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Robinson-Schensted Correspondences for Differential Posets

Thomas W. Roby

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1 Introduction

The Robinson-Schensted (R-S) correspondence and its many variations lie at the combinatorial heart of many facts from representation theory and symmetric function theory. They provide concrete bijective proofs of results that were often originally obtained in much more algebraic or abstract ways. Most of these results can be viewed as counting Hasse walks in certain partially ordered sets. Stanley was able to derive many enumerative results on the class of *differential posets* (of which Young's lattice is a member) using a highly algebraic approach which converted certain enumerative problems to (solvable) partial differential equations. Fomin (independently) defined essentially the same class of graphs and constructed a generalization of the R-S correspondence to differential posets. In this note we begin to show how Fomin's construction can be used to unify many of the R-S variants, including Knuth's generalization to semi-standard tableaux, the skew algorithms of Sagan and Stanley, the oscillating algorithms of Sundaram, and the oscillating Knuth algorithm of Gessel. It allows one to view all these variants as natural constructions. Full details and many more applications are given in the author's thesis [Rob].

2 Background

We assume some familiarity with the notion of a partition of an integer; for precise definitions and notation we refer the reader to [Mac]. We adopt the English convention of writing our Young diagrams as left justified arrays with the length of the rows (weakly) decreasing from top to bottom. The example below should help make things clear.

Definition 2.1 We may define a partial order $\subseteq$ on partitions by $\mu \subseteq \lambda$ if and only if the Young diagrams $D_\mu \subseteq D_\lambda$. Equivalently,

$$\mu \subset \lambda \iff \mu_i \leq \lambda_i \qquad \forall \;\; i \geq 1.$$

This partial order is easily seen to be a distributive lattice, Y, which we will call **Young's Lattice** and which will play a significant role in what follows. Our point of view on tableaux is that they represent chains in Young's lattice, as in the following example.

Definition 2.2 Let λ/μ be a skew shape. A **tableau** T **of shape** λ/μ is an order preserving map

$$T : D_{\lambda/\mu} \mapsto \mathbf{Z}^+$$

i.e., an assignment of positive integers to the cells of λ which is weakly increasing along the columns (from left to right) and down the rows. We define the **weight** or **type** of a tableau to be the sequence

$$w(T) = (n_1, n_2, n_3, \ldots)$$

where n_i is the number of cells of λ/μ assigned to the integer i by T. Equivalently, we may consider T to be a multichain in Young's lattice, i.e., a weakly increasing sequence of shapes

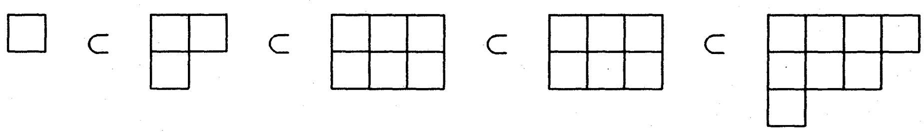
$$\mu = \lambda^0 \subseteq \lambda^1 \subseteq \ldots \subseteq \lambda^k = \lambda$$

by simply filling each skew shape λ^i/λ^{i-1} with the integer i. We write $sh(T) = \lambda/\mu$.

Example 2.3 The tableaux

$$T = \begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & 1 & 2 & 4 \\ \hline 1 & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 4 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

corresponds to the sequence of shapes



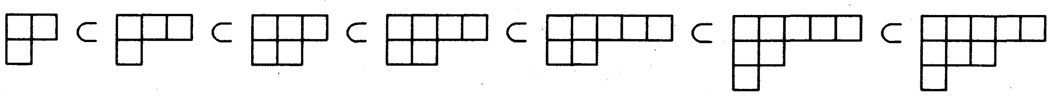
We shall be particularly interested in some special kinds of tableaux.

Definition 2.4 A tableaux T is said to be **column-strict** or **semi-standard** or **generalized** if it is strictly increasing along the columns. When the tableaux is viewed as a multichain, this says exactly that each λ^i/λ^{i-1} is a horizontal strip. T is called **standard** if it has type $(1,1,\ldots,1)$, i.e., if all the entries are distinct (so we may take them to be the numbers $1,2,\ldots,|T|$). When viewed as a multichain, this says exactly that each λ^i/λ^{i-1} is a single cell; hence, it represents a saturated chain in $\mathbf{Y}$. The sets of generalized and standard tableaux will be denoted $ST(\lambda/\mu)$ and $GT(\lambda/\mu)$, respectively.

