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A quantization of Iwasawa theory and cyclotomic extensions of Kummer fields

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Introduction

The aim of this paper is to study systematically the Iwasawa theory of Kummer p-extensions of $\mathbf{Q}$, i.e., we shall study the structure of

$$X = \mathrm{Gal}(L_\infty/\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty}))$$

as a $\mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p'}))]]$-module, where a is a rational number prime to p, L_∞ is the maximal unramified abelian p-extension of $\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})$, and $p' = p$ (if $p > 2$) $= 4$ (if $p = 2$). Let q be a topological generator of $\mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p^\infty}))$. Then the non-commutative ring

$$\Lambda_q = \mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p'}))]]$$

becomes to the Iwasawa algebra $\Lambda = \mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q}(\zeta_{p'}))]]$ under $q \to 1$, and $X/(q-1)X$ is related to $X_0 = \mathrm{Gal}(L_0/\mathbf{Q}(\zeta_{p^\infty}))$, where L_0 is the maximal unramified abelian p-extension of $\mathbf{Q}(\zeta_{p^\infty})$. Therefore, our aim can be stated as to quantize the Iwasawa theory of cyclotomic fields. Our results follow from the Iwasawa theory and the commutation relation between $q-1$ and elements of Λ_q.

First we treat general cases. We show that X is a finitely generated Λ_q-module, and that for each $n \in \mathbf{N}$, $X/(q-1)^n X$ is a finitely generated and Λ-torsion Λ-module of μ-invariant 0 whose λ-invariant satisfies asymptotically $\alpha n + \beta$ for certain integers $\alpha \geq 0$ and $\beta \geq 0$. This deduces that the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}(a^{1/p^n})$ is of μ-invariant 0, which was already known by results of Ferrero-Washington [1] and Iwasawa [2].

Next we treat special cases where Vandiver's Conjecture holds for $p \neq 2$ and $X/(q-1)X = X_0$. Then it can be shown that there exist

$$F \in \Lambda'_q = \mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q})]]$$

and $x \in X$ such that

$$F|_{q=1,\gamma=(1+p)^s} = \sum_{i=0,1,2,4,\ldots,p-3} \varepsilon_i + \sum_{i=3,5,\ldots,p-2} \varepsilon_i L_p(s,\omega^{1-i}),$$

and that $\Lambda'_q \ni \alpha \mapsto \alpha x \in X$ induces a surjective Λ'_q-homomorphism $\Lambda'_q/(\Lambda'_q \cdot F) \to X$, where $\gamma \in \mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q})$ is defined by $\gamma(\zeta_{p^n}) = {\zeta_{p^n}}^{1+p}$ ($n \in \mathbf{N}$), ω denotes the Teichmüller character, ε_i denotes the idempotent for ω^i, and $L_p(s,\omega^{1-i})$ denotes the p-adic L-function for ω^{1-i}. From this, we deduce the following inequality for the λ-invariant of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}(\zeta_p, a^{1/p^n})$:

$$\mathrm{rank}_{\mathbf{Z}_p}(X_n) \leq p^n \cdot \mathrm{rank}_{\mathbf{Z}_p}(X_0),$$

where $L_n/\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^n})$ is the maximal unramified abelian p-extension with Galois group X_n. As an analogy of the Iwasawa theory of cyclotomic fields, it seems to be interesting if there are some relations between F and the q-analogue of $L_p(s,\omega^{1-i})$ constructed by Koblitz [3].

1 Quantized Iwasawa algebra

1.1. Let p be a fixed prime number, and put $p' = p$ (if $p > 2$) and $p' = 4$ (if $p = 2$). Let Σ be the pro-p group generated by γ and q with the single relation

$$\gamma \cdot q = q^{p'+1} \cdot \gamma.$$

Let Θ be the closed subgroup of Σ generated by q, and put $\Gamma = \Sigma/\Theta$. Then Θ and Γ are isomorphic to $\mathbf{Z}_p$ with generators q and $\gamma\Theta$ respectively. Let Λ_q denote the completed group ring $\mathbf{Z}_p[[\Sigma]]$. Then by a result of Serre [4], under the correspondence $\gamma \leftrightarrow 1+T$ and $q \leftrightarrow 1+S$, Λ_q is isomorphic to, and hence is identified with, the quotient ring of $\mathbf{Z}_p[[T,S]]_{\text{n.c.}}$ (: the non-commutative power series ring over $\mathbf{Z}_p$ with variables T and S) by the single relation

$$(1.1.1) \qquad (1+T)(1+S) = (1+S)^{p'+1}(1+T),$$

which is equivalent to

$$(1.1.2) \quad TS = ST + p'S(1+T) + \sum_{i=2}^{p'+1} \binom{p'+1}{i} S^i(1+T).$$

