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Kyoto University
An infinitesimal analysis in topology

KANEDA Masaharu Niigata University Faculty of Science Department of Mathematics

1° Introduction

Algebraic topology is the study of functors from the category $\mathfrak{Top}$ of topological spaces into some algebraic categories.

Take, for example, functor $H^\cdot(\ )$ of the singular cohomology in coefficient $K$ a coommutative ring. Given a diagram in $\mathfrak{Top}$

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & Y \\
\uparrow & & \downarrow \\
X & \xleftarrow{k} & Y.
\end{array}
\]

If $f$ extends to $Y$, one will get in the category $\mathfrak{Alg}_K$ of $K$-algebras a commutative diagram

\[
\begin{array}{ccc}
H^\cdot(Z) & \xrightarrow{H^\cdot(f)} & H^\cdot(Y) \\
\downarrow & & \downarrow \\
H^\cdot(X) & \xleftarrow{k} & H^\cdot(Y).
\end{array}
\]

Further, if $\tilde{f}$ denotes an extension of $f$, then $H^\cdot(\tilde{f})$ must commute with all the natural transformations, called cohomology operations, from $H^\cdot(\ )$ into itself, hence if lucky, one can sometimes decide if an extension exists or not by algebraic means.

If $K$ is a field of positive characteristic, we have a well-known set of cohomology operations constituting a skew graded Hopf algebra $\mathcal{A}$, called the Steenrod algebra. One thus wishes to study the algebra $\mathcal{A}$ and the $\mathcal{A}$-module structures of $H^\cdot(X)$.

This is a survey to introduce an attempt [KSTY] to throw a new light on the Steenrod algebra using infinitesimal unipotent $K$-groups.

For simplicity we will fix $K = F_p$, $p$ odd prime, in what follows.

2° The Steenrod Algebra

(2.1) The Bockstein operator $\beta$ is a natural map

\[
H^n(X) \longrightarrow H^{n+1}(X) \quad \forall n \in \mathbb{N},
\]
induced by the short exact sequence $0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{F}_p \rightarrow 0$ such that

$\beta^2 = 0$

and

$\beta(xy) = (\beta x)y + (-1)^n x(\beta y) \quad \forall x \in H^n(X), y \in H^m(X),$

where the multiplication is the cup product $H^\cdot(X) \otimes K H^\cdot(X) \rightarrow H^\cdot(X)$ induced by the diagonal mapping $X \rightarrow X \times X$.

Further, one has unique natural maps \[SE\] (VI.1)

$p^i : H^n(X) \rightarrow H^{n+2i(p-1)}(X) \quad \forall i, n \in \mathbb{N},$

called the Steenrod reduced powers, such that

$P^0 = \text{id},$

$P^i = \begin{cases} x^p & \text{if } x \in H^{2i}(X) \\ 0 & \text{if } x \in H^j(X) \text{ with } j < 2i, \end{cases}$

(5) (Cartan formula)

$p^i(xy) = \sum_{i=0}^{i} p^i(x)p^{i-j}(y) \quad \forall x \in H^n(X), y \in H^m(X)$

and

(6) (Adem relations)

$\forall a < pb, \quad p^a p^b = \sum_{t=0}^{[\frac{a}{p}]} (-1)^{a+t} \binom{p-1}{a-pt} p^{a+b-t} p^t,$

$\forall a \leq pb, \quad p^a \beta p^b = \sum_{t=0}^{[\frac{a}{p}]} (-1)^{a+t} \binom{p-1}{a-pt} \beta p^{a+b-t} p^t$

$\quad + \sum_{t=0}^{[\frac{a-1}{p}]} (-1)^{a+t-1} \binom{p-1}{a-pt-1} p^{a+b-t} \beta p^t.$

With those in mind we define the Steenrod algebra $\mathcal{A}$ to be

$TK(M)/(\mathcal{B}^2, \text{Adem relations}),$
where $T_K(M)$ is the tensor algebra over $K$ of a $K$-linear space $M$ with basis $\mathcal{B}, \mathcal{P}^i, i \in \mathbb{Z}^+$, corresponding to $\beta, p^i$, respectively. We assign $\mathcal{B}$ (resp. $\mathcal{P}^i$) degree 1 (resp. $2i(p-1)$), thus making $A$ into a graded $K$-algebra. Put $\mathcal{P}^0 = 1$.

