A P-Complete Language Describable with Iterated Shuffle

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Abstract
We show that a P-complete language can be described as a single expression with the shuffle operator, shuffle closure, union, concatenation, Kleene star and intersection on a finite alphabet.

1 Introduction

In this paper, we construct a P-complete language by using shuffle operator $\triangle$, iterated shuffle $\dagger$, union $\cup$, concatenation $\cdot$, Kleene star $*$ and intersection $\cap$ over a finite alphabet. The shuffle operator was introduced by [10] to describe the class of flow expressions. Formal properties of expressions with these operators have been extensively studied from various points in the literatures [2, 3, 4, 5, 8, 9, 10, 11].

It is known that the complexity of almost classes of languages can be increased by using the iterated shuffle operator. For example, there are two deterministic context-free languages $L_1$ and $L_2$ such that $L_1 \triangle L_2$ is NP-complete [9]. Moreover, by allowing the synchronization mechanisms, any recursively enumerable set can be described [1, 3].

In [2, 11], by using the shuffle and iterated shuffle operators together with $\cup, \cdot, *, \cap$, an NP-complete language is described. We employ the same set of operators to describe our P-complete language. In the proof of P-completeness, the intersection operator plays an important role to make the language polynomial-time recognizable. However, we do not know whether the intersection operator is necessary to define a P-complete language as in the case with NP-complete [2, 11].

Recently, P-complete problems have received considerable attentions since they do not seem to allow any efficient parallel algorithms [7]. This paper gives a P-complete problem of a new kind, which is described by a single expression.
2 Preliminaries

Let $\Sigma$ be a finite alphabet and $\Sigma^*$ be $\{a_1 \cdots a_n \mid a_i \in \Sigma \text{ for } i = 1, \ldots, n \text{ and } n \geq 0\}$. A subset of $\Sigma^*$ is called a language.

**Definition 1** For languages $L$, $L_1$ and $L_2$, we define the *shuffle operator* $\triangle$, the *iterated shuffle* $\dagger$ and operators, $\cdot,*,+ \text{ as follows:}

1. $L_1 \triangle L_2 = \{x_1 y_1 x_2 y_2 \cdots x_m y_m \mid x = x_1 x_2 \cdots x_m \in L_1, y = y_1 y_2 \cdots y_m \in L_2 \text{ and } x_i, y_i \in \Sigma^* \text{ for } i = 1, \ldots, m\}$ (shuffle operator).
2. $L^\dagger = \{\epsilon\} \cup L \cup (L \triangle L) \cup (L \triangle L \triangle L) \cup \cdots$ (iterated shuffle).
3. $L_1 \cdot L_2 = \{xy \mid x \in L_1 \text{ and } y \in L_2\}$ (abbreviated to $L_1L_2$).
4. $L^* = \{\epsilon\} \cup L \cup (L \cdot L) \cup (L \cdot L \cdot L) \cdots$.
5. $L^+ = L \cdot L^*$.

We identify a language $\{w\}$ which consists of only one word with $w$. Thus, we will denote $\{w\}^*, \{w\}^+, \{w\}^\dagger, \ldots$ by $w^*, w^+, w^\dagger$, respectively.

As the basis of our reduction, we use the circuit value problem (CVP) that was shown P-complete [6]. Our definition in this paper slightly different from one in [6].

**CIRCUIT VALUE PROBLEM (CVP)**

**INSTANCE:** A circuit $C = (C_1, \ldots, C_m, C_{m+1}, \ldots, C_n)$, where each $C_i$ is either (i) $C_i = \text{true}$ or false $(1 \leq i \leq m)$, (ii) $C_i = \text{NOR}(C_j, C_k)$ $(m + 1 \leq i \leq n \text{ and } j, k < i)$.

**PROBLEM:** Decide whether the value of $C_n$ is true.

In the following section, CVP represents the set of all circuits whose output is true.

Let $\Sigma$ be a finite alphabet, $v_1, v_2, \ldots, v_m$ be symbols where $v_i \in \Sigma$ for $i = 1, \ldots, m$ and $w_1, w_2, \ldots, w_{m+1}$ be words on the alphabet $\Sigma - \{v_1, v_2, \ldots, v_m\}$. By using the iterated shuffle operator, the language $\{v_1^n v_2^n \cdots v_m^n \mid n \geq 1\}$ can be described as $(v_1 v_2 \cdots v_m)^\dagger \cap v_1^+ v_2^+ \cdots v_m^+$. Moreover, we can represent $\{w_1 v_1^n w_2 v_2^n \cdots w_m v_m^n w_{m+1} \mid n \geq 1\}$ as

$$(w_1 w_2 \cdots w_{m+1} \triangle (v_1 v_2 \cdots v_m)^\dagger) \cap w_1 v_1^+ w_2 v_2^+ \cdots w_m v_m^+ w_{m+1}.$$ 

We often use this form of languages to define our P-complete language. Whenever such languages are used in the next section, we will not describe them explicitly by using the shuffle operator and the iterated shuffle.
3 A P-complete language

The main result in this paper is the following theorem.

