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Flag-transitive extended dual polar spaces

Satoshi Yoshiara 吉荒聡
Department of Information Science
Faculty of Science
Hirosaki University 3 Bunkyo-Cho, Hirosaki
Aomori 036 JAPAN

Abstract

In this report, some recent progress of classification of flag-transitive exteded dual polar spaces (FEDP) is described, as well as an announcement of existence of three new non-classical FEDPs of rank 3.

1 FEDPs.

1.1 Terminology.

We review some standard terminology of incidence geometries (see e.g. [9] p. 2–3, [13], [8]). An (*incidence*) *geometry* $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \cdots, \mathcal{G}_{r-1}; *)$ on an ordered set $I = \{0, \ldots, r-1\}$ is an ordered sequence of r pairwise disjoint non-empty sets $\mathcal{G}_i$ $(i = 0, \ldots, r-1)$ together with a symmetric, reflexive relation $*$ (called an *incidence*) on their union $\Gamma := \mathcal{G}_0 \cup \mathcal{G}_1 \cdots \cup \mathcal{G}_{r-1}$ such that if F is any maximal subset of Γ satisfying $x * y$ for any $x, y \in F$ (called a *maximal flag*), then $|F \cap \mathcal{G}_i| = 1$ for any $i \in I$.

For a *flag* F of $\mathcal{G}$ (i.e. a subset of Γ of mutually incident elements), the subset $\{i \in I | \mathcal{G}_i \cap F \neq \emptyset\}$ (resp. its complement in I) with the induced order is called the *type* (resp. *cotype*) of F and denoted by $typ(F)$ (resp. $coty(F)$). If F is not maximal, the subsets $\mathcal{G}_i(F) := \{x \in \mathcal{G}_i | x * y (\forall y \in F)\}$ for $i \in coty(F)$ form a geometry $\mathcal{G}_F$ on $coty(F)$ by restricting the incidence relation $*$ onto

$\Gamma(F) := \cup_{i \in coty(F)} \mathcal{G}_i(F)$. This incidence geometry $\mathcal{G}_F$ is called the *residue* of $\mathcal{G}$ at a flag F.

With an incidence geometry $\mathcal{G}$ we associated a graph on Γ, called the *incidence* graph of $\mathcal{G}$, by declaring that two elements x and y of Γ are joined whenever $x * y$. If the incidence graph is connected, the geometry is called *connected.* A connected geometry $\mathcal{G}$ is called *residually connected*, if the induced graph on $\Gamma(F)$ is connected for each non-maximal flag F.

A (special) automorphism of a geometry $\mathcal{G}$ is a bijection on Γ preserving each $\mathcal{G}_i$ $(i \in I)$ and compatible with the incidence $*$. A group G of automorphisms of a geometry $\mathcal{G}$ is called *flag-transitive* if G acts transitively on the set of maximal flags of $\mathcal{G}$. A geometry $\mathcal{G}$ is called *flag-transitive* if the full automorphism group $Aut(\mathcal{G})$ is flag-transitive. Note that in a flag-transitive geometry $\mathcal{G}$, the residues at flags F and F' are isomorphic if $type(F) = type(F')$. Thus the structure of residues is determined only by their types, and so sometimes we simply use the word *J-residue* to call a residue $\mathcal{G}_F$ with $type(F) = J$ $(j \subseteq I)$. For a flag-transitive automorphism group G of a geometry $\mathcal{G}$ and a flag F of $\mathcal{G}$, the *stabilizer* of F (i.e. the subgroup of G of elements fixing any x of F) is denoted by G_F. The *kernel* of G at F is the normal subgroup of G_F fixing all elements of $\Gamma(F)$, and denoted by K_F. The group G_F/K_F ats faithfully on the residue $\mathcal{G}_F$. If $F = \{x\}$, we simply write the stabilizer and the kernel by G_x and K_x, respectively.