(2.2) Instead of $A$ itself we will consider below the graded quotient $S := A/\langle \mathcal{B} \rangle$. If $I = (i_1, \ldots, i_h) \in \mathbb{N}^h, k \geq 1$, we set in $A$

$$\mathcal{P}^I = \mathcal{P}^{i_1} \cdots \mathcal{P}^{i_h}.$$  

By abuse of notations we will denote the image of $\mathcal{P}^I$ in $S$ by the same letter. We say $I$ is admissible iff either $I = 0$ or

$$\forall \nu \in [1, k], \quad i_\nu \geq 1 \text{ and } i_\nu \geq pi_{\nu+1} \text{ with } i_{k+1} = 0$$

in which case we call $\mathcal{P}^I$ an admissible monomial.

(2.3) **Theorem (Milnor[Mil]).** We have

(i) The admissible monomials form a $K$-linear basis of $S$. In particular, each homogeneous part $S_m$ is finite dimensional.

(ii) $S$ is generated as $K$-algebra by $\mathcal{P}^{p^i}, i \in \mathbb{N}$.

(iii) With comultiplication

$$\Delta_S : \mathcal{P}^k \rightarrow \sum_{i=0}^{k} \mathcal{P}^i \otimes \mathcal{P}^{k-i}$$

and the counit $\mathcal{P}^I \rightarrow 0 \forall I \text{ admissible } \neq 0$, $S$ forms a cocommutative graded bigebral.

(2.4) If $A = \bigoplus_{i \geq 0} A_i$ is a graded $K$-bigebral with $A_0 = K$, then $A$ admits a unique antipode $\sigma_A$, making $A$ into a graded Hopf algebra, due to R. Thom [MM],(8.7): if $a \in A_n$, one defines $\sigma_A(a)$ inductively on the degree of the elements by

$$\sigma_A(a) = -a - \sum_i \sigma_A(a_i)a'_i$$

if $\Delta_A(a) = 1 \otimes a + a \otimes 1 + \sum_i a_i \otimes a'_i$ with $a_i$ and $a'_i$ homogeneous of less degrees than $a$. In particular, $S$ carries a structure of cocommutative graded Hopf algebra.

(2.5) Let $S^{*gr} = \bigoplus_{i \geq 0} S_i^*$ be the graded dual of $S$. Using the identification $(S \otimes_K S)^{*gr} \simeq S^{*gr} \otimes_K S^{*gr}$ via

$$(f \otimes g)(a \otimes b) = f(a)g(b),$$

$S^{*gr}$ comes equipped with a structure of commutative graded Hopf algebra.

Let $I_h = (p^{h-1}, p^{h-2}, \ldots, p^1, p^0), k \geq 1$, and $\xi_k \in S^{*gr}$ the dual of $\mathcal{P}^{I_h}$ with respect to the basis of admissible monomials, so $\deg(\xi_k) = 2(p^k - 1)$. 
(2.6) **Theorem (Milnor[Mil]).** $S^{*g}$ is the polynomial algebra $K[\xi_1, \xi_2, \ldots]$ in indeterminates $\xi_i$, $i \geq 1$, with the comultiplication

$$\xi_h \mapsto \sum_{i=0}^{h} \xi_{h-i}^{p^i} \otimes \xi_i$$

and the counit annihilating all $\xi_i$.

(2.7) Let $\mathcal{I}_h$, $k \geq 0$, be the ideal of $S^{*g}$ generated by

$$\xi_1^p, \xi_2^{p^{k-1}}, \ldots, \xi_h^p, \xi_{h+j}, \quad j \geq 1.$$ 

Then $\mathcal{I}_h$ is a Hopf ideal, hence $S^{*g}/\mathcal{I}_h \simeq K[\xi_1, \ldots, \xi_h]/(\xi_1^p, \ldots, \xi_h^p)$ is a finite dimensional graded Hopf algebra. In turn, its dual $S(k)$ is a Hopf subalgebra of $S$.

One can show that $S(k)$ is generated as $K$-algebra by

$$\mathfrak{P}^{p^i}, \quad 0 \leq i \leq k - 1,$$

hence $S = \bigcup_{h \geq 1} S(k)$ by (2.3). In [KSTY] $S(k)$ is denoted by $P(k - 1)$.

