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The girth of a directed distance-regular graph

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Bounding the diameter of a distance-regular graph by some function of its valency k is a long-standing open problem. See Bannai and Ito [4, 5, 6, 7], Ivanov [10] and Terwilliger [16, 17]. In this paper we consider the directed version of this problem. Not only can the diameter be bounded in terms of k, but in fact $d = g-1 \leq 7$ absolutely.

0 Introduction

A *(finite) digraph* is a pair $G = (V, E)$, with V a non-empty finite set of *vertices*, and $E \subseteq V \times V$, the set of *directed edges* (or *arcs*). A *t-path* from a vertex u to a vertex v is a sequence $u = v_0, v_1, \ldots, v_t = v$ with $(v_i, v_{i+1}) \in E$, for $0 \leq i < t$, t is the *length* of the path. If $u = v$, then this is called a *t-cycle*. The *(directed) distance* from u to v, denote $\partial(u, v)$, is the smallest t for which there is a t-path from u to v. G is *connected* if $\partial(u, v)$ is finite for all $u, v \in V$. The *diameter* is the largest distance between vertices of G, and the *girth* g is the smallest length of a cycle in G.

A connected digraph G (without loops) is said to be *distance-regular* if the size

$$s_{j,\ell,i} := |\{x : \partial(u, x) = j,\ \partial(v, x) = \ell\}|$$

depends only on j, ℓ and $\partial(u, v)$, rather than the individual vertices u, v with $\partial(u, v) = i$.

In [8], Damerell proved the following for distance-regular digraphs with $g > 2$ and diameter d:

(i) If $0 < \partial(u, v) < g$, then $\partial(u, v) + \partial(v, u) = g$,

(ii) $d = g$ (long type) or $d = g - 1$ (short type), and

(iii) every distance-regular digraph of long type can be constructed easily from a distance-regular of short type.

So we shall restrict our consideration to distance-regular digraphs of short type with $g > 2$ (since if $g = 2$, the graph can be viewed as undirected). We shall also assume that the valency $k > 1$, to avoid G being a directed cycle.

1 Preliminaries

There is no claim that the results in this section are new. They are all known or straightforward. If there are no proofs, the results are immediate from the definitions. Otherwise a short proof is given to make this paper self-contained.

Let $p_{j,\ell,i} := s_{j,g-\ell,i}$. Let $u \in V$. Let $\Gamma_i(u) = \{v \in V : \partial(u,v) = i\}$, and let $K_i := |\Gamma_i(u)|$. Note that $k_0 = 1$, $k := p_{1,g-1,0} = K_1$ is the valency, $\lambda := p_{1,1,1}$ is the number of 2-paths from u to $v \in \Gamma_1(u)$, and $\mu := p_{1,1,2}$ is the number of 2-paths from u to $v \in \Gamma_2(u)$. Let $c_i := p_{i-1,1,i}, b_i := p_{i+1,g-1,i}$, so $K_{i-1}b_{i-1} = K_i c_i, K_i = K_{g-i}$, and $K_2 = k(k-\lambda)/\mu$. Let $f_i := p_{i,1,1}$ and $h_i := p_{1,g-1,i}$. Then

Lemma 1.1 *For* $2i \leq g$,

(a) $b_i \leq b_{i-1}$,

(b) $c_i \geq c_{i-1}$,

(c) $K_i \geq K_{i-1}$.

Proof: Fix $x \in \Gamma_1(u), y \in \Gamma_i(u) \bigcap \Gamma_{i-1}(x)$. Then b_i counts $z \in \Gamma_{i+1}(u) \bigcap \Gamma_i(x) \bigcap \Gamma_1(y)$ and b_{i-1} counts $z \in \Gamma_i(x) \bigcap \Gamma_1(y)$. The proof of (b) and (c) follows from $b_i = c_{g-i}$.

Lemma 1.2

$$k \cdot f_i = K_i \cdot h_i.$$

Proof: Count edges from $\Gamma_i(u)$ to $\Gamma_1(u)$ in two different ways.