1.2 FEDPs and FEQs

An *extended dual polar space* (abbreviated to EDP) is a residually connected incidence geometry $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \cdots, \mathcal{G}_r; *)$ on $I = \{0, \ldots, r\}$ $(2 \leq r)$ if the residue $\mathcal{G}_F$ at a flag F with $|coty(F)| = 2$ satisfies the following conditions. (See [9] p.1 and p.3 for generalized n-gons and a circle geometry. We call (s, t) in [9] p.1 the *order* of a generalized polygons, instead parameters):

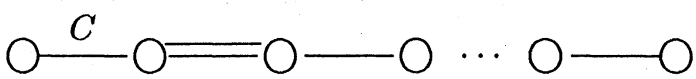
(0) If $coty(F) = \{0, 1\}$, the residue $\mathcal{G}_F$ is a *circle geometry* (i.e. there are bijections ρ_0 and ρ_1 from $\mathcal{G}_0(F)$ and $\mathcal{G}_1(F)$ onto the sets of vertices and edges of a complete graph, respectively, such that $x_0 * x_1$ $(x_i \in \mathcal{G}_i(F)$, $i = 0, 1)$ iff $\rho_0(x_0)$ is a vertex on an edge $\rho_1(x_1)$).

(1) If $coty(F) = \{1, 2\}$, the residue $\mathcal{G}_F$ is a *generalized quadrangle* (i.e. the incidence graph of $\mathcal{G}_F$ is of diameter 4 and girth 8).

(i) If $coty(F) = \{i, i+1\}$ with $3 \le i \le r-1$, the residue $\mathcal{G}_F$ is a *projective plane* (i.e. the incidence graph of $\mathcal{G}_F$ is of diameter 3 and girth 6).

(ij) Otherwise, the residue $\mathcal{G}_F$ is a *generalized digon* (i.e. $x*y$ for any $x \in \mathcal{G}_i(F)$ and $y \in \mathcal{G}_j(F)$ for $i < j$ with $coty(F) = \{i, j\}$).

An EDP is nothing more than an incidence geometry belonging to the following diagram (see e.g. [9] p.3 for the formal definition of a diagram).



Elements of $\mathcal{G}_i$ are called *points, lines* and *planes*, respectively for $i = 0, 1$ and 2. We abbreviate a flag-transitive EDP to an FEDP. An FEDP is called *linear* if we may identify a line with the two points incident with it, that is, there is at most one line incident with two distinct points. A linear FEDP of rank 3 is called an FEQ (flag-transitive extended generalized quadrangle) [2].

An EDP $\mathcal{G} = (\mathcal{G}_0, \ldots, \mathcal{G}_r; *)$ of rank $r+1$ is called *classical* if its 0-residue $(\mathcal{G}_1(P), \ldots, \mathcal{G}_{r-1}(P); *)$ at each point P is a dual polar space for a classical geometry: That is, if there is a vector space V and a non-degenerate form f of Witt index r on V listed in the table below such that $\mathcal{G}_i(P)$ is the set of totally isotropic (or singular) subspaces of V of projective dimension $r-i$ $(i = 1, \ldots, r)$ and $*$ is given by inclusion.

Symbol	Space V	Form f
$C_r(q)$	$\mathbf{F}_q^{2r}$	Symplectic
$B_r(q)$	$\mathbf{F}_q^{2r+1}$	Quadratic
$D_r(q)$	$\mathbf{F}_q^{2r}$	Quadratic (+ type)
${}^2D_{r+1}(q)$	$\mathbf{F}_q^{2(r+1)}$	Quadratic (− type)
${}^2A_{2r-1}(q^2)$	$\mathbf{F}_{q^2}^{2r}$	Hermitian
${}^2A_{2r}(q^2)$	$\mathbf{F}_{q^2}^{2r+1}$	Hermitian

In particular, classical GQs (EDP of rank $r = 2$) consist of the following five families: the GQ $W(q) = C_2(q)$ of order (q,q) and its dual $Q(4,q) = B_2(q)$ admitting the simple group $S_4(q) \cong O_5(q)$, the GQ $Q^-(5,q) = {}^2D_2(q)$ of order (q,q^2) and its dual $H(3,q^2) = {}^2A_3(q^2)$ admitting the simple group $O_6^-(q) \cong U_4(q^2)$, and the GQ $H(4,q^2) = {}^2A_4(q^2)$ of order (q^2,q^3) admitting the simple group $U_5(q^2)$ (see [7] 3.1.1 p.36). (Since the GQ $D_2(q)$ is of order $(q,1)$, which is not *thick*, we usually remove this GQ from our list of classical GQs.)

