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SCHREIER COSET GRAPHS AND THEIR APPLICATIONS

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Abstract

Schreier coset graphs depict the standard permutation representation of a finitely-generated group on the cosets of a subgroup, and accordingly they may be used to provide diagrammatic interpretations of several aspects of combinatorial group theory, such as the Reidemeister-Schreier procedure, as well as a proof of the Ree-Singerman theorem (on the cycle structures of generating-permutations for a transitive group). Also they can often be applied to construct factor groups of given finitely-presented groups, with interesting results. Some of these uses are described, with applications to the construction of arc-transitive graphs and maximal automorphism groups of Riemann surfaces.

§1. INTRODUCTION

If G is a group with generating-set $\{x_1, x_2, ..., x_d\}$, and H is a subgroup of finite index n in G, define $\Gamma = \Gamma(x_1, x_2, ..., x_d\,;H)$ to be the graph whose vertices are the right cosets of H in G, and whose edges are of the form Hg—Hgx_i for $1 \le i \le d$. For obvious reasons this is called a *Schreier coset graph* (or *coset diagram*) for G; see [**1**]. The action of G on cosets of H by right multiplication may also be described by a *coset table*, similar to a Cayley table, with rows indexed by the cosets and columns indexed by the group generators x_i and their inverses. Both the graph and the table can be useful for the systematic enumeration of cosets (see [**1**] again).

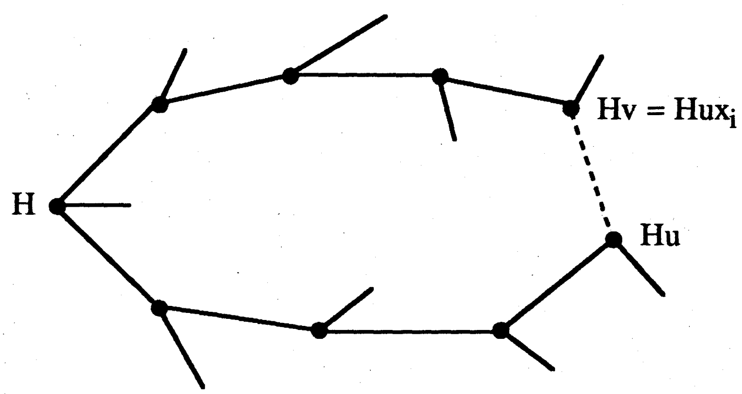
More generally if G has a (transitive) permutation representation on a set Ω of size n, we may form a graph whose vertices are the points of Ω and whose edges are of the form $\alpha — \alpha^{x_i}$ for $1 \le i \le d$. This is also a coset graph, the same as the one associated with the cosets of the stabilizer of a point. In fact these things are essentially interchangeable: the coset table, the coset graph, and the permutations induced by the group generators.

Before discussing some important applications of coset graphs, we shall make a few elementary observations about them, as follows:

First, any directed path in a coset graph $\Gamma = \Gamma(x_1, x_2, ..., x_d; H)$ may be described by a word in the generators x_i and their inverses, corresponding to the labels on the edges traversed by the path. In particular, any circuit in the graph based at the vertex labelled H corresponds to an element w of H (noting that $Hw = H$ if and only if $w \in H$), and this element is easily identifiable by tracing the edges used.

Next, a Schreier transversal for H in G may be associated with a spanning-tree in the coset graph: any path based at the vertex H in a spanning-tree corresponds to a word, say $u = x_1^{e_1} x_2^{e_2} \ldots x_s^{e_s}$, which can be used to label the vertex at the end of that path, and then all the intermediate vertices are labelled with initial sub-words $x_1^{e_1} x_2^{e_2} \ldots x_r^{e_r}$ (for $r \leq s$).

In particular, the edges of the coset graph not used by the spanning-tree correspond to a Schreier generating-set for H in G: if u is any element of the Schreier transversal, and $v = \overline{ux_i}$ is the transversal's representative of the coset Hux_i, then the x_i-edge from vertex Hu to vertex Hv completes a circuit in the coset graph corresponding to the Schreier generator $ux\overline{ux_i}^{-1}$, as illustrated by the broken edge in the diagram below.



§2. SOME APPLICATIONS

The observations made above lead quite naturally to a graphical interpretation of the Reidemeister-Schreier process: to find a presentation for a subgroup H of finite index n in a finitely-presented group $G = \langle X \mid R \rangle$, choose a spanning-tree in the associated coset graph, label the unused edges with (irredundant) Schreier generators, and then apply the relators in R to each of the n vertices in turn to obtain the Reidemeister-Schreier relations!