Example 2.5 The standard tableau

$$T = \begin{array}{cccc} & & \boxed{1} & \boxed{3}\,\boxed{4} \\ & \boxed{2} & \boxed{6} & \\ \boxed{5} & & & \end{array}$$

corresponds to the saturated chain



We will be interested in counting the number of chains in Young's lattice which satisfy certain conditions. Hence, we define

Definition 2.6 $f^{\lambda/\mu} := \#\{\text{standard tableaux } T \text{ of shape } \lambda/\mu\}$

The basic Robinson-Schensted algorithm is a bijection between permutations and pairs of standard tableaux of the same shape. More precisely, we have the following

Theorem 2.7 Fix an integer n. There is a bijection between permutations in S_n and pairs of standard Young tableaux (P,Q) of the same shape λ, where λ runs over all partitions of n:

$$\pi \quad \overset{\text{R-S}}{\longleftrightarrow} \quad (P,Q)$$

The usual approach to the R-S algorithm is through repeated use of a process called *insertion*, which the interested reader may find discussed in many of the references, especially [Knu] and [Sag].

The Robinson-Schensted correspondence has many important properties which are not immediately obvious from the above description. Some of these become much easier to prove using Fomin's approach. For example, Schützenberger's well-known theorem that interchanging the P and Q tableaux is equivalent to taking the inverse of the permutation.

3 Growths

In any graded poset, if y covers x (i.e., $x \leq y$ and $\rho(y) = \rho(x) + 1$) we will sometimes write "$x \lessdot y$".

Definition 3.1 Let P and Q be graded posets. A map $g : P \mapsto Q$ is called a *growth* if it preserves the relation $\lessdot$:

$$x \lessdot y \longrightarrow g(x) \lessdot g(y)$$

Not every order preserving map is a growth, and not every growth is one-to-one.

Example 3.2 **(1)** The rank function $\rho : P \mapsto \mathbf{Z}$ is a growth.

(2) A multichain in P is a growth from $\mathbf{Z}$ to P.

(3) The composition of two growths is a growth.

We will be concerned primarily with growths on skew diagrams.

Definition 3.3 If S is a skew diagram (i.e., a finite convex subset of $\mathbf{Z}^2$), then a growth on S is called *two-dimensional.* For a skew diagram S we define the *upper and lower boundaries* of S by

$$\partial^+(S) = \{(x, y) \in S : (x + 1, y + 1) \notin S\}$$

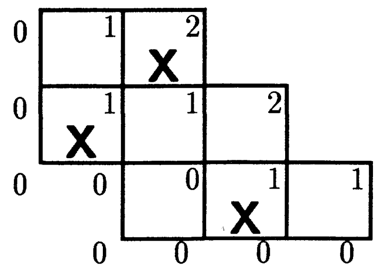
$$\partial^-(S) = \{(x, y) \in S : (x - 1, y - 1) \notin S\}$$

There are two important caveats associated with this seemingly innocuous definition. First, although a skew diagram is the same thing as a skew shape, we will view our skew diagrams with the usual coordinate geometry orientation rather than the matrix one, i.e., upside down. Second, we view the preimage of the growth as the *vertices* of the diagram, rather than the cells. We will reserve the cells for other (related) uses. Since we will be using skew shapes and skew diagrams in completely different ways, no confusion should result.

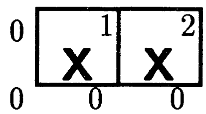
Example 3.4 In the skew diagram below, the growth $g : S \mapsto \mathbf{Z}$ is given by

$$g(x) = \#\text{cells marked with an X which are below and to the left of } v.$$

The image of g on a vertex v of S is given just slightly below and to the left of v.



The above example reflects a general class of two-dimensional growths; the cells marked with an X cannot share a row or column of S since then we could find a pair of adjacent vertices whose g values differed by at least two. A prototypical example is



where the right edge of the diagram is disallowed in the definition of "growth".

Definition 3.5 A *generalized permutation* is a finite set of cells in the skew diagram S which do not share any row or column. Other commonly used terms include *nontaking rook placement* or *permutation with restricted positions.*

We can turn any generalized permutation σ on the cells of a skew diagram S into a poset in a natural way as follows. If the coordinates of the cells are given by (i_1, j_1) and (i_2, j_2), then

$$(i_1, j_1) \leq (i_2, j_2) \iff i_1 \leq i_2 \quad \text{and} \quad j_1 \leq j_2.$$

In other words, a cell is bigger if it is above and to the right of another in S. Next, we turn a poset into a partition via the following theorem, due independently to C. Greene [Gre1] and Fomin [Fom1].