The algebra Λ_q is a complete local ring with maximal ideal (p, T, S). Let Λ denote the Iwasawa algebra $\mathbf{Z}_p[[\Gamma]] = \mathbf{Z}_p[[T]]$. Then by putting $q = 1$, Λ_q becomes to Λ, so we call Λ_q the *quantized Iwasawa algebra.* By (1.1.2), any $\alpha \in \Lambda_q$ can be uniquely expressed as

$$\alpha = \sum_{n=0}^{\infty} S^n \alpha_n \ (\alpha_n \in \Lambda).$$

In what follows, Λ_q (resp. Λ)-modules mean topological additive groups on which Λ_q (resp. Λ) acts continuously. Since Λ_q contains Λ naturally, any Λ_q-module can be regarded as a Λ-module.

1.2. **Lemma.** *If M is a compact left Λ_q-module such that $v_1, ..., v_n \in M$ generate $M/(p,T,S)M$ over $\mathbf{F}_p$, then they generate M over Λ_q. In particular,*

$$M/(p,T,S)M = \{0\} \iff M = \{0\}.$$

Proof. One can prove this in the same way as for Lemma 13.16 of [4].

1.3. **Lemma.** *For any $\alpha \in \Lambda_q$, there exists a unique $\alpha' \in \Lambda_q$ such that $\alpha S = S\alpha'$.*

Proof. It follows from (1.1.2).

1.4. **Corollary.** *For any left Λ_q-module M and $n \in \mathbf{N}$, $S^n M$ is a left sub Λ_q-module of M.*

1.5. **Lemma.** *For any $\alpha \in \Lambda_q$, there exists a unique $\alpha' \in \Lambda_q$ such that $S\alpha = \alpha' S$. Then*

$$\min\{n \mid p \not| \alpha_{0,n}\} \;=\; \min\{n \mid p \not| \alpha'_{0,n}\},$$

where $\alpha_0 = \sum_{n=0}^{\infty} \alpha_{0,n} T^n$ ($\alpha_{0,n} \in \mathbf{Z}_p$) and $\alpha'_0 = \sum_{n=0}^{\infty} \alpha'_{0,n} T^n$ ($\alpha'_{0,n} \in \mathbf{Z}_p$).

Proof. By (1.1.2),

$$(1 + p' + \sum_{i=2}^{p'+1} \binom{p'+1}{i} S^{i-1})ST \;=\; (T - p' - \sum_{i=2}^{p'+1} \binom{p'+1}{i} S^{i-1})S.$$

Therefore, if $\alpha = T$, then

$$\alpha' \;=\; (1 + p' + \sum_{i=2}^{p'+1} \binom{p'+1}{i} S^{i-1})^{-1}(T - p' - \sum_{i=2}^{p'+1} \binom{p'+1}{i} S^{i-1}),$$

and hence

$$\alpha'_0 \;=\; (1+p')^{-1}(T - p') \;\equiv\; T \bmod (p\Lambda).$$

1.6. **Lemma.** *For each integer* $n \geq 0$, *put* $\sigma_n = (1+S)^{p^n} - 1$. *Then for any* $\alpha \in \Lambda_q$, *there exists a unique* $\alpha' \in \Lambda_q$ *such that* $\alpha\sigma_n = \sigma_n\alpha'$.

Proof. It follows from

$$T\sigma_n = \sigma_n\{(\sum_{i=0}^{p'-1}(\sigma_n+1)^i)(1+T)-1\}.$$

1.7. **Corollary.** *For any left* Λ_q*-module* M *and* $n \geq 0$, $\sigma_n M$ *is a left sub* Λ_q*-module of* M.