Another application of coset graphs provides an elegant proof of the following theorem, first obtained independently by Ree and Singerman using the Riemann-Hurwitz formula:
If G *is the group generated by permutations* $x_1, x_2, \ldots, x_d$ *of a set* Ω *of size* n*, such that* $x_1 x_2 \ldots x_d$ *is the identity permutation, and* c_i *is the number of orbits of* $\langle x_i \rangle$ *on* Ω*, then* G *is transitive on* Ω *only if* $c_1 + c_2 + \ldots + c_d \leq (d-2)n + 2$.

This result (which puts an upper bound on the total number of cycles of the generators of a transitive permutation group) has been used recently to investigate generating systems for the sporadic finite simple groups, and may be proved simply as follows: embed the associated coset graph in a surface of genus $g \geq 0$ by orienting anti-clockwise the edges at each vertex in the order given by the sequence $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_d, x_d^{-1}$, and then apply the Euler polyhedral formula $2-2g = \chi = V - E + F$. See [2] for the details.

On a more practical note, coset diagrams can often be used to prove certain groups are infinite, by joining diagrams together to construct permutation representations (of the given group) of arbitrarily large degree. This is essentially equivalent to the Abelianized form of the Reidemeister-Schreier process, and is described in [3]. The same sort of method is also useful for the construction of infinite families of finite quotients of a given finitely-presented group, as illustrated in the next Section.

In fact some quite strange things can happen when these diagrams are joined together. For example, consider the (2, 3, 7) triangle group $\Delta_{237} = \langle x, y \mid x^2 = y^3 = (xy)^7 = 1 \rangle$. This group has a number of finite quotients — including all but finitely many alternating groups, infinitely many PSL(2, q)'s, and several of the sporadic simple groups — and the groups PSL(2, 13), A_{64} and A_{22} in particular. Correspondingly there are transitive permutation representations of Δ_{237} of degrees 14, 64, and 22, as given by the actions of the generators x and y described below.

Degree 14:

$$x \longmapsto (\alpha_3, \alpha_4)(\alpha_5, \alpha_7)(\alpha_6, \alpha_{10})(\alpha_8, \alpha_{12})(\alpha_9, \alpha_{13})(\alpha_{11}, \alpha_{14})$$
$$y \longmapsto (\alpha_1, \alpha_2, \alpha_3)(\alpha_4, \alpha_5, \alpha_6)(\alpha_7, \alpha_8, \alpha_9)(\alpha_{10}, \alpha_{11}, \alpha_{12})$$

