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Max-Flow Problem of Strang’s Type

Takeshi MUKU, Maretsugu YAMASAKI

1. Introduction

The celebrated duality theorem called max-flow min-cut theorem on a finite network due to Ford and Fulkerson [1] has been generalized to many directions. Among them, we shall be interested in Strang’s work [4]. Strang’s results were further generalized by Nozawa [3] in the continuous case. Strang gave a max-flow min-cut theorem on a finite network as a motivation of his theory. Here we shall be concerned with the Strang’s max-flow problem on an infinite network. Related to this max-flow problem, we shall discuss several mathematical programming problems as in [5].

More precisely, let $X$ be the countable set of nodes, $Y$ be the countable set of arcs and $K$ be the node-arc incidence matrix. We always assume that the graph $G = \{X, Y, K\}$ is connected and locally finite and has no self-loop. For a strictly positive real function $r$ on $Y$, the pair $N = \{G, r\}$ is called an infinite (discrete) network in this paper. In case $r = 1$, we can identify $G$ with $N = \{G, 1\}$, and we may call $G$ an infinite network.

Denote by $L(X)$ the set of real valued functions on $X$. For $u \in L(X)$, let $Su$ be its support, i.e.,

$$Su = \{x \in X; u(x) \neq 0\},$$

and let $L_0(X)$ be the set of $u \in L(X)$ such that $Su$ is empty or a finite set. For notation and terminology, we mainly follow [5] and [6].

For a given $f \in L(X)$, we call $w \in L(Y)$ a $f$-flow if there exists a number $t$ which satisfies the condition

$$\sum_{y \in Y} K(x, y) w(y) = tf(x) \text{ on } X.$$

Denote by $F(f)$ the set of all $f$-flows. In case $f \neq 0$, the number $t$ in the above definition is uniquely determined by $w$, so we call it the strength of $w$ and denote it by $I(w)$.

Given a non-negative real function $C$ on $Y$ which is called a capacity, we consider the following max-flow problem which was studied by Strang in the case where $G$ is a finite network:

(1.1) Find $M(F(f); C) = \sup \{I(w); w \in F(f), \vert w(y) \vert \leq C(y) \text{ on } Y\}$.

For a subset $A$ of $X$, denote by $\varphi_A$ the characteristic function of $A$, i.e., $\varphi_A(x) = 1$ for $x \in A$ and $\varphi_A(x) = 0$ for $x \in X - A$. Let $a, b$ two distinct nodes and consider the special case where $f = \varphi_{\{b\}} - \varphi_{\{a\}}$. Then $w \in F(f)$ implies

$$\sum_{y \in Y} K(x, y) w(y) = 0 \text{ on } X - \{a, b\},$$
\[ I(w) = - \sum_{y \in Y} K(a, y) w(y) = \sum_{y \in Y} K(b, y) w(y). \]

Namely every $f$-flow is a usual flow from the source $a$ to the sink $b$ and Problem (1.1) is the usual max-flow problem.

To state a dual problem of Problem (1.1), let us recall the definition of a cut. For mutually disjoint nonempty subsets $A$ and $B$ of $X$, denote by $A \ominus B$ the set of all arcs which connect directly $A$ with $B$. A subset $Q$ of $Y$ is a cut if there exists a nonempty proper subset $A$ of $X$ such that $Q = A \ominus (X - A)$.

Let us define a quasi-norm $\|u\|_{C}$ of $u \in L(X)$ by
\[ \|u\|_{C} = \sum_{y \in Y} C(y) \left| \sum_{x \in X} K(x, y) u(x) \right|. \]

For $Q = A \ominus (X - A)$, we have
\[ \|\varphi_{A}\|_{C} = \|1 - \varphi_{A}\|_{C} = \sum_{y \in Q} C(y). \]

Let us define an inner product $<u, v>$ of $u, v \in L(X)$ by
\[ <u, v> = \sum_{x \in X} u(x) v(x) \]
whenever the sum is well-defined.

Let $U(X)$ be the set of all functions $u \in L(X)$ taking values only 0 and 1, i.e., the range $u(X)$ of $u$ is equal to $\{0, 1\}$. Notice that for every cut $Q = A \ominus (X - A)$, both $\varphi_{A}$ and $1 - \varphi_{A}$ belong to $U(X)$.

Now we consider the general case where $f$ satisfies the condition
\[ (1.2) \quad f \neq 0, \quad <|f|, 1 > < \infty \quad \text{and} \quad <f, 1> = 0. \]

This condition holds if $G$ is a finite network and $F(f)$ contains $w$ such that $I(w) \neq 0$.