2 General Case

2.1. Let $\{\zeta_{p^n}\}_{n\in\mathbf{N}}$ be a set of primitive p^n-th roots ζ_{p^n} of 1 such that ${\zeta_{p^{n+1}}}^p = \zeta_{p^n}$ $(n \in \mathbf{N})$, and put

$$\mathbf{Q}(\zeta_{p^\infty}) = \bigcup_{n\in\mathbf{N}} \mathbf{Q}(\zeta_{p^n}).$$

Let $l_i \neq p$ be prime numbers, m_i positive integers $(i = 1, ..., k)$ prime to p, and put $a = \prod_{i=1}^k l_i^{m_i}$. Let $\{a^{1/p^n}\}_{n\in\mathbf{N}}$ be a set of p^n-th roots a^{1/p^n} of a such that $(a^{1/p^{n+1}})^p = a^{1/p^n}$ $(n \in \mathbf{N})$, and put

$$\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty}) = \bigcup_{n\in\mathbf{N}} \mathbf{Q}(\zeta_{p^\infty}, a^{1/p^n}).$$

Let γ and q be elements of $\mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p'}))$ defined by

$$\gamma(\zeta_{p^n}) = {\zeta_{p^n}}^{1+p'},\ \gamma(a^{1/p^n}) = a^{1/p^n} \quad (n \in \mathbf{N})$$

and

$$q(\zeta_{p^n}) = \zeta_{p^n},\ q(a^{1/p^n}) = \zeta_{p^n} \cdot a^{1/p^n} \quad (n \in \mathbf{N})$$

respectively. Then $\mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p'}))$ is isomorphic to, and hence is identified with, the group Σ defined in 1.1, and via this identification, $\Theta = \mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p^\infty}))$ and $\Gamma = \mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q}(\zeta_{p'}))$. Let L_∞

be the maximal unramified abelian p-extension of $\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})$, and put $X = \mathrm{Gal}(L_\infty/\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty}))$. Let $\sigma \in \Sigma = \mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p'}))$ act on X as

$$\sigma \cdot x \;=\; \tilde{\sigma} x \tilde{\sigma}^{-1} \;(x \in X),$$

where $\tilde{\sigma} \in \mathrm{Gal}(L_\infty/\mathbf{Q}(\zeta_{p'}))$ is a lifting of σ. Then this action is well-defined, and hence we can regard X as a left $\Lambda_q(= \mathbf{Z}_p[[\Sigma]])$-module.

2.2. **Lemma.** *Let $l \neq p$ be a rational prime. Then the set of primes of $\mathbf{Q}(\zeta_{p^\infty})$ lying above l is a finite set whose cardinality is equal to the index of $\langle l\rangle$ in $\mathbf{Z}_p^\times$, where $\langle l\rangle$ denotes the closed subgroup of $\mathbf{Z}_p^\times$ generated by l.*

2.3. Let $\{l_i\}$ be as above. Then by Lemma 2.2, there exist finitely many primes of $\mathbf{Q}(\zeta_{p^\infty})$ lying above $l_1, ..., l_k$, which we denote by $\lambda_1, ..., \lambda_m$. For each $j = 1, ..., m$, let $\tilde{\lambda}_j$ be a prime of L_∞ lying above λ_j, and $I_j \subset \mathrm{Gal}(L_\infty/\mathbf{Q}(\zeta_{p^\infty}))$ the inertia group of $\tilde{\lambda}_j/\lambda_j$. Since $L_\infty/\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})$ is unramified and $\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p^\infty})$ is totally ramified at λ_j, the inclusion $I_j \hookrightarrow \mathrm{Gal}(L_\infty/\mathbf{Q}(\zeta_{p^\infty}))$ induces a bijection

$$I_j \xrightarrow{\sim} \mathrm{Gal}(L_\infty/\mathbf{Q}(\zeta_{p^\infty}))/X \cong \Theta,$$

and hence

$$\mathrm{Gal}(L_\infty/\mathbf{Q}(\zeta_{p^\infty})) \;=\; XI_j \;(j = 1, ..., m).$$

Let $\sigma_j \in I_j$ maps to q. Then σ_j is a topological generator of I_j and there exists a unique $x_j \in X$ such that $\sigma_j = x_j\sigma_1$.

2.4. **Proposition.** *Let L_n be the maximal unramified abelian p-extension of $\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^n})$, and put*

$$X' = SX + \sum_{j=2}^{m} \mathbf{Z}_p x_j, \;\; X_n = X/(\sigma_n/S)X'.$$

Then

$$\mathrm{Gal}(L_n/\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^n})) \cong X_n.$$

Proof. One can prove this in the same way as for Lemma 13.15 of [5].

2.5. **Proposition.** *X/SX is a finitely generated and Λ-torsion Λ-module with $\mu = 0$.*

Proof. In Proposition 2.4, let $n = 0$. Then by the Iwasawa theory of $\mathbf{Z}_p$-extensions and a result of [1], $\mathrm{Gal}(L_0/\mathbf{Q}(\zeta_{p^\infty}))\cong X/X'$ is a finitely generated and Λ-torsion Λ-module with $\mu = 0$. On the other hand,

$$X'/SX \cong N/(N \cap SX) \ (N := \sum_{j=2}^{m} \mathbf{Z}_p x_j)$$

is isomorphic modulo a finite group to a free $\mathbf{Z}_p$-module of finte rank, and hence this μ-invariant is equal to 0. Therefore, X/SX is a finitely generated and Λ-torsion Λ-module with $\mu = 0$.