**Theorem 1** A P-complete language can be described with operators $\cdot,+,\cup,\cap,\triangle,\uparrow$.

3.1 Definition of the language

We will describe a P-complete language $L$ with the alphabet $\Sigma = \{0, 1, a, b, u, v, x, y, z\}$. This language is defined stepwise.

At first, a language $L$ is defined as follows:

$$L_a = a^+0 \cup a^+1 = \{a^i\beta \mid i \geq 1 \text{ and } \beta \in \{0,1\}\}.$$

$$L_{bba} = (b^+1b^+1a^+0) \cup (b^+0b^+1a^+1) \cup (b^+1b^+0a^+1) \cup (b^+0b^+0a^+1)$$

$$= \{b^i\beta'\beta''a^i\beta \mid i, j, k \geq 1 \text{ and } (\beta', \beta'', \beta) \in \{(1,1,0), (0,1,1), (1,0,1), (0,0,1)\}\}.$$  

$$L_b = b^+1 = \{b^i1 \mid i \geq 1\}.$$  

$$L = L_a^+L_{bba}^+L_b.$$  

The following language $T$ (resp. $F$) is used for a distribution of true (resp. false) value.

$$T_x = \{1zxu^i \mid i \geq 1\}, \quad T_y = \{1y^iv^i \mid i \geq 1\}.$$  

$$T_{xy} = \{1zxu^i1y^iv^i \mid i \geq 1\}, \quad T_{yy} = \{1y^iv^i1y^iv^i \mid i \geq 1\}.$$  

$$T_{odd} = T_xT_{yy}^*T_y \cap T_xT_{yy}^* = \{1zxu^i(1y^iv^i)^j \mid i \geq 1, j \geq 1 \text{ and } j \text{ is odd}\}.$$  

$$T_{even} = T_xT_{yy}^*T_y \cap T_xT_{yy}^* = \{1zxu^i(1y^iv^i)^j \mid i \geq 1, j \geq 1 \text{ and } j \text{ is even}\}.$$  

$$T = T_x \cup T_{odd} \cup T_{even} = \{1zxu^i(1y^iv^i)^j \mid i \geq 1 \text{ and } j \geq 0\}.$$  

$F$ is defined in a similar way by simply replacing a symbol with 0 in the definition of 1.

$$F = \{0zxu^i(0y^iv^i)^j \mid i \geq 1 \text{ and } j \geq 0\}.$$  

Subwords $1y^iv^i$ (resp. $0y^iv^i$) of a word in $T$ (resp. $F$) are combined with $b^i0$ (resp. $b^i1$) of words in $L$ and determines the value of the $i$th variable. These three languages $L, T$ and $F$ are combined one another by using the shuffle operator and the iterated shuffle.

$$\mathcal{J} = L \triangle (T \cup F)^\dagger.$$
A language $\mathcal{K}$ is used to make our language $\mathcal{L}$ polynomial time decidable. We construct the language $\mathcal{K}$ stepwise as follows:

\[ A_{11} = \{a^i1zx^i | i \geq 1\}. \]
\[ A_{00} = \{a^00zx^i | i \geq 1\}. \]
\[ A_{01} = \{a^01zx^i | i \geq 1\}. \]

In a similar way, the following languages are defined:

\[ B_{01} = \{b^i01v^i | i \geq 1\}. \]
\[ B_{11} = \{b^i11v^i | i \geq 1\}. \]

\[ M = (A_{11} \cup A_{00})^+(B_{01}B_{01}A_{01})^+B_{11}. \]

The language $M$ contains a word $w$ in which $zx^i u^i$ occurs more than once in $w$ for some $i$, where $zx^i u^i$ corresponds to the $i$th gate. We will remove such words $w$ from $M$ so that each $zx^i u^i$ occurs exactly once for all $1 \leq i \leq n$.

\[ N_{s} = (zxuzx^2) \triangle (xuxu) \cap (zx^+u^+zx^+u^+) = \{zx^i u^i | i \geq 1\}. \]

\[ N_{odd} = zxux^* \cap N_{s}^* = \{zxuzx^2 \cdots zx^i u^i | i \geq 1 \text{ and } i \text{ is odd}\}. \]
\[ N_{even} = zxux^* \cap N_{s}^* = \{zxuzx^2 \cdots zx^i u^i | i \geq 1 \text{ and } i \text{ is even}\}. \]

\[ N = N_{odd} \cup N_{even} = \{zxuzx^2 \cdots zx^i u^i | i \geq 1\}. \]