Theorem 3.6 Let P be any finite poset. For k a positive integer, set $c_k(P)$ (resp. $a_k(P)$) to be the size of the largest number of elements which is the union of k chains (resp. antichains) of P. Now, let $\lambda_k(P) = c_k(P) - c_{k-1}(P)$ and $\mu_k(P) = a_k(P) - a_{k-1}(P)$. Then $\lambda(P) = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ and $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ are *partitions*, and μ is the conjugate of λ.

The interested reader may find a proof of this in either of the references cited above. Two examples of posets with eight elements and their corresponding partitions are given in Figure 1.

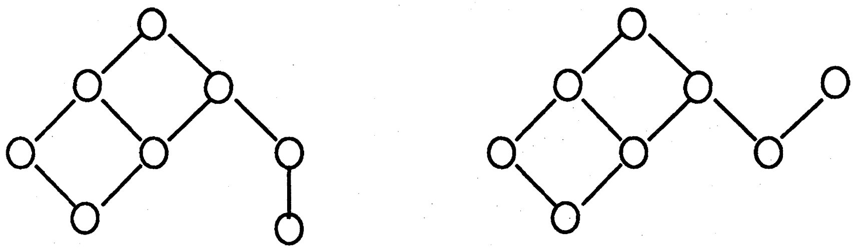


Figure 1: Two posets and their partitions

Definition 3.7 If σ is a generalized permutation of the cells of a skew diagram S, then we let $\lambda(\sigma)$ be the partition given by the above theorem applied to σ (regarded as a poset in the way described above).

We can extend the above example in a natural way to obtain a two-dimensional growth on Sq_n.

Example 3.8 In the above example we take define a growth g as follows. For each vertex of v of S, let $\mathcal{C}(v)$ denote the set of cells of S below and to the left of v. If we restrict out attention to those cells which are in π and take the poset corresponding to this generalized permutation, then we get a map

$$v \mapsto \lambda(\mathcal{C}(v) \cap \pi)$$

which is a two dimensional growth $g : S \to \mathbf{Y}$:

$\emptyset$	1	2	21	22	32	321	3211
		X					
$\emptyset$	1	1	11	21	31	32	321
					X		
$\emptyset$	1	1	11	21	21	22	221
				X			
$\emptyset$	1	1	11	11	11	21	22
						X	
$\emptyset$	1	1	11	11	11	11	21
							X
$\emptyset$	1	1	11	11	11	11	11
	X						
$\emptyset$	$\emptyset$	$\emptyset$	1	1	1	1	1
			X				
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

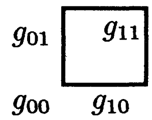


Figure 2: Close up of the growth around one cell

What is particularly interesting about this growth is to consider its restriction to the upper boundary $\partial^+(Sq_n)$. The top edge T and the right edge R can each be interpreted as a saturated chain in $\mathbf{Y}$ from $\emptyset$ to $(3,2,1,1)$, i.e., as a standard tableau. We have

$$R = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array} \qquad T = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}$$

This is the same pair of tableaux we would have obtained by applying the usual insertion algorithm to π.

4 Differential Posets

So far we have had no indication that the preceding map might be a bijection, just that we can obtain a growth on the upper boundary (equivalently a pair of Young tableaux) from a permutation. But now we notice that if we know the value of the growth at all but the lower left vertex, then it is determined by the value at the other three vertices. Precisely, g_{00} is determined by g_{11}, g_{10}, and g_{01}; see Figure 2. For Young's lattice this works as follows. If $g_{01} \neq g_{10}$, then because Young's lattice is modular, we must have $g_{00} = g_{01} \cap g_{10}$ (and also $g_{11} = g_{01} \cup g_{10}$). Otherwise, if $g_{11} \backslash g_{01}$ is a box in row i, then set g_{00} so that $g_{01} \backslash g_{00}$ is a box in row $i-1$. When $i = 1$, we let $g_{01} = g_{00}$ and we mark an X in the cell defined by the four vertices.

In fact, the above way of viewing the usual Robinson-Schensted correspondence, leads one to wonder what properties of Young's lattice were essential. It turns out that the following axioms on a partially ordered set allow one to define nice analogues of R-S. In Fomin's original paper "differential posets" are called "Y graphs", but here we follow Stanley's terminology.