Strang introduced the following min-cut problem:
\[ (1.3) \quad \text{Find } M^*(U(f); C) = \inf \{ \|\varphi\|_{C} / |<f, \varphi>|; \varphi \in U(f) \}, \]
where $U(f) = \{ \varphi \in U(X); <\varphi, f> \neq 0 \}$.

In the special case where $f = \varphi_{\{b\}} - \varphi_{\{a\}}$ as above, it is easily seen that Problem (1.3) is reduced to the usual min-cut problem.

Strang stated the following duality theorem [4; p.128]:

**THEOREM 1.1.** Let $G$ be a finite network. Then $M(F(f); C) = M^*(U(f); C)$ holds and both Problems (1.1) and (1.3) have optimal solutions.

In the next section, we shall begin with proving this theorem which was roughly stated in [4]. We shall study whether this theorem is valid or not on an infinite network. Related
to the $f$-flows, we shall consider an extremum problem which is analogous to the extremal width of $a$ and $b$ (cf. [5]).

2. Max-flow min-cut theorem on a finite network

In this section, we always assume that $G$ is a finite network, i.e., $X$ and $Y$ are finite sets. To apply the duality theory in [2], we shall formulate Problem (1.1) as a usual linear programming problem on paired spaces.

Let us take

$$
\begin{align*}
\mathcal{X} &= Y = L(Y) \times R, \\
\mathcal{Y} &= Y = L(X) \times L(Y) \times L(Y), \\
\mathcal{P} &= L(Y) \times R, \\
\mathcal{Q} &= \{0\} \times L^+(Y) \times L^+(Y), \\
Tx &= T(w, t) = (\sum_{y \in Y} K(\cdot, y)w(y) - tf, w, -w), \\
y_0 &= (0, -1), \\
z_0 &= (0, -C, -C).
\end{align*}
$$

Define bilinear functionals:

$$(x, y)_1 = ((w, t), (w', t'))_1 = \sum_{y \in Y} w(y)w'(y) + tt'$$

for $x = (w, t), y = (w', t') \in L(Y) \times R$;

$$(z, w)_2 = ((u, v, w), (u', v', w'))_2 = <u, u'> + \sum_{y \in Y} v(y)v'(y) + \sum_{y \in Y} w(y)w'(y)$$

for $z = (u, v, w), w = (u', v', w') \in L(X) \times L(Y) \times L(Y)$. Then $\mathcal{X}$ and $\mathcal{Y}$ (resp. $\mathcal{Z}$ and $\mathcal{W}$) are paired linear spaces with respect to $\langle \cdot, \cdot \rangle_1$ (resp. $\langle \cdot, \cdot \rangle_2$). We see that the quintuple $\{T, P, Q, y_0, z_0\}$ is a linear program and

$$-M(\mathcal{F}(f); C) = \inf\{(x, y_0)_1; x \in P, Tx - z_0 \in Q\}.$$

Denote by $T^*$ the adjoint of $T$. Then

$$T^*(u, w_1, w_2) = (\sum_{x \in X} K(x, \cdot)u(x) + w_1 - w_2, -< u, f >).$$

The dual problem is to find the value

$$\tilde{M}^* = \sup\{(z_0, w)_2; w \in Q^+, y_0 - T^*w \in P^+\},$$

where $\mathcal{P}^+$ and $\mathcal{Q}^+$ are dual cones of $\mathcal{P}$ and $\mathcal{Q}$ respectively and given by

$$\mathcal{P}^+ = \{0\} \times \{0\}, \\
\mathcal{Q}^+ = L(X) \times L^+(Y) \times L^+(Y).$$

Rewriting the right hand side of $\tilde{M}^*$, we see that $-\tilde{M}^*$ is equal to the value of the following extremum problem: Minimize the objective function

$$\sum_{y \in Y} C(y)[w_1(y) + w_2(y)]$$
subject to $w_1, w_2 \in L^+(Y)$, $<u, f> = 1$ and
\[ \sum_{x \in X} K(x, y)u(x) + w_1(y) - w_2(y) = 0 \text{ on } Y. \]
Therefore we have
\[ -\tilde{M}^* = V := \inf\{||u||_C; u \in L(X), <u, f> = 1\}. \]

Since $X$ and $Z$ are finite dimensional and $\mathcal{P}$ and $\mathcal{Q}$ are polyhedral cones, there is no duality gap (cf. [2]), i.e., $M(\mathbf{F}(f); C) = \tilde{M}^*$. It follows that $M(\mathbf{F}(f); C) = V$. By an easy calculation, we obtain
\[
(2.1) \quad V = \min\{||u||_C/|<u, f>|; u \in L(X), <u, f> \neq 0\},
\]
and hence
\[
(2.2) \quad V = \min\{||u||_C/|<u, f>|; u \in \mathbf{V}(f)\},
\]
where $\mathbf{V}(f) = \{u \in L(X); 0 \leq u(x) \leq 1 \text{ on } X, <u, f> \neq 0\}$.