Lemma 1.3

$$f_i = f_{g-i} \quad \textit{and} \quad h_i = h_{g-i}.$$

Proof:

$$h_i = p_{1,g-1,i} = p_{g-1,1,i} = p_{1,g-1g-i} = h_{g-i}.$$

Lemma 1.4

$$f_0 = 1, h_0 = k, \quad \textit{and} \quad f_1 = \lambda.$$

Lemma 1.5

$$k - 1 - 2\lambda = \sum_{i=2}^{g-2} f_i.$$

Lemma 1.6

$$\sum_{i=2}^{g-2} K_i(h_i - 1) = k(k - 1 - 2\lambda) - (n - 1 - 2k).$$

Lemma 1.7

$$\sum_{i=2}^{g-2} h_i \leq \mu - 1.$$

Proof:

$$\sum_{i=2}^{g-2} h_i = \sum_{i=2}^{g-2} \frac{kf_i}{K_i} \leq \sum_{i=2}^{g-2} \frac{kf_i}{K_2} = \sum_{i=2}^{g-2} \frac{\mu f_i}{k-\lambda} = \mu \frac{k-1-2\lambda}{k-\lambda} < \mu.$$

Lemma 1.8

$$K_j \sum_r p_{i,g-r,j} \cdot p_{\ell,g-m,r} = \sum_t p_{i,\ell,t} \cdot p_{j,m,t} \cdot K_t.$$

Proof: Fix a vertex w, and count the number of sets $\{x, y, z\}$ with $\partial(w,x) = i, \partial(w,y) = j$, $\partial(x,z) = \ell$, and $\partial(y,z) = m$ by choosing them in the order (y,x,z) and then in the order (z,x,y).

Corollary 1.9

$$\sum_{i=1}^{g-1} f_i h_i = (k-\lambda)(\mu-1) + \lambda(\lambda-1).$$

Proof: Set $i = j = \ell = m = 1$ above.

Corollary 1.10

$$\sum_{i=2}^{g-2} f_i(\mu - 1 - h_i) = (\lambda+1)(\lambda+1-\mu).$$

Corollary 1.11 *If $k > 1$, then*

$$\lambda > 0, \mu > 1, \text{ and } n \leq 1 + k(k+1-2\lambda).$$

Proof: From corollary 1.10, $(\lambda+1)(\lambda+1-\mu) \geq 0$. So if $\lambda = 0$, then $\mu = 1$. From lemma 1.7, $\mu - 1 \geq h_i \geq 0$ for $2 \leq i \leq g-2$. So if $\mu = 1$, then $h_i = 0$ for $2 \leq i \leq g-2$. But then from lemma 1.2, $f_i = 0$ for $2 \leq i \leq g-2$. Then corollary 1.10 forces $\lambda = 0$, and lemma 1.5 forces $k = 1$.

2 Graph-theoretic lemmas about Γ_1

Throughout this paper, let $u \in V(G)$ and $x \in \Gamma_1(u)$ be fixed. Let $X_i := \Gamma_i(x) \bigcap \Gamma_1(u)$. Then $f_i = |X_i|$. Let $Y_i := \bigcup_{j=0}^{i} X_j$ for $i < g-1$.

Lemma 2.1

$$X_i \neq \emptyset, \text{ so } f_i \neq 0 \text{ and } h_i \neq 0.$$

Proof: Since $\Gamma_1(u)$ is λ-regular, there are the same number of edges out of Y_i as into it. Those out of Y_i must go from X_i to X_{i+1}. But there are $\lambda > 0$ edges from $X_{g-1} \subseteq Y_i^c$ to $\{x\} = X_0 \subseteq Y_i$.

Lemma 2.2

$$\mu f_2 \geq \binom{\lambda+1}{2}$$

Proof: Count the number of edges $(y, w), y \in X_1$. There are $\lambda \cdot f_1 = \lambda^2$ edges with $w \in \Gamma_1(u)$. These end in either X_1 or X_2, but there are at most $\binom{\lambda}{2}$ that end in X_1 because $g \geq 3$. There are at most μ such edges ending at any $w \in X_2$.