In general if P is any graded poset, then we let ρ denote its rank function, i.e., if $x \in P$ then $\rho(x)$ is the length l of the longest chain $x_0 < x_1 < \cdots < x_l = x$ in P with top element x. Write

$$P_i = \{x \in P : \rho(x) = i\}$$

So $P = P_0 \uplus P_1 \uplus P_2 \uplus \cdots$ (disjoint union).

Definition 4.1 Let r be a positive integer. A poset P is called **r-differential** if it satisfies the following three conditions:

(D1) P is locally finite, graded, and has a $\hat{0}$ element.

(D2) If $x \neq y$ in P and there are exactly k elements of P which are covered by both x and y, then there are exactly k elements of P which cover both x and y.

(D3) If $x \in P$ and x covers exactly k elements of P, then x is covered by exactly $k + r$ elements of P.

When $r = 1$, we will sometimes omit the r in r-differential.

Property (D2) is essentially a modularity condition. For a lattice L satisfying (D1), condition (D2) is equivalent to L being modular. Young's lattice is the prototypical example, but there is also a class of Fibonacci Posets, and a few less interesting examples. But much of the theory can also be carried out for so called "**r**-differential posets", where the value of r is allowed to vary at each level of the poset. This allows one to expand considerably the class of exmaples. See [Rob] and the references for more information.

It turns out that differential posets have exactly the right structure to extend growths nicely from either boundary of a skew diagram. In order to do this we first look at what can happen locally as we attempt to extend a growth up or down.

Definition 4.2 A two-dimensional growth $g : S \mapsto \mathbb{Z}$ is called *semimodular* if the values of g on the vertices of any cell, say g_{00}, g_{01}, g_{10}, g_{11} (see Figure 2) satisfy the following inequality:

$$g_{00} + g_{11} \geq g_{01} + g_{10} \tag{1}$$

When the inequality is strict, we call the cell in question an *atom* of g.

Let $g : S \to P$ be a growth in any differential poset P. Consider the value of g at the four vertices of a cell as in Figure 2. If we attempt to compute the value of g_{00} given the other three vertices, we have several cases:

Case 1: $|g_{01}| < |g_{10}|$. This forces $g_{00} = g_{01}$ and $g_{11} = g_{10}$ by the definition of growth.

Case 2: $|g_{10}| < |g_{01}|$. This forces $g_{00} = g_{10}$ and $g_{11} = g_{01}$ as in case 1.

Case 3: $|g_{01}| = |g_{10}|$ but $g_{01} \neq g_{10}$. By modularity (D2) of P, g_{11} determines g_{00} and vice-versa.

Case 4: $g_{01} = g_{10} := x \in P$. This is the interesting case. Set $C^+(x) = \{y \in P : x \lessdot y\}$ and $C^-(x) = \{y \in P : y \lessdot x\}$. By axiom (D3) $C^+(x)$ has one more element than $C^-(x)$, so there is not a bijection between these sets. But there almost is. In fact, we can construct a bijection Φ_x between $C^+(x)$ and $C^-(x) \cup \{x\}$. Now set $\Phi_x(x) = x$. If we are extending down from g_{11} we set $g_{00} = \Phi_x(g_{11})$. This is well-defined since g_{11} must be something which covers or equals x. In fact, the growth will be semimodular at this cell; for in the generic case, $|g_{00}| + 1 = |g_{01}| = |g_{11}| - 1$. If $g_{11} = x$, then all four vertices will have the same value. Finally, if $x \lessdot g_{11}$ and $\Phi_x(g_{11}) = x$, then we have

$$|g_{00}| + |g_{11}| > |g_{01}| + |g_{10}|$$

and the cell is an atom of the semimodular growth $|g|$.

To extend a growth upwards we will examine the same four cases. The first three are the same. In case 4 we find that our correspondence Φ gives a well-defined answer as long as $g_{00} \neq x$. If $g_{00} = x$, then $\Phi_x^{-1}(g_{00})$ could be either x or a certain element of $C^+(x)$. To decide between these we need an extra piece of information, which is whether or not the cell in question contains an X, i.e., whether it should be an atom of the semimodular growth $|g|$.

Definition 4.3 Let P be a differential poset. At each $x \in P$ define a correspondence Φ as in case 4 above. We call the collection $\Phi := \{\Phi_x : x \in P\}$ an *R-correspondence.* A growth $g : S \to P$ is said to be *compatible* with the R-correspondence if at each cell such that $g_{01} = g_{10}$, we have $g_{00} = \Phi_{g_{01}}(g_{11})$. We call such a g a *Φ-growth.*

An R-correspondence is defined in a purely local way, and a differential poset will have many different ones.