### 3° Infinitesimal unipotent groups

(3.1) An (affine) $K$-group (scheme) $\emptyset$ is a representable functor from the category $K\text{Alg}$ of commutative $K$-algebras into the category of groups: there is commutative Hopf $K$-algebra $K[\emptyset]$ such that

(1) \hspace{1cm} $\emptyset(\_ ) = K\text{Alg}(K[\emptyset], \_).$

If $m_\emptyset$ (resp. $\Delta_\emptyset$, $\epsilon_\emptyset$, $\sigma_\emptyset$) is the multiplication (resp. comultiplication, counit, antipode) of $K[\emptyset]$, then for each $R \in K\text{Alg}$, $\emptyset(R)$ is a group under the multiplication

\[
\emptyset(R) \times \emptyset(R) \longrightarrow \emptyset(R)
\]

and inversion

\[
\emptyset(R) \longrightarrow \emptyset(R)
\]
with the identity element defined by

$$
\varepsilon_K(R) \xrightarrow{\sim} \mathcal{O}(R)
$$

$$
K\text{Alg}(K, R) \xrightarrow{\varepsilon_K} K\text{Alg}(K[\mathcal{O}], R).
$$

A $\mathcal{O}$-module is a $K$-linear space $M$ together with a map $\Delta_M : M \to M \otimes_K K[\mathcal{O}]$, called a $K[\mathcal{O}]$-comodule map, such that for each $R \in K\text{Alg}$, the map $\mathcal{O}(R) \times (M \otimes_K R) \to M \otimes_K R$ via

$$
(z, m \otimes r) \mapsto ((M \otimes_K z) \circ \Delta_M(m))r = \sum_i m_i \otimes rz(a_i)
$$

if $\Delta_M(m) = \sum_i m_i \otimes a_i$, makes $M \otimes_K R$ into a $\mathcal{O}(R)$-module over $R$.

(3.2) We say a $K$-group $\mathcal{O}$ is algebraic iff the algebra $K[\mathcal{O}]$ is of finite type over $K$. In this note we will consider only algebraic $K$-groups.

Let $\sigma_\emptyset = \text{ker}(\varepsilon_\emptyset)$, called the augmentation ideal of the Hopf algebra $K[\mathcal{O}]$, and set

$$
\text{Dist}_m(\emptyset) = \{\mu \in K[\mathcal{O}]^* \mid \mu(7_{\emptyset}^{m+1}) = 0\}, \quad m \in \mathbb{N}.
$$

Then $\text{Dist}(\emptyset) := \bigcup_{m \in \mathbb{N}} \text{Dist}_m(\emptyset)$ carries a structure of cocommutative Hopf algebra, called the algebra of distributions of $\emptyset$, with the multiplication given by

$$
(\mu \nu)(a) = (\mu \otimes \nu) \circ \Delta_\emptyset(a) = \sum_i \mu(a_i) \nu(a'_i) \quad \text{if} \quad \Delta_\emptyset(a) = \sum_i a_i \otimes a'_i,
$$

comultiplication $\Delta'_\emptyset$ such that $\Delta'_\emptyset(\mu)(a \otimes b) = \mu(ab)$, counit $\varepsilon'_\emptyset$ such that $\varepsilon'_\emptyset(\mu) = \mu(1)$, and the antipode $\sigma'_\emptyset$ such that $\sigma'_\emptyset(\mu) = \mu \circ \sigma_\emptyset$, using natural isomorphisms $[J], (1.7.4)$

$$
\text{Dist}(\emptyset) \otimes_K \text{Dist}(\emptyset) \xrightarrow{\sim} \text{Dist}(\emptyset \times \emptyset)
$$

$$
\sum_{i=0}^m \text{Dist}_i(\emptyset) \otimes_K \text{Dist}_{m-i}(\emptyset) \xrightarrow{\sim} \text{Dist}_m(\emptyset \times \emptyset).
$$

In particular,

$$
\text{Dist}^+_1(\emptyset) := \{\mu \in \text{Dist}_1(\emptyset) \mid \mu(1) = 0\}
$$

forms a Lie algebra over $K$, called the Lie algebra of $\emptyset$, with $[\mu, \nu] := \mu \nu - \nu \mu$.

Any $\mathcal{O}$-module $M$ carries a structure of $\text{Dist}(\emptyset)$-module such that

$$
\mu m = (M \otimes_K \mu) \circ \Delta_M(m) = \sum_i \mu(a_i)m_i
$$

$$
\forall \mu \in \text{Dist}(\emptyset) \text{ and } m \in M \text{ if } \Delta_M(m) = \sum_i m_i \otimes a_i.
$$
We say a $K$-group $\mathfrak{G}$ is infinitesimal iff $K[\mathfrak{G}]$ is finite dimensional over $K$ with the nilpotent augmentation ideal $J_{\mathfrak{G}}$.