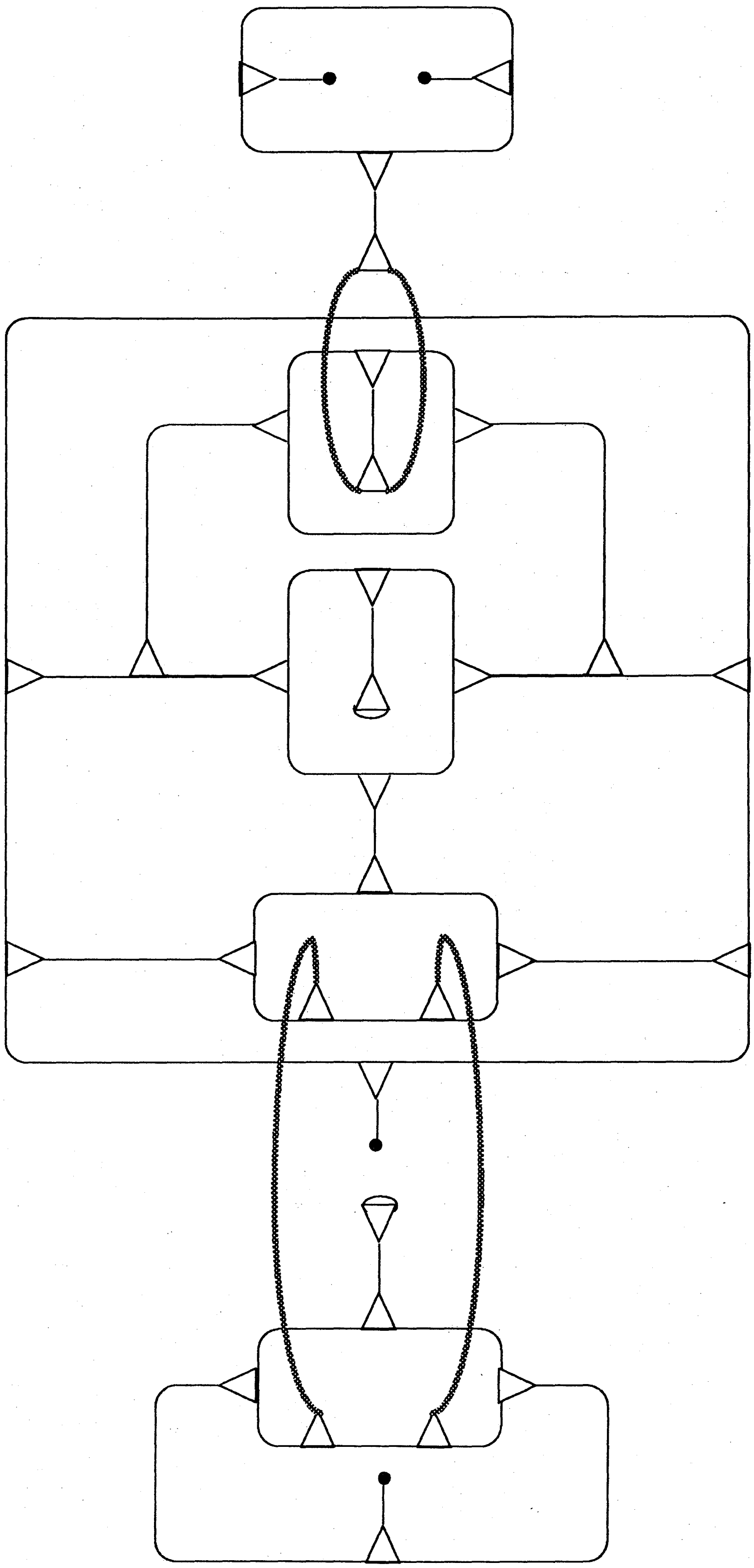
Degree 64:

$$\begin{aligned} x \longmapsto\ & (\beta_3, \beta_4)(\beta_5, \beta_7)(\beta_6, \beta_{10})(\beta_8, \beta_{12})(\beta_9, \beta_{13})(\beta_{11}, \beta_{16})(\beta_{14}, \beta_{19})(\beta_{15}, \beta_{22}) \\ & (\beta_{17}, \beta_{25})(\beta_{18}, \beta_{28})(\beta_{20}, \beta_{31})(\beta_{21}, \beta_{42})(\beta_{23}, \beta_{34})(\beta_{24}, \beta_{26})(\beta_{27}, \beta_{37})(\beta_{29}, \beta_{40}) \\ & (\beta_{30}, \beta_{32})(\beta_{33}, \beta_{43})(\beta_{35}, \beta_{46})(\beta_{36}, \beta_{49})(\beta_{38}, \beta_{50})(\beta_{39}, \beta_{52})(\beta_{41}, \beta_{56})(\beta_{44}, \beta_{45}) \\ & (\beta_{47}, \beta_{55})(\beta_{48}, \beta_{58})(\beta_{51}, \beta_{64})(\beta_{53}, \beta_{61})(\beta_{54}, \beta_{57})(\beta_{60}, \beta_{62}) \end{aligned}$$
$$\begin{aligned} y \longmapsto\ & (\beta_1, \beta_2, \beta_3)(\beta_4, \beta_5, \beta_6)(\beta_7, \beta_8, \beta_9)(\beta_{10}, \beta_{11}, \beta_{12})(\beta_{13}, \beta_{14}, \beta_{15})(\beta_{16}, \beta_{17}, \beta_{18}) \\ & (\beta_{19}, \beta_{20}, \beta_{21})(\beta_{22}, \beta_{23}, \beta_{24})(\beta_{25}, \beta_{26}, \beta_{27})(\beta_{28}, \beta_{29}, \beta_{30})(\beta_{31}, \beta_{32}, \beta_{33}) \\ & (\beta_{34}, \beta_{35}, \beta_{36})(\beta_{37}, \beta_{38}, \beta_{39})(\beta_{40}, \beta_{41}, \beta_{42})(\beta_{43}, \beta_{44}, \beta_{45})(\beta_{46}, \beta_{47}, \beta_{48}) \\ & (\beta_{49}, \beta_{50}, \beta_{51})(\beta_{52}, \beta_{53}, \beta_{54})(\beta_{55}, \beta_{56}, \beta_{57})(\beta_{58}, \beta_{59}, \beta_{60})(\beta_{61}, \beta_{62}, \beta_{63}) \end{aligned}$$

