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A duality for finite t-modules

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The duality mentioned in the title is the $\mathbb{F}_q[t]$-analogue of the Cartier duality, replacing the multiplicative group $\mathbb{G}_m$ by the Carlitz module C. Finite t-modules are, roughly, finite locally free group schemes which are $\mathbb{F}_q[t]$-submodules of abelian t-modules ([1]) with scalar t-action on their tangent spaces. The duality is expected to play a fundamental role in the theory of Drinfeld motives. Throughout the article, $\mathcal{O}_S$ denotes the structure sheaf of a scheme S.

1. Definitions

Let A be any commutatie ring. For an A-scheme S, we denote by $\alpha : A \to \Gamma(S, \mathcal{O}_S)$ the structure morphism.

DEFINITION. An *A-module scheme* over an A-scheme S is a pair (G, Ψ) consisting of a commutative group scheme G over S and a ring homomorphism $\Psi : A \to \mathrm{End}(G/S)$; $a \mapsto \Psi_a$ such that, for each $a \in A$, Ψ_a induces multiplication by $\alpha(a)$ on the $\mathcal{O}_S$-module $\mathrm{Lie}(G/S)$.

EXAMPLE. A vector bundle G on S can be naturally regarded as a $\Gamma(S, \mathcal{O}_S)$-module scheme. We shall mean by a *vector group scheme* such a $\Gamma(S, \mathcal{O}_S)$-module scheme.

Now we define several "modules" and "sheaves" (Mi and Si for $i = 1, 2, 3$ below). The morphisms are defined naturally for them (though we omit the definition). Each "modules" and "sheaves", for $i = 1, 2, 3$, are in anti-equivalence of categories.

First, let $A = \mathbb{F}_q$, and let S be an $\mathbb{F}_q$-scheme. For an $\mathbb{F}_q$-module scheme (G, Ψ) over S, set $\mathcal{E}_G := \underline{\mathrm{Hom}}_{\mathbb{F}_q, S}(G, \mathbb{G}_a)$. ($\underline{\mathrm{Hom}}_{\mathbb{F}_q, S}$ denotes the Zariski sheaf on S of $\mathbb{F}_q$-linear homomorphisms.)

DEFINITION M1. An $\mathbb{F}_q$-module scheme (G, Ψ) over S is called a *finite φ-module* if $\mathcal{O}_G$ and $\mathcal{E}_G$ are locally free of finite rank over $\mathcal{O}_S$ with rank $\mathcal{O}_G = q^{\mathrm{rank}\ \mathcal{E}_G}$, and $\mathcal{E}_G$ generates the $\mathcal{O}_S$-algebra $\mathcal{O}_G$.

DEFINITION S1. (Drinfeld [2], §2) A *φ-sheaf* is a pair $(\mathcal{E}, \varphi)$ consisting of a locally free $\mathcal{O}_S$-module $\mathcal{E}$ on S of finite rank and an $\mathcal{O}_S$-module homomorphism $\varphi : \mathcal{E}^{(q)} \to \mathcal{E}$. (Here $\mathcal{E}^{(q)}$ denotes the base extension of $\mathcal{E}$ by the q-th power map $\mathcal{O}_S \to \mathcal{O}_S$.)

In the rest, A is the polynomial ring $\mathbb{F}_q[t]$ in one variable t over $\mathbb{F}_q$, and S is an A-scheme. Set $\theta := \alpha(t)$, the image of t in $\mathcal{O}_S$.

DEFINITION M2. A *finite t-module* (G, Ψ) over S is an A-module scheme over S such that
(1) G is killed by some $a \in A - \mathbb{F}_q$; and
(2) $(G, \Psi |_{\mathbb{F}_q})$ is a finite φ-module over S.

DEFINITION S2. A *t-sheaf* $(\mathcal{E}, \varphi, \psi_t)$ on S is the pair of a φ-sheaf $(\mathcal{E}, \varphi)$ and an endomorphism ψ_t of $(\mathcal{E}, \varphi)$ which induces multiplication by θ on Coker(φ). (Recall that Coker(φ) is canonically isomorphic to Lie*Gr$(\mathcal{E}, \varphi)$ ([2], Proposition 2.1, 2)).)

We note here that a finite φ-module G can be canonically embedded into a vector group scheme $E_G := \underline{\mathrm{Spec}}(\mathrm{Sym}^{\cdot}_{\mathcal{O}_S} \mathcal{E}_G)$.

