \sum_{j=1}^2 (|u_j|^2 + |v_j|^2) = 1) \).
Let \( h_1: S_1^7 \to S_7^q \) be the homeomorphism defined by
\[ h_1([u_1 + jv_1, u_2 + jv_2, u_3 + jv_3]) \]
\[ = (1/(u_1u_1^* + v_1v_1^*))(u_2u_1^* + v_2v_1^*) + j(v_2u_1^* - u_2v_1^*), v_3u_1^* + v_3v_1^* + j(v_3u_1^* - u_3v_1^*)) \]
for every \([u_1 + jv_1, u_2 + jv_2, u_3 + jv_3] \in S_1^7 \). Set
\[ \Sigma^7 = \{(r_1z_1, r_2z_2, r_3z_3, r_4z_4) \mid \sum_{j=1}^4 r_j = 1, r_j \geq 0, z_j \in S^1 \}. \]
Let \( f_q: S_7^q \to \Sigma^7 \) be the homeomorphism defined by
\[ f_q((u_1 + jv_1, u_2 + jv_2) = (u_1/s, v_1/s, u_2/s, v_2/s) \]
for every \((u_1 + jv_1, u_2 + jv_2) \in S_7^q \) with \( s = \sum_{j=1}^4 (|u_j| + |v_j|) \). Set
\[ S_7^q = ([w_0, w_1, w_2, w_3, w_4] \in \mathbb{C}^5 | \sum_{i=0}^4 |w_i|^2 = 1), \]
and \( S_1^q = (z \in \mathbb{C} | |z|^2 = 1) \). Then \( CP(4) \) is defined as the orbit space \( S_7^q/S_1^q \), where the \( S_1^q \)-action on \( S_7^q \) is defined by
\[ z \cdot ([w_0, w_1, w_2, w_3, w_4]) = (zw_0, zw_1, zw_2, zw_3, zw_4) \]
for every \((z, [w_0, w_1, w_2, w_3, w_4]) \in S_1 \times S_7^q \). For each
\[(a_0, a_1, a_2, a_3, a_4) \in \mathbb{Z}^5, \Phi(a_0, a_1, a_2, a_3, a_4): G \times CP(4) \to CP(4) \]
denotes the \( G \)-action on \( CP(4) \) defined by
\[ \Phi(a_0, a_1, a_2, a_3, a_4)(z, [w_0, w_1, w_2, w_3, w_4]) \]
\[ = [z^{a_0}w_0, z^{a_1}w_1, z^{a_2}w_2, z^{a_3}w_3, z^{a_4}w_4] \]
for every \((z, [w_0, w_1, w_2, w_3, w_4]) \in G \times CP(4) \). Set
\[ D_2^8 = ([w_0, w_1, w_2, w_3, w_4] \in CP(4) | |w_0|^2 > 1/2), \]
\[ S_2^7 = ([w_0, w_1, w_2, w_3, w_4] \in CP(4) | |w_0|^2 = 1/2) \]
and \( S^7_c = \{(w_1, w_2, w_3, w_4) \in \mathbb{C}^4 | \sum_{i=1}^4 |w_i|^2 = 1\}\). Let \( h_2 : S^7_c \to S^7\) be the homeomorphism defined by

\[
h_2([w_0, w_1, w_2, w_3, w_4]) = (w_1/w_0, w_2/w_0, w_3/w_0, w_4/w_0)
\]

for every \([w_0, w_1, w_2, w_3, w_4] \in S^7_c\). Let \( f_c : S^7_c \to \Sigma^7\) be the homeomorphism defined by

\[
f_c(w_1, w_2, w_3, w_4) = (w_1/s, w_2/s, w_3/s, w_4/s)
\]

for every \((w_1, w_2, w_3, w_4) \in S^7_c\) with \(s = \sum_{j=1}^4 |w_j|\). For each \((d_1, d_2, d_3, d_4) \in \mathbb{Z}^4\), let \( f(d_1, d_2, d_3, d_4) : \Sigma^7 \to \Sigma^7\) be the map defined by

\[
f(d_1, d_2, d_3, d_4)((r_1z_1, r_2z_2, r_3z_3, r_4z_4)) = (d_1z_1, d_2z_2, d_3z_3, d_4z_4)
\]