Degree 22:

$$x \longmapsto (\gamma_1,\gamma_2)(\gamma_3,\gamma_4)(\gamma_5,\gamma_7)(\gamma_6,\gamma_{10})(\gamma_8,\gamma_{19})(\gamma_9,\gamma_{13})(\gamma_{11},\gamma_{16})(\gamma_{12},\gamma_{20})(\gamma_{14},\gamma_{18})(\gamma_{21},\gamma_{22})$$
$$y \longmapsto (\gamma_1,\gamma_2,\gamma_3)(\gamma_4,\gamma_5,\gamma_6)(\gamma_7,\gamma_8,\gamma_9)(\gamma_{10},\gamma_{11},\gamma_{12})(\gamma_{13},\gamma_{14},\gamma_{15})(\gamma_{16},\gamma_{17},\gamma_{18})(\gamma_{19},\gamma_{20},\gamma_{21})$$

Simplified versions of the coset diagrams associated with these representations appear on the next page. Fixed points of y are denoted by heavy dots; loops corresponding to fixed points of x or y are omitted; double x-edges are replaced by single ones; and 3-cycles of y are denoted by small triangles (whose vertices are permuted anticlockwise by y). Also the vertices have not been labelled, to keep the diagram uncluttered. Now if these three diagrams are joined together by the fuzzy bold x-edges as indicated, we obtain a new coset diagram for Δ_{237} on 100 points, corresponding to juxtaposition of the permutations given for x and y, and the introduction of additional 2-cycles (α_1, β_1), (α_2, β_2), $(\beta_{59}, \gamma_{17})$ and $(\beta_{63}, \gamma_{15})$ to the permutation induced by x. The new permutations still satisfy the relations $x^2 = y^3 = (xy)^7 = 1$, but this time generate the Hall-Janko group J_2!



§3. FURTHER RESULTS

Note that arbitrarily many copies of the middle (64-vertex) diagram on the previous page may be linked together in an alternating chain, with new x-edges joining common β_1-, β_2-, β_{59}- and β_{63}-vertices in successive copies. In particular, this produces a transitive permutation representation of Δ_{237} of degree 64m (or equivalently, a subgroup of index 64m in Δ_{237}) for every positive integer m, giving a simple proof that Δ_{237} is infinite.

In fact the same method of composition of coset diagrams (and even the same diagram on 64 vertices, along with some others) can be used to show that all but finitely many of the alternating groups A_n are factor groups of Δ_{237}, and even of its quotient $(2, 3, 7; 84) = \langle x, y \mid x^2 = y^3 = (xy)^7 = [x, y]^{84} = 1 \rangle$; see [4]. This in turn gives rise to an infinite family of compact Riemann surfaces (of genus $g_n = \frac{n!}{168} + 1$), each having the maximum possible number of conformal automorphisms, by Hurwitz's theorem.

Similarly, for all but finitely many n the symmetric group S_n is a homomorphic image of the amalgamated product

$$\begin{aligned} G_5 = \langle\, h, p, q, r, s, a \mid\ & h^3 = p^2 = q^2 = r^2 = s^2 = a^2 = 1, \\ & [p, q] = [p, r] = [p, s] = [q, r] = [q, s] = 1, \quad (rs)^2 = pq, \\ & ph = hp, \quad qh = hr, \quad rh = hpqr, \quad shs = h^{-1}, \quad a^{-1}pa = q, \quad a^{-1}ra = s \,\rangle. \end{aligned}$$

This can be proved by adding new edges for the involutory generator a to chains of coset diagrams for the subgroup $\langle h, p, q, r, s\rangle$ (of order 48), and gives rise to an infinite family of 5-arc-transitive cubic graphs; see [5]. Also the same sort of approach (but with a different group in place of G_5) can be used to prove the existence of infinitely many finite 4-arc-transitive cubic graphs containing circuits of specified lengths; see [6].

Finally, recent use of coset diagrams to find torsion-free subgroups of certain finitely-presented groups has been instrumental in the construction of small volume hyperbolic 3-orbifolds and other hyperbolic 3-manifolds with interesting properties; see [7].

References

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