If $\mathfrak{G}$ is infinitesimal, then $\mathfrak{G}(R)$ is a singleton for any integral domain $R$. Also $\text{Dist}(\mathfrak{G}) = K[\mathfrak{G}]^*$.

Let $\mathfrak{G}$ be an arbitrary $K$-group again. The map

$\phi : K[\mathfrak{G}] \rightarrow K[\mathfrak{G}]$ via $a \mapsto a^p$

is a homomorphism of Hopf algebras, inducing a morphism of $K$-groups $F_{\mathfrak{G}} := K\mathfrak{Alg}(\phi, \_): \mathfrak{G} \rightarrow \mathfrak{G}$, called the Frobenius morphism of $\mathfrak{G}$. Then $\mathfrak{G}^1 := \ker(F_{\mathfrak{G}}) = \mathfrak{G} \times_{\mathfrak{G}} \epsilon_{K}$ is a normal subgroup of $\mathfrak{G}$, with

(2) $K[\mathfrak{G}^1] \simeq K[\mathfrak{G}] \otimes_{K[\mathfrak{G}]} (K[\mathfrak{G}]/J_{\mathfrak{G}}) \simeq K[\mathfrak{G}]/(a^p | a \in J_{\mathfrak{G}})$,

hence $\mathfrak{G}^1$ is infinitesimal. More generally, $\mathfrak{G}^r := \ker(F_{\mathfrak{G}}^r), r \in \mathbb{Z}^+$, is an infinitesimal normal subgroup of $\mathfrak{G}$ with $K[\mathfrak{G}^r] \simeq K[\mathfrak{G}]/(a^{pr} | a \in J_{\mathfrak{G}})$, called the $r$-th Frobenius kernel of $\mathfrak{G}$.

(3.4) We now focus on the unipotent $K$-group $\mathfrak{U}_n$ such that $K[\mathfrak{U}_n] = K[x_{ij}]_{1 \leq j < i \leq n}$ polynomial algebra in indeterminates $x_{ij}, 1 \leq j < i \leq n$, with the comultiplication

(1) $x_{ij} \mapsto \sum_{k=0}^{i} x_{ik} \otimes x_{kj}$

and the counit $x_{ij} \mapsto 0 \forall i, j$, where we agree that $x_{ii} = 1 \forall i$. If $R \in K\mathfrak{Alg}$, $\mathfrak{U}_n(R)$ is isomorphic to the group of $n \times n$ lower triangular unipotent matrices with the entries in $R$. Also if each $x_{ij}$ is assigned degree 1, then

(2) $\text{Dist}(\mathfrak{U}_n) \simeq K[\mathfrak{U}_n]^*\mathfrak{g} \quad \text{as } K\text{-linear spaces.}$

Let $\mathfrak{U}_{ij}$ be a subgroup of $\mathfrak{U}_n$ with $K[\mathfrak{U}_{ij}] = K[\mathfrak{U}_n]/(x_{ij})_{(i,j) \neq (i,j)} \simeq K[x_{ij}], \text{ hence } \mathfrak{U}_{ij}(R)$ consists of the $(i,j)$-th elementary unipotent matrices with the entries in $R$. With $\text{deg}(x_{ij}) = 1, K[\mathfrak{U}_{ij}]$ is a graded commutative cocommutative Hopf algebra, and

(4) $\text{Dist}(\mathfrak{U}_{ij}) \simeq K[\mathfrak{U}_{ij}]^*\mathfrak{g} \simeq S_K(x_{ij})^*\mathfrak{g}$

the graded dual of the symmetric algebra in $x_{ij}$ [B],(III.11), with the dual monomial basis $X_{ijk} : X_{ijk}(x_{ij}^l) = \delta_{kl} \quad \forall k, l \in \mathbb{N}$. Hence

(5) $X_{ijk}X_{ij\ell} = \binom{k+\ell}{k} X_{ijk+\ell}$,

and

(6) $\Delta_{\mathfrak{U}_{ij}}(X_{ijk}) = \sum_{\ell=0}^{k} X_{ij\ell} \otimes X_{ij\ell-k}$. 
Under the multiplication one has a natural bijection

\[ \prod_{1 \leq j < i \leq n} \iota t_{j}(R) \sim 1t_{\mathfrak{n}}(R) \quad \forall R \in K\mathfrak{U}lg, \]

where the product is taken in any order, hence also a $K$-linear isomorphism

\[ \bigotimes_{i,j} Dist(14_{i}) \sim Dist(\ ) \]

which is, however, not an isomorphism of Hopf algebras if $n \geq 3$.