for every \((r_1z_1, r_2z_2, r_3z_3, r_4z_4) \in \Sigma^7\). Now we set

\[
X = X(d_1, d_2, d_3, d_4; b_1, b_2, b_3, b_4) = \Sigma^7 \cup f_q(d_1, d_2, d_3, d_4) \cup f_c(b_1, b_2, b_3, b_4) \cup (\text{CP}(4) \setminus \text{D}_2^8),
\]

where \( f_q(d_1, d_2, d_3, d_4) \) (resp. \( f_c(b_1, b_2, b_3, b_4) \)) is the composition

\[
f(d_1, d_2, d_3, d_4) \circ f_q \circ h_1 : S^7 \to \Sigma^7
\]

(resp. \( f(b_1, b_2, b_3, b_4) \) \circ f_c \circ h_2 : S^7 \to \Sigma^7\). Suppose \( d_1(c_2 - c_1) = b_1(a_1 - a_0), d_2(-c_2 - c_1) = b_2(a_2 - a_0), d_3(c_3 - c_1) = b_3(a_3 - a_0)\) and \( d_4(-c_3 - c_1) = b_4(a_4 - a_0)\). Then \(X\) has the \( G\)-action compatible with \(\Psi(c_1, c_2, c_3)\) \( \cup \) \(\text{HP}(2) \setminus \text{D}_1^8\) and \(\Phi(a_0, a_1, a_2, a_3, a_4) \cup \text{CP}(4) \setminus \text{D}_2^8\); that is, \(X\) has a \( G\)-CW-complex structure. It is easy to see that we have

\[
H^*(X; \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2]/(x_1 x_2, d_1d_2d_3d_4 x_2 x_1^2 + b_1 b_2 b_3 b_4 x_1^4, x_2^3, x_1^5).
\]

If \( b_1 b_2 b_3 b_4 d_1 d_2 d_3 d_4 = -n^2 \) (resp. \( b_1 b_2 b_3 b_4 d_1 d_2 d_3 d_4 = n^2 \)) for some positive integer \( n\), then we have \( X \sim_{\mathbb{Q}} \text{HP}(2) \# \text{CP}(4)\).
(resp. \( X \sim_{\Omega} \mathrm{HP}(2) \# (-\mathrm{CP}(4)) \)). Thus we obtain following examples.

\[(4.2)\] If \( c_1 = c_2 = c_3 = c \neq 0, a_1 = a_3 = a_0, a_2 = a_4 = a_0 - 2c, d_i = 1\) \((1 \leq i \leq 4)\), \(-b_1 = b_2 = b_3 = b_4 = 1\), then we have

\[ X = \mathrm{HP}(2) \# \mathrm{CP}(4) \quad \text{and} \quad X^G = \mathrm{CP}(2) \# \mathrm{CP}(2) + \mathbb{S}^2. \]

\[(4.2)\] If \( c_1 = c_2 = c_3 = c \neq 0, a_1 = a_3 = a_2 + 2c = a_4 - a = a_0 \) \((a(a+2c) \neq 0)\), \(d_i = b_i = 1\) \((i = 1, 2)\), \(d_3 = -b_4 = 2c\) and \(d_4 = b_3 = a\), then we have

\[ X \sim_{\Omega} \mathrm{HP}(2) \# \mathrm{CP}(4) \quad \text{and} \quad X^G = F_1 + S^0, \]

where \( H^*(F_1; \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2]/(x_1 x_2, 2c x_2^2 + ax_1^2) \) and \( \deg x_1 = \deg x_2 = 2 \).