By joining works by Tits, Brouwer and Aschbacher (see [1]), flag-transitive polar spaces of rank ≥ 3 are either classical or of rank 3 (and non-classical). There is a unique known example of a flag-transitive non-classical polar space of rank. It is called the *sporadic A_7-geometry*, which has 7 points, 35 lines and 15 planes and the full automorphism group A_7. It is conjectured that a non-classical flag-transitive polar space of rank 3 is isomorphic to the sporadic A_7-geometry, but so far no proof exists [1]. It seems to me that the following conjecture is much more easy to prove.

Conjecture. *If $\mathcal{G}$ is a non-classical FEDP of rank 4, 0-residues are isomorphic to the A_7-geometry.*

Thus, assuming the above conjecture is true, one of the following occurs for an FEDP $\mathcal{G}$:

(1) $\mathcal{G}$ is of rank 3 and classical.

(2) $\mathcal{G}$ is of rank 3 and non-classical.

(3) $\mathcal{G}$ is of rank 4 with point-residues isomorphic to the sporadic A_7-geometry.

(4) $\mathcal{G}$ is of rank ≥ 4 and classical.

2 Classification.

2.1 Cases (1) and (3).

The classical FEDP of rank 3 are completely classified and all of them turn out to be FEQs (see [10],[12], [8] for the precise results and terminology). There are 13 isomorphism classes of such geometries, including one with full automorphism group $HS.2$ found by the author [11].

In the table below, we summarize the fundamental information of these 13 isomorphism classes of FEQs. In the table, G is the full automorphism group

[1] In my talks in Kyoto and Matsuyama (Oct.1991, [16] p.105 line -8), I mistakenly stated that the classification has completed, but it is not true. If $\mathcal{G}$ is a non-classical flag-transitive polar space of rank 3 and not isomorphic to the sporadic A_7-geometry, it is known that $\{0,1\}$-residues are non-Desarguesian projective planes of order satisfying many strong (and strange) conditions. See e.g. [5]

of $\mathcal{G}$, v and c are the number of points and planes, respectively, and (s,t) is the order of the GQ $\mathcal{G}_P$ for a point P. We use the notation in §1.2 to denote the classical GQs. We set $k := s+2$, the number of points on a circle. We also set $X_P := G_P/K_P$ and $X_C := G_C/K_C$ for a point P and a circle C, where G_x and K_x $(x = P, C)$ denote the stabilizer and the kernel in G of x (see 1.1). The symbol d means the diameter of the point-line graph of $\mathcal{G}$, defined on the set of points by declaring that two distinct points form an edge whenever they are incident with a line.

$\mathcal{G}$	G	v	c	X_P	$\mathcal{G}_P$	(s,t)	X_C	k	d
$\mathcal{A}_\infty$	$2^5:S_6$	32	120	S_6	$W(2)$	$(2,2)$	S_4	4	3
$\overline{\mathcal{A}}$	$2^4:S_6$	16	60	S_6	$W(2)$	$(2,2)$	S_4	4	1
$\mathcal{A}_+$	S_8	28	105	S_6	$W(2)$	$(2,2)$	S_4	4	2
$\mathcal{A}_-$	$U_4(2).2$	36	135	S_6	$W(2)$	$(2,2)$	S_4	4	2
$\mathcal{K}^+$	$2^6.U_4(2).2$	64	720	$U_4(2).2$	$Q_5^-(2)$	$(2,4)$	S_4	4	2
$\mathcal{K}^-$	$S_6(2)\times 2$	56	630	$U_4(2).2$	$Q_5^-(2)$	$(2,4)$	S_4	4	3
$\overline{\mathcal{K}}$	$S_6(2)$	28	315	$U_4(2).2$	$Q_5^-(2)$	$(2,4)$	S_4	4	1
$\mathcal{M}$	$McL.2$	275	15400	$U_4(3).2$	$Q_5^-(3)$	$(3,9)$	S_5	5	2
$\mathcal{O}$	$3.O_6^-(3).2^2$	378	1701	$U_4(2).2$	$H_3(4)$	$(4,2)$	S_6	6	4
$\overline{\mathcal{O}}$	$O_6^-(3).2^2$	126	567	$U_4(2).2$	$H_3(4)$	$(4,2)$	S_6	6	2
$\mathcal{U}$	$U_5(2).2$	176	1408	$S_4(3).2$	$W(3)$	$(3,3)$	S_5	5	2
$\mathcal{S}$	$Suz.2$	22880	232960	$U_4(3).2$	$H_3(9)$	$(9,3)$	M_{11}	11	4
$\mathcal{Y}$	$HS.2$	1100	11200	$L_3(4).2^2$	$H_3(9)$	$(9,3)$	M_{11}	11	2