So let $f_2 := \dfrac{(\lambda+1)(\lambda+\epsilon)}{2\mu}$ for some $\epsilon \geq 0$. And hence

$$\begin{aligned}\sum_{i=3}^{g-3} f_i &= k - 1 - 2\lambda - 2f_2 = \frac{\mu f_2}{h_2} - (\lambda+1) - 2f_2 \\ &= (\lambda+1)\frac{(\mu - 2h_2)(\lambda+\epsilon) - 2h_2\mu}{2h_2\mu}.\end{aligned}$$

Lemma 2.3 *If $g > 6$, then*

$$f_3 + f_4 \geq \frac{1}{2}(f_2 + 1).$$

Proof: If $g > 4$, then for any pair of distinct vertices $\{y, z\}$, at most one of the following holds:

$$y \in \Gamma_1(z),\ y \in \Gamma_2(z),\ z \in \Gamma_1(y),\ z \in \Gamma_2(y).$$

Let $d_{i,j}(y) := |\{z : z \in X_i \bigcap \Gamma_j(y)\}|$. Then

$$\sum_{i=0}^{g-1} d_{i,j}(y) = f_j,\ \sum_{j=0}^{g-1} d_{i,j}(y) = f_i.$$

Count pairs $\{y, z\}$ with $y, z \in X_2$ to get

$$\binom{f_2}{2} \geq \sum_{y \in X_2} [d_{2,1}(y) + d_{2,2}(y)].$$

But

$$d_{1,1}(y) + d_{2,1}(y) + d_{3,1}(y) = \lambda$$

and

$$d_{1,2}(y) + d_{2,2}(y) + d_{3,2}(y) + d_{4,2}(y) = f_2.$$

So, for some $y \in X_2$,

$$\begin{aligned} f_3 + f_4 &\geq d_{3,2}(y) + d_{4,2}(y) + d_{3,1}(y) \\ &= (f_2 - d_{1,2}(y) - d_{2,2}(y)) + (\lambda - d_{1,1}(y) - d_{2,1}(y)) \\ &\geq (f_2 + \lambda) - \lambda - \frac{1}{2}(f_2 - 1) \\ &= \frac{1}{2}(f_2 + 1) \end{aligned}$$

3 Counting

Lemma 3.1 *If $g \geq 6$, then*

$$\sum_{i=3}^{g-3} h_i \leq \mu - 1 - 2h_2 < h_2.$$

Proof: The first inequality is from lemma 1.7. From corollary 1.10,

$$2 \cdot \frac{(\lambda+\epsilon)(\lambda+1)}{2\cdot\mu} \cdot (\mu - 1 - h_2) + (\lambda+1)\frac{(\mu - 2h_2)(\lambda+\epsilon) - 2h_2\mu}{2h_2\mu} \cdot (2h_2 + z)$$

$$\leq (\lambda+1)(\lambda+1-\mu),$$

for some z with $0 < z < \mu - 1 - 2h_2$. This simplifies to

$$(\lambda+\epsilon)[-2h_2(1+h_2) + (2h_2+z)(\mu - 2h_2)] + 2h_2\mu(\epsilon + \mu - 1 - 2h_2 - z) \leq 0.$$

But then $-2h_2(1+h_2) + (2h_2+z)(\mu - 2h_2) < 0$, because all the other terms are positive. So

$$\mu - 2h_2 < \frac{2h_2(1+h_2)}{2h_2+z} = h_2 + 1.$$

Corollary 3.2

$$\sum_{i=3}^{g-3} f_i \leq f_2 - (\lambda + 1).$$

Proof:

$$\begin{aligned}\sum_{i=3}^{g-3} f_i &= k - 1 - 2\lambda - 2f_2 \\ &= \frac{\mu f_2}{h_2} - (\lambda + 1) - 2f_2 \\ &= \frac{(\mu - 2h_2)}{h_2} f_2 - (\lambda + 1) \\ &\leq f_2 - (\lambda + 1).\end{aligned}$$

Theorem 3.3 *Let G be a directed distance-regular diagraph of short type (that is, with the diameter d equal to $g - 1$ where g is the girth). Then $d = g - 1 \leq 7$.*

Proof: If $g - 4 > 4$, then from corollary 3.2 and lemma 2.3,

$$f_2 - (\lambda + 1) \geq \sum_{i=3}^{g-3} f_i \geq 2f_3 + 2f_4 \geq f_2 + 1.$$

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