\[(4.2)\] If \( c_1 = 5c \neq 0, c_2 = 4c, c_3 = 13c, a_i = a_0 - 6c \) \((1 \leq i \leq 4)\), \(d_i = 6\) \((1 \leq i \leq 4)\), \(b_1 = 1, b_2 = 9, b_3 = -8\) and \(b_4 = 18\), then we have

\[ X \sim_{\Omega} \mathrm{HP}(2) \# \mathrm{CP}(4) \quad \text{and} \quad X^G = \mathrm{CP}(3) + S^0. \]

\[(4.2)\) \((k=2)\] If \( c_1 = c \neq 0, c_2 = c_3 = 0, a_i = a_0 - c \) \((1 \leq i \leq 3)\), \(a_4 = a_0 + c\), \(d_i = b_i = 1\) \((1 \leq i \leq 3)\), \(d_4 = 1\) and \(b_4 = -1\), then we have

\[ X = \mathrm{HP}(2) \# \mathrm{CP}(4) \quad \text{and} \quad X^G = \mathrm{CP}(2) + S^4 + \text{point}. \]

\[(4.2)\) \((k=1)\] If \( c_1 = c_2 = 0, c_3 = c \neq 0, a_1 = a_2 = a_0, a_3 = a_4 = a_0 - c\), \(d_i = (-1)^i b_i = 1\) \((2 \leq i \leq 4)\) and \(d_1 = b_1 = 1\), then we have

\[ X = \mathrm{HP}(2) \# \mathrm{CP}(4) \quad \text{and} \quad X^G = \mathrm{CP}(2) + S^2 + \text{point}. \]

\[(4.2)\) \((k=0)\] If \( c_1 = c_2 = 0, c_3 = c \neq 0, a_1 = a_2 = a_0, a_3 = a_0 + c, a_4 = a_0 - c, d_i = b_i = 1\) \((2 \leq i \leq 4)\) and \(d_1 = -b_1\).
= 1, then we have

\[ X = \text{HP}(2) \# \text{CP}(4) \quad \text{and} \quad X^G = \text{CP}(2) \times 3 \text{ points.} \]

\[(4.2)(6) \quad k=2, \ m=1 \] If \( c_1 = 2c \neq 0, \ c_2 = c_3 = 0, \ a_1 - c \)
\[= a_3 - c = a_2 + c = a_0, \ a_4 = a_0 + 16c, \ 2d_i = (-1)^i b_i = 2 \ (1 \leq i \leq 3), \]
\(d_4 = 8 \quad \text{and} \quad b_4 = -1, \) then we have

\[ X \sim \text{HP}(2) \# \text{CP}(4) \quad \text{and} \quad X^G = S^4 + S^2 + S^0. \]

\[(4.2)(6) \quad k=2, \ m=0 \] If \( c_1 = 2c \neq 0, \ c_2 = c_3 = 0, \ a_1 + c \)
\[= a_2 - c = a_3 + 2c = a_4 + 8c = a_0, \ d_i = 1 \ (1 \leq i \leq 3), \ d_4 = 4, \ b_1 \]
\[= -b_2 = 2 \quad \text{and} \quad b_3 = b_4 = 1, \] then we have

\[ X \sim \text{HP}(2) \# \text{CP}(4) \quad \text{and} \quad X^G = S^4 + 4 \text{ points.} \]

\[(4.2)(6) \quad k=m=1 \] If \( c_1 = c_2 = c \neq 0, \ c_3 = 0, \ a_1 = a_0, \ a_2 \)
\[= a_0 + 2c, \ a_3 = a_4 = a_0 - c, \ d_i = 1 \ (1 \leq i \leq 4) \quad \text{and} \quad b_1 = -b_2 = b_3 \]
\[= b_4 = 1, \] then we have

\[ X = \text{HP}(2) \# \text{CP}(4) \quad \text{and} \quad X^G = S^2 + S^2 + 2 \text{ points.} \]

\[(4.2)(6) \quad k=1, \ m=0 \] If \( c_1 = c_2 = c \neq 0, \ c_3 = 0, \ a_1 = a_2 + 2c \)
\[= a_3 - c = a_4 + c = a_0, \ d_i = 1 \ (1 \leq i \leq 4), \ b_i = (-1)^i \ (2 \leq i \leq 4) \]
\[\text{and} \quad b_1 = 1, \] then we have

\[ X = \text{HP}(2) \# \text{CP}(4) \quad \text{and} \quad X^G = S^2 + 4 \text{ points.} \]