As for the case (3) in the last section, the following result was proved by the author [15].

Theorem 2.1 *There is a unique isomorphism class of FEDPs of rank* 4 *with* 0*-residues isomorphic to the sporadic* A_7*-geometry. It is the one point extension of the sporadic* A_7*-geometry with the full automorphism group* $2^4 : A_7$.

2.2 Case (4).

As for the case (4), we first consider FEDPs of rank 4. Note that possible isomorphism types of $\{0,3\}$-residues are restricted to $W(2)$, $Q_5^-(2)$, $Q_4(3)$, $W(3)$,

$H_3(2^2)$ and $H_3(3^2)$, since the point-residue is isomorphic to one of the 13 classes of EGQs above.

FEDPs with $\{0,3\}$-residues $Q_5^-(2)$ are classified by the author [14]. T. Meixener [6] also characterized the geometry below for $Co.2\times 2$ as an FEDP satisfying an additional assumption.

Theorem 2.2 *Let $\mathcal{G}$ be a simply connected FEDP of rank 4 with $\{0,3\}$-residues the GQ $Q_5^-(2)$. Then one of the following holds.*

(1) *$\mathcal{G}$ is a geometry on 6300 points with the full automorphism group $Co.2\times 2$.*

(2) *There is a normal subgroup N of $Aut(\mathcal{G})$ with $Aut(\mathcal{G})/N \cong U_6(2).2$.*

As for FEDPs with $\{0,3\}$-residues $W(2)$, the author proved the following result [14], [17].

Theorem 2.3 *Let $\mathcal{G}$ be a simply connected classical FEDP of rank 4 with $\{0,3\}$-residues the GQ $W(2)$, admitting a flag-transitive group G. Then the kernel K_P of the action of the stabilizer G_P of a point P on the residue $\mathcal{G}_P$ at P is either trivial or the natural module for $S_6(2)$ or $O_7(2)$. Furthermore,*

(1) *If $K_P = 1$, then $\mathcal{G}$ is either a geometry on 2^{16} points with $Aut(\mathcal{G}) \cong 2(2^6 \times 2_+^{1+8})S_6(2)$, or a geometry on 32640 points with $Aut(\mathcal{G}) \cong S_8(2)$.*

(2) *If $K_P \cong 2^6$, then we get two possible sets of relations presenting G, one of which contains a normal subgroup N with $G/N \cong F_{22}$ or $F_{22}.2$.*

Two new FEDPs of rank 4 admitting F_{24} and F_{22} are constructed by the author [17]. The latter is a subgeometry of the former, and the $\{0,3\}$-residues of the former (resp. the latter) is isomorphic to $H_3(2^2)$ (resp. $W(2)$).

The former geometry is constructed as follows: Take a maximal subgroup $O_{10}^-(2)$ of F_{24}'. One of the maximal parabolic subgroup P of $O_{10}^-(2)$ is isomorphic to $2^8 : O_8^-(2)$, in which $O_2(P) \cong 2^8$ consists of the identity and $2B$-involutions (products of four mutually commuting 3-transpositions in F_{24}). Furhtermore, there is a non-singular quadratic form q of negative type on $O_2(P)$ preserved by a complement $O_8^-(2)$. Let $E_3 \subset E_2 \subset E_1$ be a chain of isotropic subspaces of $E_0 = O_2(P)$ with respect to q of dimension $\dim E_3 = 1$, $\dim E_2 = 2$ and $\dim E_2 = 3$. We let $\mathcal{G}_i$ be the conjugacy class of E_i in F_{24}' $(i = 0,\ldots,3)$, and define an incidence by inclusion. Then we may verify that the resulting geometry

$\mathcal{G} = (\mathcal{G}_0, \ldots, \mathcal{G}_3; *)$ is an FEDP with point residues the dual polar space ${}^2D_4(2)$ for $O_8^-(2)$, admitting a flag-transitive group F_{24}.