2.6. **Theorem.** *X is a finitely generated left Λ_q-module.*

Proof. It follows from Lemma 1.2 and Proposition 2.5.

2.7. **Lemma.** *Let M be a left Λ_q-module such that there exists a Λ-homomorphism with finite cokernel*

$$\varphi : \bigoplus_{i=1}^{n}(\Lambda/(g_i)) \longrightarrow M/SM$$

for some $g_i \in \Lambda$ $(i = 1, ..., n)$. Then there exists a Λ-homomorphism with finite cokernel

$$\psi : \bigoplus_{i=1}^{n}(\Lambda/(h_i)) \longrightarrow SM/S^2M,$$

where $h_i \in \Lambda$ $(i = 1, ..., n)$ such that $Sg_i \equiv h_i S \ \mathrm{mod}(S^2\Lambda_q)$.

Proof. By the assumption, there exist $\alpha_i \in M$ such that $g_i\alpha_i \in SM$ and that $\sum_{i=1}^{n} \Lambda\alpha_i + SM$ is finite index in M. Hence there exist $e_j \in M$ $(j = 1, ..., m)$ such that

$$M = \bigcup_{j=1}^{m} (\sum_{i=1}^{n} \Lambda\alpha_i + SM) + e_j.$$

By Propositions 1.3 and 1.5, $S\Lambda\alpha_i \subset \Lambda S\alpha_i + S^2M$. Hence we have

$$SM = \bigcup_{j=1}^{m} (\sum_{i=1}^{n} \Lambda S\alpha_i + S^2M) + Se_j$$

and

$$h_iS\alpha_i \equiv Sg_i\alpha_i \equiv 0 \bmod(S^2M).$$

This completes the proof.

2.8. **Lemma.** *Let N be a finitely generated Λ-module. Then N is a Λ-torsion Λ-module with $\mu = 0$ if and only if N/pN is a finite group.*

Proof. It follows from the structure theorem of finitely generated Λ-modules ([5], Theorem 13.12).

2.9. **Theorem.** *For each $n \in \mathbf{N}$, X/S^nX, X/σ_nX and X_n are finitely generated and Λ-torsion Λ-modules with $\mu = 0$. Moreover, there exist integers $\alpha \geq 0$ and $\beta \geq 0$ independent of n, and an integer n_0 such that for all $n \geq n_0$,*

$$\operatorname{rank}_{\mathbf{Z}_p}(X/S^nX) = \alpha n + \beta.$$

Proof. By Corollary 1.4 and Lemma 2.7, X/S^nX is a finitely generated and Λ-torsion Λ-module with $\mathbf{Z}_p$-rank satisfying the above asymptotic behavior. Hence by Lemma 2.8 and that $\sigma_nX+pX = S^{p^n}X+pX$, X/σ_nX and X_n are finitely generated and Λ-torsion Λ-modules with $\mu = 0$.

3 Special Case

3.1. Let p be an odd prime not dividing the class number of $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$, and let l be a prime congruent modulo p^2 to a topological generator of $\mathbf{Z}_p^\times$ (there exist infinitely many such primes by Dirichlet's theorem on arithmetic progressions). Then by Lemma 2.2, there exists only one prime of $\mathbf{Q}(\zeta_{p^\infty})$ lying above l. Put $a = l$ and let the notation be as in §2. Then $X_n = X/\sigma_n X$. Let Λ'_q (resp. Λ') be the completed group algebra $\mathbf{Z}_p[[\Sigma']]$ (resp. $\mathbf{Z}_p[[\Gamma']]$) of $\Sigma' = \mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q})$ (resp. $\Gamma' = \mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q})$). Then $\Gamma' = \Sigma'/\Theta$, and hence Λ'_q becomes to Λ by putting $q = 1$. Regard Γ' as a subgroup of Σ' by

$$\gamma(a^{1/p^n}) \;=\; a^{1/p^n} \; (\gamma \in \Gamma', n \in \mathbf{N}).$$

Then Λ'_q contains Λ' naturally. Put $\Delta = \mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})$, and regard Δ as a subgroup of Γ' by the Teichmüller character $\omega : \mathbf{F}_p^\times \to \mathbf{Z}_p^\times$ and the identifications $\Delta = \mathbf{F}_p^\times$, $\Gamma' = \mathbf{Z}_p^\times$. For each $i = 0, 1, ..., p-2$, put

$$\varepsilon_i \;=\; \frac{1}{|\Delta|}\sum_{\delta\in\Delta} \omega^{-i}(\delta)\cdot\delta \;\in \mathbf{Z}_p[\Delta],$$