We now arrange the $\iota t_{ij}$ in (7) and (8) in the increasing order such that

\[ (i,j) \succ (s,t) \text{ iff } i > s \text{ or } i = s \text{ with } j < t, \]

and fix the arrangement in taking the product once and for all. Then under (8)

\[ (\prod_{i,j} X_{ijk})_{h \in N} \frac{n(n-1)}{2} i,j \]

forms a $K$-linear basis of $Dist(\iota t_{n})$ dual to the monomial basis $\prod_{i,j} x_{ij}^{h}$ of $K[\iota t_{\mathfrak{n}}]$ in the sense of (2).

\section{The Steenrod Algebra Revisited}

(4.1) In 1988 Tezuka M. found a homomorphism of bigebras

\[ \psi : K[\iota t_{n}] = K[x_{ij}]_{1 \leq j < i \leq n} \rightarrow K[\xi_{1}, \xi_{2}, \ldots] = S^{gr} \quad \text{via} \quad x_{ij} \mapsto \xi_{i-j}^{p^{j-1}}. \]

Further, by assigning $x_{ij}$ degree $2(p^{j-i} - 1)p^{i-1}$ we can make $K[\iota t_{n}]$ into a graded bigebra. Then by the unicity of the antipode on graded bigebras (2.4), $\psi$ is actually a homomorphism of graded Hopf algebras.

Now $\text{im} \psi = K[\xi_{1}, \ldots, \xi_{n}]$ is a Hopf subalgebra of $S^{gr}$, so let us write $\psi$ again for the homomorphism of Hopf algebras $K[\iota t_{n}] \rightarrow K[\xi_{1}, \ldots, \xi_{n}]$ induced by $\psi$. If

\[ \pi_{1} : K[\iota t_{n}] \rightarrow K[x_{ij}]_{i,j} / (x_{ij}^{p^{j-i}})_{i,j} = K[\iota t_{n}^{p}] \]

and

\[ \pi_{2} : K[\xi_{1}, \ldots, \xi_{n-1}] \rightarrow K[\xi_{1}, \ldots, \xi_{n-1}] / (\xi_{1}^{p^{n-1}}, \xi_{2}^{p^{n-2}}, \ldots, \xi_{n-1}^{p}) = S(n-1)^{*} \]

are the natural maps, we get a commutative diagram of surjective homomorphisms of Hopf algebras

\[ \begin{array}{ccc}
K[\iota t_{n}] & \xrightarrow{\psi} & K[\xi_{1}, \ldots, \xi_{n}] \\
\pi_{1} \downarrow \quad \psi_{n} \downarrow \quad \pi_{2} \\
K[\iota t_{n}^{n}] & \rightarrow & S(n-1)^{*}.
\end{array} \]

Dualizing $\psi_{n}$,
(4.2) THEOREM [KSTY], (3.3). We have an imbedding of Hopf algebras \( S(n-1) \) into \( \text{Dist}(\mathfrak{U}_n) \).

(4.3) Hence any \( \mathfrak{U}_n \)-module carries a structure of \( S(n-1) \)-module upon restriction, enabling one to exploit the representation theory of \( \mathfrak{U}_n \) in the study of \( S(n-1) \)-modules, where \( \mathfrak{U}_n \) is the \( K \)-group such that \( \mathfrak{U}_n(R) \) is the group of \( n \times n \) invertible matrices with the entries in \( R, R \in K \mathfrak{U} \).

(4.4) To illustrate an application, let us first recall some representation theory of \( \mathfrak{U}_n \).

Let \( E \) be the natural \( n \)-dimensional \( \mathfrak{U}_n \)-module of basis \( e_1, \ldots, e_n \). If \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \) is a partition of \( r = \sum_{i=1}^{n-1} \lambda_i \), let \( (\lambda_1', \ldots, \lambda_m') \) be the transposed partition of \( \lambda \), and put

\[
(1) \quad \Phi_{\lambda} = \left( \sum_{\sigma \in S_{\lambda_1'}} \text{sgn}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(\lambda_1')} \right) \otimes \cdots \otimes \left( \sum_{\sigma \in S_{\lambda_m'}} \text{sgn}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(\lambda_m')} \right)
\]

in \( E^{\otimes r} \). After R. Carter and G. Lusztig [CL] we call \( \text{Dist}(\mathfrak{U}_n) \Phi_{\lambda} \) in \( E^{\Phi^r} \) the Weyl module of highest weight \( \lambda \), and denote it by \( V(\lambda) \).