\[(4.2)(6) \quad k=m=0 \] If \( c_1 = c \neq 0, \ c_2 = 2c, \ c_3 = 3c, \ a_1 - c \)
\[= a_2 + 3c = a_3 + 2c = a_4 + 4c = a_0, \ d_i = b_i = 1 \ (i \neq 3) \quad \text{and} \quad d_3 = -b_3 \]
\[= 1, \] then we have

\[ X = \text{HP}(2) \# \text{CP}(4) \quad \text{and} \quad X^G = 6 \text{ points.} \]

\[(4.3)(1) \] If \( c_1 = c_2 = c_3 = c \neq 0, \ a_1 = a_3 = a_0, \ a_2 = a_4 \)
\[= a_0 - 2c, \ d_i = b_i = 1 \ (1 \leq i \leq 4), \] then we have

\[ X = \text{HP}(2) \# (-\text{CP}(4)) \]

and \( X^G = \text{CP}(2) \# (-\text{CP}(2)) + S^2 \sim \text{CP}(2) \times S^2 + S^2. \)

\[(4.3)(2) \] If \( c_1 = c_2 = c_3 = c \neq 0, \ a_1 = a_3 = a_2 + 2c = a_4 - a \)
\[ a_0 \quad (a(a+2c) \neq 0), \quad d_i = b_i = 1 \quad (i = 1, 2), \quad d_3 = -b_4 = 2c \quad \text{and} \quad d_4 = -b_3 = a, \text{then we have} \]

\[ X \sim \#_{C}(\mathbb{P}^2) \# (-\mathbb{C}P^4) \quad \text{and} \quad X^G = F_1 + S^0, \]

where \[ H^*(F_1; \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2]/(x_1x_2, 2cx_2^2 - ax_1^2) \quad \text{and} \quad \deg x_1 = \deg x_2 = 2. \]

\[ ((4.3)(3) \text{ k=2}) \quad \text{If} \quad c_1 = c \neq 0, \quad c_2 = c_3 = 0, \quad a_i = a_0 - c \quad (1 \leq i \leq 4) \quad \text{and} \quad d_i = b_i = 1 \quad (1 \leq i \leq 4), \text{then we have} \]

\[ X = \mathbb{P}^2 \# (-\mathbb{C}P^4) \quad \text{and} \quad X^G = \mathbb{C}P^3 + S^4. \]

\[ ((4.3)(3) \text{ k=1}) \quad \text{If} \quad c_1 = 0, \quad c_2 = c_3 = c \neq 0, \quad a_i = a_0 - c \quad (1 \leq i \leq 4), \quad d_i = 1 \quad (1 \leq i \leq 4) \quad \text{and} \quad b_i = (-1)^i \quad (1 \leq i \leq 4), \text{then we have} \]

\[ X = \mathbb{P}^2 \# (-\mathbb{C}P^4) \quad \text{and} \quad X^G = \mathbb{C}P^3 + S^2. \]

\[ ((4.3)(3) \text{ k=0}) \quad \text{If} \quad c_1 = 0, \quad c_2 = c \neq 0, \quad c_3 = 4c, \quad a_i = a_0 + 2c \quad (1 \leq i \leq 4), \quad d_i = 2 \quad (1 \leq i \leq 4), \quad b_1 = -b_2 = 1 \quad \text{and} \quad b_3 = -b_4 = 4, \text{then we have} \]

\[ X \sim_{\mathbb{Q}} \#_{C}(\mathbb{P}^2) \# (-\mathbb{C}P^4) \quad \text{and} \quad X^G = \mathbb{C}P^3 + S^0. \]

\[ ((4.3)(5) \text{ k=2}) \quad \text{If} \quad c_1 = 2c \neq 0, \quad c_2 = c_3 = 0, \quad a_i = a_0 - c \quad (1 \leq i \leq 3), \quad a_4 = a_0 - 16c, \quad 2d_i = b_i = 2 \quad (1 \leq i \leq 3), \quad d_4 = 16 \quad \text{and} \quad b_4 = 2, \text{then we have} \]

\[ X \sim_{\mathbb{Q}} \#_{C}(\mathbb{P}^2) \# (-\mathbb{C}P^4) \quad \text{and} \quad X^G = \mathbb{C}P^2 + S^4 + \text{point}. \]