Then, we define the language $\mathcal{K}$ which will be used for allowing a language $\mathcal{J}$ to be in $P$.

\[ \mathcal{K} = M \cap (N \Delta \Sigma^*), \text{ where } \Sigma' = \Sigma - \{u, x, z\}. \]

Finally, we defined the language $\mathcal{L}$ as follows:

\[ \mathcal{L} = \mathcal{J} \cap \mathcal{K}. \]

### 3.2 Proof of the P-completeness

Theorem 1 follows from the next lemma.

**Lemma 1** $\mathcal{L}$ is log-space equivalent to CVP, i.e., $\mathcal{L}$ is log-space reducible from CVP and CVP is log-space reducible from $\mathcal{L}$. 
$w = a^{11}zxua^{2}11zx^{2}u^{2}a^{3}00zx^{3}u^{3}b^{0}yvb^{2}01y^{2}v^{2}a^{4}01zx^{4}u^{4}b^{2}01y^{2}v^{2}b^{3}01y^{3}v^{3}a^{5}01zx^{5}u^{5}b^{4}01y^{4}v^{4}b^{5}01y^{5}v^{5}a^{6}01zx^{6}u^{6}b^{6}11y^{6}v^{6}$.

Figure 1: This above circuit is transformed to the word $w$.

Proof. We will define a function $f$ from CVP to $\Sigma^*$. $f$ is a function which transforms $C = (C_1, \ldots, C_n) \in$ CVP to $f(C) = w_1 \cdots w_n w_{n+1} \in \Sigma^*$, where

$$w_i = \begin{cases} a^{i}11zx^{i}u^{i} & (C_i = true) \\ a^{i}00zx^{i}u^{i} & (C_i = false) \\ b^{i}01y^{i}v^{i}b^{k}01y^{k}v^{k}a^{i}01zx^{i}u^{i} & (C_i = NOR(C_j, C_k)) \\ b^{n}11y^{n}v^{n} & (i = n + 1). \end{cases}$$

It is easy to see that this function is computable in log-space.

We show following two claims.

Claim 1. $f(C) \in \mathcal{L}$ for every $C \in$ CVP.

Proof. Let $w = w_1 \cdots w_m w_{m+1} \cdots w_n w_{n+1}$ be a word transformed from some $n$-gates instance $C = (C_1, \ldots, C_m, C_{m+1}, \ldots, C_n)$ where $C_i$ is an input gate for $1 \leq i \leq m$, an NOR gate for $m + 1 \leq i \leq n$ and an output of this circuit is true. Let $\beta_i = 1$ (resp. $\beta_i = 0$) if the value of $C_i$ is true (resp. false) for $1 \leq i \leq n$.

According to $B = (\beta_1, \ldots, \beta_n)$, we divide $w_i$ into two words $w_i'$ and $w_i''$ as follows:

1. For $i = 1, \ldots, m$, $w_i' = a^i \beta_i$, $w_i'' = \beta_i z x^i u^i$.
2. For $i = m + 1, \ldots, n$, $w_i' = b^i \beta_j b^k \beta_k a^i \beta_i$, $w_i'' = \beta_j y^i v^j \beta_k y^k v^k \beta_i z x^i u^i$.
3. $w_{n+1}' = b^n 1$, $w_{n+1}'' = 1 y^n v^n$. We note that $w_i'$ is in $L_{bba}$ since $C_i = NOR(C_j, C_k)$. We show following two claims.

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According to $B = (\beta_1, \ldots, \beta_n)$, we divide $w_i$ into two words $w_i'$ and $w_i''$ as follows:

1. For $i = 1, \ldots, m$, $w_i' = a^i \beta_i$, $w_i'' = \beta_i z x^i u^i$.
2. For $i = m + 1, \ldots, n$, $w_i' = b^i \beta_j b^k \beta_k a^i \beta_i$, $w_i'' = \beta_j y^i v^j \beta_k y^k v^k \beta_i z x^i u^i$.
3. $w_{n+1}' = b^n 1$, $w_{n+1}'' = 1 y^n v^n$. We note that $w_i'$ is in $L_{bba}$ since $C_i = NOR(C_j, C_k)$.
It is easy to see that a word $w' = w_1' \cdots w_{n+1}'$ is in $L = L_a^+ L_{ba}^+ L_b$. 