DEFINITION M3. A *finite v-module* (G, Ψ, V) over S is a finite t-module scheme (G, Ψ) over S together with a morphism $V : E_G^{(q)} \to E_G$ of $\mathbb{F}_q$-module schemes such that $\Psi_t = (\theta + V \circ F_{E_G}) |_G$. (Here θ means multiplication by $\theta = \alpha(t) \in \Gamma(S, \mathcal{O}_S)$ on E_G, and F_{E_G} is the Frobenius morphism of E_G.)

DEFINITION S3. A *v-sheaf* $(\mathcal{E}, \varphi, v)$ on S is the pair of a φ-sheaf $(\mathcal{E}, \varphi)$ on S and an $\mathcal{O}_S$-module homomorphism $v : \mathcal{E} \to \mathcal{E}^{(q)}$ such that $(\mathcal{E}, \varphi, \psi_t)$ with $\psi_t := \theta + \varphi \circ v$ is a t-sheaf on S. (Here θ means multiplication by θ on $\mathcal{E}$.)

EXAMPLE. Let (E, Ψ) be a Drinfeld A-module of rank r over $S = \mathrm{Spec}\, R$, where R is an A-algebra. Assume the action of t is given by

$$\psi_t(X) = \theta X + a_1 X^q + \cdots + a_r X^{q^r}, \quad a_i \in R, \; a_r \in R^\times,$$

with respect to a trivialization $E \simeq \mathbb{G}_a = \mathrm{Spec}\, R[X]$. Then for $a \in A - 0$, the finite t-module $G = \mathrm{Ker}(\Psi_a)$ is furnished with a v-module structure by

$$\begin{aligned} &v : \mathcal{E}_G \to \mathcal{E}_G^{(q)}, \\ &X^{q^i} \mapsto X^{q^{i-1}} \otimes (\theta^{q^i} - \theta) + X^{q^i} \otimes a_1^{q^i} + \cdots + X^{q^{r+i-1}} \otimes a_r^{q^i}. \end{aligned}$$

(Here $X^{q^{i-1}} \otimes (\theta^{q^i} - \theta) := 0$ if $i = 0$.) But this v-module structure is *not* unique unless the Frobenius morphism is injective on $\mathcal{E}_G^{(q)}$.

In fact, finite v-modules over "mixed characteristic" bases are not so far from finite t-modules, since we have:

PROPOSITION. *Let (G, Ψ) be a finite t-module which is étale over the generic points of S. Then (G, Ψ) has a unique v-module structure V_G extending the given t-module structure; $\Psi_t = (\theta + V_G \circ F_{E_G}) |_G$. If G and G' are two such finite t-modules, then a morphism $G \to G'$ of finite t-modules preserves this v-module structure. In particular, if $\alpha : A \to \mathcal{O}_S$ is injective, then the two concepts, a finite t-module and a finite v-module, are equivalent.*

The same is valid for a t-sheaf $(\mathcal{E}, \varphi, \psi_t)$ such that $\varphi : \mathcal{E}^{(q)} \to \mathcal{E}$ is injective.

2. The duality

For an $\mathcal{O}_S$-module $\mathcal{E}$, put $\mathcal{E}^* := \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}_S)$. If $(\mathcal{E}, \varphi, v)$ is a v-sheaf on S, then φ and v induce respectively the $\mathcal{O}_S$-module homomorphisms

$$\varphi^* : \mathcal{E}^* \to \mathcal{E}^{*(q)} \quad \text{and} \quad v^* : \mathcal{E}^{*(q)} \to \mathcal{E}^*.$$

It is easy to check that $(\mathcal{E}^*, v^*, \varphi^*)$ is a v-sheaf on S.

DEFINITION. We define the *dual* $(\mathcal{E}, \varphi, v)^*$ of a v-sheaf $(\mathcal{E}, \varphi, v)$ to be the v-sheaf $(\mathcal{E}^*, v^*, \varphi^*)$. If a finite v-module G corresponds to a v-sheaf $(\mathcal{E}, \varphi, v)$, then we define its *dual* G^* to be the finite v-module which corresponds to the v-sheaf $(\mathcal{E}^*, v^*, \varphi^*)$.

We have clearly the following

PROPOSITION. *Let G be a finite v-module.*
(i) G^* has the same rank as G.
(ii) The correspondence $G \mapsto G^*$ is functorial. This functor is exact.
(iii) G^{**} is canonically isomorphic to G.
(iv) $(G \times_S T)^* \simeq G^* \times_S T$ for any S-scheme T.

The same is true for the duality of v-sheaves.