For each $i = 3, 5, ..., p-2$, let $L_p(s, \omega^{1-i})$ denote the p-adic L-function with character ω^{1-i}, and f_i the element of Λ such that

$$f_i|_{T=(1+p)^s-1} \;=\; L_p(s, \omega^{1-i}).$$

Let $\sigma \in \Sigma'$ act on X as

$$\sigma\cdot x \;=\; \tilde{\sigma}x\tilde{\sigma}^{-1} \; (\sigma \in \Sigma', x \in X),$$

where $\tilde{\sigma} \in \mathrm{Gal}(L_\infty/\mathbf{Q})$ is a lifting of σ. Then this action is an extension of the above action of Σ on X.

3.2. **Proposition.** *There exist $F \in \Lambda'_q$ and $x \in X$ such that*

$$F|_{S=0} = \sum_{i=0,1,2,4,\ldots,p-3} \varepsilon_i + \sum_{i=3,5,\ldots,p-2} \varepsilon_i f_i$$

and that $\Lambda'_q \ni \alpha \mapsto \alpha x \in X$ induces a surjective Λ'_q-homomorphism

$$\Lambda'_q/(\Lambda'_q \cdot F) \longrightarrow X.$$

Proof. Put

$$f = \sum_{i=0,1,2,4,\ldots,p-3} \varepsilon_i + \sum_{i=3,5,\ldots,p-2} \varepsilon_i f_i.$$

Then it is known (cf. [5], Theorem 10.14 and 10.16) that there exists $x \in X$ such that $\Lambda' \ni a \mapsto ax \bmod(SX) \in X_0$ induces an Λ'-isomorphism $\Lambda'/(f) \tilde{\to} X_0$. Hence x generates X over Λ'_q (cf. Lemma 1.2) and there exist $F \in \Lambda'_q$ satisfying the above conditions.

3.3. **Theorem.** $\mathrm{rank}_{\mathbf{Z}_p}(X_n) \leq p^n \cdot \mathrm{rank}_{\mathbf{Z}_p}(X_0)$.

Proof. Let $x \in X$ and $F \in \Lambda'_q$ be as in Proposition 3.2. Then $S^k \varepsilon_i x$ $(i = 0, \ldots, p-2, k = 0, \ldots, p^n - 1)$ generate X_n over Λ. Since $S^l \varepsilon_j F x = 0$, $\varepsilon_j S \in S\Lambda'_q$, and $S^{p^n} \in \sigma_n \Lambda_q + p\Lambda_q$, for each $j = 0, \ldots, p-2$ and $l = 0, \ldots, p^n - 1$, there exist $a_{ijkl} \in \Lambda$ such that

$$\begin{aligned} a_{ijkl} &= 0 \ (k < l), \\ a_{ij00} &= \begin{cases} \delta_{ik} & (i = 0, 1, 2, 4, \ldots, p-3) \\ \delta_{ik} f_i & (i = 3, 5, \ldots, p-2), \end{cases} \\ S a_{ijkk} &\equiv a_{ijk+1k+1} S \bmod(S^2 \Lambda_q), \end{aligned}$$

and

$$\sum_{i=0}^{p-2} \sum_{k=0}^{p^n-1} a_{ijkl} S^k \varepsilon_i x \in \sigma_n X + pX.$$

Let d_i $(i = 3, 5, \ldots, p-2)$ be the minimal degree of non-zero terms of $f_i \bmod(p)$. Then by Lemma 1.5, the minimal degree of non-zero terms of

a_{iikk} mod(p) is also d_i, and hence

$$\begin{aligned}\mathrm{rank}_{\mathbf{Z}_p}(X_n) &\leq \mathrm{rank}_{\mathbf{F}_p}(X_n \otimes_{\mathbf{Z}_p} \mathbf{F}_p) \\ &= p^n \cdot \sum_{i=3,5,\ldots,p-2} d_i \\ &= p^n \cdot \mathrm{rank}_{\mathbf{Z}_p}(X_0).\end{aligned}$$

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