On the other hand, since $w'' = w_1'' \cdots w_{n+1}''$ is constructed with subwords of the form $\beta_i z x^i u^i$ or $\beta_i y^i v^i$ and for each NOR gate, input gate numbers of this gate are always smaller than its number, we can describe the word $w''$ as word in $t_1 \Delta t_2 \Delta \cdots \Delta t_n$, where $t_i = \beta_i z x^i u^i \beta_i y^i v^i \cdots \beta_i y^i v^i$. Since $t_i \in T$ or $F$ for $i = 1, \ldots, n$, \( f(C) = w_1 \cdots w_m w_{m+1} \cdots w_n w_{n+1} \) is in \( w' \Delta t_1 \Delta \cdots \Delta t_n \subset L \Delta (T \cup F) \dagger = L \). \( \square \)

Since every word $w$ of $L$ is contained in $M$, $w$ is of the form $w_1 \cdots w_m w_{m+1} \cdots w_n w_{n+1}$, where, for $i = 1, \ldots, n + 1$,

\[
\begin{align*}
    w_i = & \begin{cases} 
    a^i \beta_i z x^i u^i t_i & (1 \leq i \leq m, \beta_i \in \{0,1\}) \\
    b^i 0 1 y^i v^i t_i' & (m + 1 \leq i \leq n) \\
    b^{n+1} a^i 0 1 z x^i u^i t_i & (i = n + 1)
    \end{cases}
\end{align*}
\]

We transform a word $w \in L$ to a circuit $C = (C_1, \ldots, C_m, C_{m+1}, \ldots, C_n)$ as follows:

1. For $i = 1, \ldots, m$, if $\beta_i = 1$ then $C_i = true$ else $C_i = false$.
2. For $i = m + 1, \ldots, n$, $C_i = \text{NOR}(C_j, C_k)$ where $j = \ell_i'$ and $k = \ell_i''$.

It is easy to see that this transformation, say $g$, is a well-defined function computable in log-space.

**Claim 2.** $g(w) \in \text{CVP}$ for every $w \in L$.

**Proof.** For $w \in L$, let $w''$ be the word obtained by dropping off the contribution from $L$. Then $w''$ is in $(T \cup F) \dagger$ and has the form $c_1 c_2 \cdots c_{3n-2m+1}$ where $c_r = \beta_r z x^r u^r$ or $\beta_r y^r v^r \beta_r \in \{0,1\}$, $p_r \geq 1$ and $1 \leq r \leq 3n-2m+1$. Since $w''$ contains $n$ $z$’s, there exist $n$ words $t_1, t_2, \ldots, t_n \in L \cup F$ such that $w''$ is in $t_1 \Delta t_2 \Delta \cdots \Delta t_n \dagger$. It is easy to see that each $c_r$ $(1 \leq r \leq 3n-2m+1)$ is a subword of some $t_i$ $(1 \leq i \leq n)$. Thus, without loss of generality, we may assume that for each $i = 1, \ldots, n$, $t_i$ is of the form $\beta_i z x^i u^i \beta_i y^i v^i \cdots \beta_i y^i v^i \beta_i \in \{0,1\}$, since $w''$ is also in $N \Delta \Sigma'^*$ and for $1 \leq i \leq n$, a subword $\beta_i y^i v^i$ of $w''$ does not occur before a subword $\beta_i z x^i u^i$ of $w''$, we have $j, k < i$.

We claim that for $i = 1, \ldots, n$, $t_i \in T$ if and only if the value of $C_i$ is true. This is shown by the induction. For $i = 1, \ldots, m$, if $\beta_i = 1$, then $t_i$ must be in $T$. Thus, by definition of $g$, $C_i = true$. For $i \geq m+1$, suppose that for $j, k < i$, this claim is true. We only discuss the case of $t_j \in T$ and $t_k \in T$. By the assumption, the values of $C_j$ and $C_k$ are true. We remove contributions of $t_j$ and $t_k$ from $w_i$. The remaining word is $b^i 0 b^i 0 a^i 0 1 z x^i u^i$. Moreover, $w_i$ must have a contribution from $L_{ba}$. This contribution must be of the form $b^i 0 b^i 0 a^i 1$. Thus, the remaining word after removing this contribution is $0 z x^i u^i$. Therefore, $t_i$ must be in $F$. On the other hand, the value of $C_i = \text{NOR}(C_j, C_k)$ is false. Other case is shown in a similar way. Thus, this claim holds.

Since $t_n$ must be in $T$, the value of $C_n$ is true. Thus $g(w) \in \text{CVP}$. \( \square \)

By the discussion above, we can say that $L$ is log-space reducible to $\text{CVP}$ via $f$ and $\text{CVP}$ has a log-space reduction $g$ (inverse of $f$) from $L$. \( \square \)
References