Our main result is the following

THEOREM. *Let C be the Carlitz module over Spec A, and let G be a finite v-module over S.*
(i) The functor

$$\underline{Hom}_{v,S} : (S\text{-schemes}) \to (A\text{-modules})$$
$$T \mapsto Hom_{v,T}(G \times_S T,\ C \times_{\mathrm{Spec}\ A} T)$$

is represented by (the underlying finite t-module of) G^*.
(ii) There exists an A-bilinear pairing of A-module schemes:

$$\Pi : G \times_S G^* \to C$$

such that:

(ii-1) If G' is a finite t-module over S sitting in an A-bilinear pairing Π' : $G \times_S G' \to C$, then there exists a unique morphism $M : G' \to G^*$ of finite t-modules which makes the diagram

$$\begin{array}{ccc} G \times_S G' & \xrightarrow{\Pi'} & C \\ {\scriptstyle 1\times M}\downarrow & & \parallel \\ G \times_S G^* & \xrightarrow[\Pi]{} & C \end{array}$$

commute.

(ii-2) If $\alpha : A \to \mathcal{O}_S$ is injective and S is integral with function field K, then Π *induces a non-degenerate A-bilinear pairing between the A-modules of geometric points:*

$$G(K^{\text{sep}}) \times G^*(K^{\text{sep}}) \to C(K^{\text{sep}}).$$

If we consider only the t-module structure, we will have the following:
(i) The functor

$$\begin{aligned} \underline{\text{Hom}}_{t,S}(G, C) : (S\text{-schemes}) &\to (A\text{-modules}) \\ T &\mapsto \text{Hom}_{t,T}(G \times_S T,\ C \times_{\text{Spec}A} T) \end{aligned}$$

is represented by an A-module scheme $\widetilde{G}^*$ over S.
(ii) If G is étale over the generic points of S, then $\widetilde{G}^*$ is of the form $G^* \cup \widetilde{G}_0^*$, where G^* is (the underlying finite t-module of) the dual finite v-module of G with the unique v-module structure, and $\widetilde{G}_0^*$ is supported on the locus in S where G is not étale. In general, $\widetilde{G}_0^*$ has a positive dimension.

Finally, we mention the Frobenius-Verschiebung relation over a "finite characteristic" base.

PROPOSITION. *Let (G, Ψ, V) be a finite v-module over S.*
(i) Let d be a positive integer, and $F_G^d : G \to G^{(q^d)}$ the q^d-th power Frobenius morphism. Then $G^{(q^d)}$ (resp. F_G^d) is a finite v-module (resp. a morphism of finite v-modules) if and only if $Im(\alpha) \subset \mathbb{F}_{q^d}$.
(ii) Assume $Ker(\alpha : A \to \mathcal{O}_S) = (\mathfrak{p})$ with $\mathfrak{p} \in A$ being a monic prime element of degree d. Let $V_{G,\mathfrak{p}} : G^{(q^d)} \to G$ be the dual morphism of $F_{G^*,\mathfrak{p}} := F_{G^*}^d : G^* \to G^{*(q^d)}$. Then we have

$$\Psi_{\mathfrak{p}} = V_{G,\mathfrak{p}} \circ F_{G,\mathfrak{p}} \quad \text{and} \quad \Psi_{\mathfrak{p}}^{(q^d)} = F_{G,\mathfrak{p}} \circ V_{G,\mathfrak{p}}.$$

In particular, we have an exact sequence of finite t-modules

$$0 \to Ker(F_{G,\mathfrak{p}}) \to Ker(\Psi_{\mathfrak{p}}) \to Ker(V_{G,\mathfrak{p}}) \to 0.$$

3. Comments

(1) Our theory is almost the "Dieudonné theory" for finite t-modules. It might be possible to develope the theory of universal extensions of abelian t-modules from our point of view.

(2) The relation between our duality and the dual isogeny of abelian t-modules remains to be worked out.

(3) I have not considered the case of general A, the ring of elements of a function field over a finite field which are regular outside a fixed place.

(4) Does there exist such a duality for "finite π-modules", with target in arbitrary Lubin-Tate group? To what extent are the coefficients $\mathbb{Q}_p(r)$ celestial (or godgiven)? That is, in which cases can one replace the coefficients $\mathbb{Q}_p(r)$ by other local fields with other "Lubin-Tate twists"?

(5) Our duality will be used to prove the "Carlitz-Hodge-Tate decomposition", which will be reported elsewhere.

References

[1] G. W. Anderson, t-motives, Duke math. J. 53 (1986), 457–502

[2] V. G. Drinfeld, Moduli variety of F-sheaves, Funktsional'nyi Analiz i Ego Prilozheniya 21 (1987), 23– 41

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