It is easy to show that there is no FEDP with $\{0,3\}$-residues $Q_4(3)$ or with point residues the EGQ for $HS.2$.

However, as for the case when the $\{0,3\}$-residues are isomorphic to $H_3(2^2)$, $W(3)$ or $H_3(3^2)$, there is no classification so far. There are known examples of FEDPs admitting F_{24}, F_{24} and M for those with the $\{0,3\}$-residues $H_3(2^2)$, $W(3)$ and $H_3(3^2)$, respectively.

The classification in these cases should be most interesting, but may require some new methods. Because the target geometries have too many points and ranks, not only to handle by hand but also for computers with an ordinary storage capacity at the present time.

As for classical FEDPs of rank greater than 4, nothing is known. However, T. Meixner and the author conjecture that they can be constructed as subgeometries of some (possibly infinite) buildings.

2.3 Case (2).

How about the remaining case (2) ? Unfortunately, flag-transitive GQs have not yet been classified. Among thick GQs, there are four known flag-transitive GQs except the classical GQs and the duals of $H(4,q^2)$ for prime powers q (see [4] p.98, Summary). They are $T_2^*(O_q)$ for some oval O_q in the projective plane $\mathbf{PG}(2,q)$ for $q = 4$ (of order $(3,5)$) and 16 (of order $(15,17)$) and their duals (see [7] 3.1.3 p.38 for $T_2^*(O)$), where a (hyper) *oval* means a set of $q+2$ projective points such that no three points lie on a line in common.

By the argument used in Lemma 12 in [12], it is easy to verify that there is no FEQ with 0-residues the dual of $H(4,q^2)$ for any q. However, there are new FEQs with point-residues $T_2^*(O_4)$ and its dual, which are characterized as follows in [15]. Note that the full automorphism group of $T_2^*(O_4)$ is isomorphic to $2^6 3S_6$, in which 2^6 acts regularly on the set of points of $T_2^*(O_4)$.

Theorem 2.4 *Up to isomorphism, there is a unique simply connected FEQ with point residues isomorphic to $T_2^*(O_4)$, admitting an automorphism group G in which the stabilizer of a point P contains a normal subgroup inducing a regular permutation group on the lines incident with P.*

This new EGQ $\mathcal{G}$ is defined on 160 points, having 3072 planes and the full automorphism group $2_+^{1+8}\cdot(A_5 \times A_5)2$ (the extension $Aut(\mathcal{G})/2_+^{1+8}$ does not split). Taking the quotient by the unique central involution of $Aut(\mathcal{G})$, we have an FEQ on 80 points. So far the only construction of this geometry known to the author is one in terms of coset geometry.

Theorem 2.5 *Up to isomorphism, there are two simply connected FEQs with point residues isomorphic to the dual of $T_2^*(O_4)$, admitting an automorphism group G in which the stabilizer of a point P contains a normal subgroup inducing a regular permutation group on the lines incident with P.*

One of these new FEQs (denoted by $\mathcal{G}^{(0)}$) is defined on 896 points, having 8192 planes and the full automorphism group $2_+^{1+12} : 3S_7$. The other FEQ (denoted by $\mathcal{G}^{(1)}$) is defined on 448 points, having 4096 planes and the full automorphism group $2^{6+6} : L_3(2)$.

So far the only construction of the geometry $\mathcal{G}^{(1)}$ known to the author is one in terms of coset geometry. An explicit construction of $\mathcal{G}^{(0)}$ was given in [16] §3 in terms of isotropic 1, 2, 4-spaces of an 8-dimensional unitary space over $\mathbf{F}_4$ (for the detail, [15] 5.4). Taking the quotient by the unique central involution of $Aut(\mathcal{G}^{(0)})$, we have an FEQ on 448 points.

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