In case \( \lambda \) is column \( p \)-regular, i.e.,

\[
(2) \quad 0 \leq \lambda_i - \lambda_{i+1} \leq p - 1 \quad \forall i \in [1, n-1] \quad \text{with} \quad \lambda_n = 0,
\]

one can show [KSTY],(3.7)

\[
(3) \quad V(\lambda) = S(n-1) \Phi_{\lambda}.
\]

(4.5) Let \( Y \) be the complex \((p^n - 1)\)-projective space. Then

\[
(1) \quad H^r(Y) \cong K[z]/(x^{p^n}) \quad \text{as graded } K\text{-algebras},
\]

where \( z \) is an indeterminate of degree 2. Hence \( H^r(Y) \) admits a structure of \( S \)-module. Explicitly,

\[
(2) \quad \theta^i(z^j) \equiv \binom{j}{i} x^{j+i(p-1)} \mod x^{p^n}.
\]

If \( V \) is a \( K \)-linear span of \( z, x, \ldots, z^{p^n-1} \) in \( H^r(Y) \), \( V \) is stable under the action of \( S(n-1) \). Further, there is an isomorphism

\[
(3) \quad \theta : E \simto V \quad \text{via} \quad e_i \mapsto z^{p^{i-1}}, \quad i \in [1, n]
\]

of \( S(n-1) \)-modules [KSTY],(4.3), which induces by the Künneth formula or by the Cartan formula an imbedding of \( S(n-1) \)-modules

\[
(4) \quad \theta^{\otimes r} : E^{\otimes r} \hookrightarrow H^r(Y^r) \quad \text{via} \quad e_{i_1} \otimes \cdots e_{i_r} \mapsto z^{p^{i_1-1}} \otimes \cdots z^{p^{i_r-1}} \quad \forall r \in \mathbb{Z}^+.
\]
In particular, the Weyl module $V(\lambda)$ with $r = \sum_{i=1}^{n-1} \lambda_i$ imbeds in $H^*(Y^r)$ as an $S(n-1)$-submodule.

(4.6) Fix a column $p$-regular partition $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ of $r = \sum_{i=1}^{n-1} \lambda_i$ with its transpose $(\lambda'_1, \ldots, \lambda'_m)$.

Composed several times with the cup product, $\theta^{\otimes r}$ of (4.4) yields an $S(n-1)$-homomorphism

\[ \phi : E^{\otimes r} \longrightarrow H^*(Y^{\lambda'_1}) \] such that $e_i \otimes \cdots \otimes e_{i_r} \mapsto \otimes_{j=1}^r x_j^{e(j)}$, where $e(j) = p^{i(j)-1} + p^{i(j+1)-1} + \cdots + p^{i(j)+\lambda'_i+\cdots+\lambda'_m-1}$ with $k(j) = \max\{i | 1 \leq i \leq m, \lambda'_i \geq j\} - 1$. Set $\theta_\lambda = \phi|_{V(\lambda)}$. One finds

(2) $\theta_\lambda(\Phi_\lambda) \neq 0$.

(4.7) Theorem (Smith, Mitchell[MIT], [KSTY], (4.10)). If $\lambda = ((n-1)(p-1), (n-2)(p-1), \ldots, p-1)$, then $\theta_\lambda$ imbeds $V(\lambda)$, called the Steinberg module that is free over $S(n-1)$, into $H^*(Y^{n-1})$ as $S(n-1)$-modules.

(4.8) Further, we have a curious

Proposition [KSTY],(4.10). If $\lambda_1 \leq p - 1$ and if $V(\lambda)$ is $\mathfrak{S}_n$-simple, then $\theta_\lambda$ imbeds $V(\lambda)$ into $H^*(Y^{\lambda'_1})$ as a $S(n-1)$-submodule.

(4.9) To further our speculation, Mabuchi [Ma] has verified that in case $n \geq 3$ and $\lambda = (p, 1, \ldots, 1)$ with 1 appearing $(n-2)$-times,

(1) $\theta_\lambda$ is injective iff $V(\lambda)$ is $\mathfrak{S}_n$-simple.

Computer work by his fellow student Takeno S. has also checked (1) for all column $p$-regular $\lambda$ in case $n = 3$ and $p \leq 7$.

References