\[ ((4.3)(5) \text{ k=1}) \quad \text{If} \quad c_1 = c_2 = 0, \quad c_3 = c \neq 0, \quad a_1 = a_2 = a_0, \quad a_3 = a_4 = a_0 - c \quad \text{and} \quad d_i = (-1)^i b_i = 1 \quad (1 \leq i \leq 4), \text{then we have} \]

\[ X = \mathbb{P}^2 \# (-\mathbb{C}P^4) \quad \text{and} \quad X^G = \mathbb{C}P^2 + S^2 + \text{point}. \]

\[ ((4.3)(5) \text{ k=0}) \quad \text{If} \quad c_1 = c_2 = 0, \quad c_3 = c \neq 0, \quad a_1 = a_2 = a_0, \quad a_3 = a_0 + c, \quad a_4 = a_0 - c \quad \text{and} \quad d_i = b_i = 1 \quad (1 \leq i \leq 4), \text{then we have} \]
$X = \text{HP}(2) \# (\text{-CP}(4))$ and $X^G = \text{CP}(2) + 3$ points.

[(4.3)(6) $k=2, m=n=1$] If $c_1 = c \neq 0$, $c_2 = c_3 = 0$, $a_1 = a_3 = a_0 + c$, $a_2 = a_4 = a_0 - c$ and $d_i = (-1)^ib_i = 1$ $(1 \leq i \leq 4)$, then we have

$X = \text{HP}(2) \# (\text{-CP}(4))$ and $X^G = S^4 + S^2 + S^2$.

[(4.3)(6) $k=2, m=1, n=0$] If $c_1 = 2c \neq 0$, $c_2 = c_3 = 0$, $a_1 + c = a_2 - c = a_3 + 2c = a_4 - 8c = a_0$, $d_i = 1$ $(1 \leq i \leq 3)$, $d_4 = 4$, $b_i = -b_2 = 2$ and $b_3 = -b_4 = 1$, then we have

$X \sim \text{HP}(2) \# (\text{-CP}(4))$ and $X^G = S^4 + 4$ points.

[(4.3)(6) $k=m=n=1$] If $c_1 = 0$, $c_2 = c_3 = c \neq 0$, $a_1 = a_3 = a_0 + c$, $a_2 = a_4 = a_0 - c$ and $d_i = b_i = 1$ $(1 \leq i \leq 4)$, then we have

$X = \text{HP}(2) \# (\text{-CP}(4))$ and $X^G = S^2 + S^2 + S^2$.

[(4.3)(6) $k=m=1, n=0$] If $c_1 = c_2 = c \neq 0$, $c_3 = 0$, $a_1 = a_0$, $a_2 = a_0 + 2c$, $a_3 = a_4 = a_0 - c$, $d_i = 1$ $(1 \leq i \leq 4)$ and $-b_1 = -b_2 = b_3 = b_4 = 1$, then we have

$X = \text{HP}(2) \# (\text{-CP}(4))$ and $X^G = S^2 + S^2 + S^0$.

[(4.3)(6) $k=1, m=n=0$] If $c_1 = c_2 = c \neq 0$, $c_3 = 0$, $a_0 = a_1 = a_2 + 2c = a_3 - c = a_4 + c$ and $d_i = (-1)^ib_i = 1$ $(1 \leq i \leq 4)$, then we have

$X = \text{HP}(2) \# (\text{-CP}(4))$ and $X^G = S^2 + 4$ points.

[(4.3)(6) $k=m=n=0$] If $c_1 = c \neq 0$, $c_2 = 2c$, $c_3 = 3c$, $a_1 - c = a_2 + 3c = a_3 - 2c = a_4 + 4c = a_0$ and $d_i = b_i = 1$ $(1 \leq i \leq 4)$,
then we have

\[ X = \mathrm{HP}(2) \ # \ (\neg \mathrm{CP}(4)) \] and \[ X^G = 6 \] points.

References


[2] K. Hokama: On the fixed point set of \( S^1 \)-actions on \( \mathrm{CP}^n \times \mathrm{CP}^n \), Math. J. Okayama Univ. 17 (1975), 181-185.