The main result of Fomin is proved using the algorithmic method above:

Theorem 4.4 (Fomin) Let P be a differential poset and S any skew diagram. Fix an R-correspondence Φ on S. We have a bijection between growths $g^+ : \partial^+(S) \mapsto P$ and pairs (g^-, σ) where $g^- : \partial^-(S) \mapsto P$, and σ is a generalized permutation on S. Each of these is also associated with a uniquely defined two-dimensional Φ-growth $g : S \mapsto P$, whose restriction to $\partial^+(S)$ and $\partial^-(S)$ is g^+ and g^-, respectively, and whose modulus $|g|$ is the semimodular growth whose atoms are σ.

5 Skew tableaux

Definition 5.1 A **biword,** π, is a sequence of vertical pairs of positive integers

$$\pi = \begin{matrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{matrix}$$

with $i_1 \leq i_2 \leq \dots \leq i_k$. We denote the top and bottom lines of π by $\hat{\pi} = i_1 i_2 \dots i_k$ and $\check{\pi} = j_1 j_2 \dots j_k$. **Partial permutations of n** have no entries greater than n, and within each line the entries are distinct. A tableaux of shape λ/μ is called *partial* if its elements are distinct (but not necessarily the numbers $1, 2, \dots, n$). Let $PT(\lambda/\mu)$ denote the set of all partial tableaux of shape λ/μ.

The basic result of [SS] follows. In the following, "$\uplus$" denotes "disjoint union".

Theorem 5.2 Let n be a fixed positive integer and α a fixed partition (not necessarily of n). Then there is a map

$$(\pi, T, U) \longleftrightarrow (P, Q)$$

defined below which is a bijection between $\pi \in PS_n$ with $T, U \in PT(\alpha/\mu)$ such that $\check{\pi} \uplus T = \hat{\pi} \uplus U = \{1, 2, \dots, n\}$, on the one hand, and $P, Q \in ST(\lambda/\alpha)$ such that $\lambda/\alpha \vdash n$, on the other.

Example 5.3 This is the same example Sagan and Stanley give in their paper [SS, p. 165], but we reinterpret it using Fomin's approach. Let $n = 5$, $\alpha = (2, 2, 1)$, $\pi = \begin{matrix} 1 & 2 & 4 \\ 4 & 2 & 3 \end{matrix}$,

$$T = \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & 5 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \quad \text{and} \quad U = \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & 3 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Then we get

$$P = \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & 2 & 3 \\ \hline \cdot & \cdot & \multicolumn{2}{c}{} \\ \cline{1-2} \cdot & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} 5 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \text{and} \quad Q = \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & 1 & 4 \\ \hline \cdot & \cdot & \multicolumn{2}{c}{} \\ \cline{1-2} \cdot & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} 5 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}.$$

In the picture below, P is the right edge Q the upper edge, T the left edge, and U the lower edge. The partial permutation is represented by X's, as usual.

221	321	322	3221	4221	42211
211	311 X	321	322	422	4221
211	211	311	321	421 X	4211
211	211	311 X	321	321	3211
211	211	211	221	221	2211
21	21	21	22	22	221

Proof: All the hard work has already been done, and this is just a simple application of Theorem 4.4. The skew tableaux P and Q represent the growth on the upper boundary, while T and U represent the lower boundary and π represents the atoms of the semi-modular growth $|g|$. The condition that $\check{\pi} \uplus T = \hat{\pi} \uplus U = \{1, 2, \ldots, n\}$ insures that the tableaux on the upper boundary are standard, and vice-versa. □

The enumerative corollary of the above is

Corollary 5.4 Let n be a fixed positive integer and α be a fixed partition. Then

$$\sum_{\lambda/\alpha \vdash n} f^2_{\lambda/\alpha} = \sum_{k=0}^{n} \binom{n}{k}^2 k! \sum_{\alpha/\mu \vdash n-k} f^2_{\alpha/\mu}.$$

These results reduce to the original Robinson-Schensted results when $\alpha = \emptyset$. It turns out that this algorithm also enjoys the property that inverting the permutation (and interchanging T and U) is equivalent to interchanging P and Q. The original proof was somewhat lengthy, but from our point of view it's almost obvious—just transpose the entire diagram.

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