Our next step is to show that $\mathbf{V}(f)$ can be replaced by $\mathbf{U}(f)$ in (2.2). To do this, we need a discrete analogue to the coarea formula.

**Lemma 2.1.** Let $u \in L^+(X)$ and $u(X) = \{\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n\}$ with $\alpha_0 = 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$ and put $A_k = \{x \in X; u(x) \geq \alpha_k\}$. Then
\[ \sum_{x \in X} u(x)f(x) = \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{x \in A_k} f(x). \]

**Proof.** Put $\beta_k = \sum_{x \in A_k} f(x)$ for $0 \leq k \leq n$ and let $A_{n+1} = \emptyset$ and $\beta_{n+1} = 0$. By the relation
\[ B_k := A_k - A_{k+1} = \{x \in X; u(x) = \alpha_k\}, \]
we see that
\[ \sum_{x \in X} u(x)f(x) = \sum_{k=1}^{n+1} \sum_{x \in B_{k-1}} u(x)f(x) \]
\[ = \sum_{k=1}^{n+1} \alpha_{k-1}(\beta_{k-1} - \beta_k). \]
Changing the order of summation, we obtain the desired relation.

**Lemma 2.2.** Let $u, \{\alpha_k\}$ and $A_k$ be the same as above and put $Q_k = A_k \ominus (X - A_k)$ for $k = 1, \cdots, n$. Then
\[ \sum_{y \in Y} C(y) \mid \sum_{x \in X} K(x, y)u(x) \mid = \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{y \in Q_k} C(y). \]
PROOF. Note that \( B_j \cap B_k = \emptyset \) if \( j \neq k \) and
\[
| \sum_{x \in X} K(x, y) u(x) | = \alpha_k - \alpha_j
\]
if \( y \in B_j \ominus B_k \) and \( j < k \). Note that if the endpoints of \( y \) belong to \( B_j \), i.e., \( \{x \in X; K(x, y) \neq 0\} \subset B_j \), then
\[
| \sum_{x \in X} K(x, y) u(x) | = 0.
\]
Put
\[
\mu_{jk} = \sum_{y \in B_j \ominus B_k} C(y) \quad \nu_j = \sum_{y \in Q_j} C(y)
\]
with \( \nu_{n+1} = 0 \). Then it is easily seen that
\[
\sum_{k=0}^{J} \mu_{kj} = \sum_{y \in A_j \ominus (X-A_j)} C(y) = \sum_{y \in Q_j} C(y) = \nu_j
\]
and similarly
\[
\sum_{k=j+1}^{n} \mu_{jk} = \sum_{y \in Q_{j+1}} C(y) = \nu_{j+1}.
\]
By the above observation, we have
\[
\sum_{y \in Y} C(y) \left| \sum_{x \in X} K(x, y) u(x) \right| = \sum_{j=1}^{n} \sum_{k=j+1}^{n} \mu_{jk} (\alpha_k - \alpha_j)
\]
\[
= \sum_{j=1}^{n} \alpha_j \sum_{k=0}^{j} \mu_{kj} - \sum_{j=0}^{n} \alpha_j \sum_{k=j+1}^{n} \mu_{jk}
\]
\[
= \sum_{j=1}^{n} \alpha_j \nu_j - \sum_{j=0}^{n} \alpha_j \nu_{j+1}.
\]
Now we shall prove a fundamental lemma.

**Lemma 2.3.** The relation \( V = M^*(U(f); C) \) holds and there exists \( \varphi \in U(f) \) such that \( M^*(U(f); C) = \|\varphi\|_C / |<\varphi, f>| \).

**Proof.** Let us put \( M^* = M^*(U(f); C) \). Clearly, \( V \leq M^* \). Suppose that \( V < M^* \), i.e., there exists \( \epsilon > 0 \) such that \( M^* \geq V + \epsilon \). Then
\[
(2.3) \quad \|\varphi\|_C \geq (V + \epsilon) |<\varphi, f>|.
\]
holds for all $\varphi \in \mathbf{U}(f)$. Since (2.3) holds trivially for $\varphi \in \mathbf{U}(X) - \mathbf{U}(f)$, (2.3) holds for all $\varphi \in \mathbf{U}(X)$. For any proper subset $A$ of $X$, we have $\varphi_A \in \mathbf{U}(X)$ and by (2.3)

$$\sum_{y \in A \ominus (X - A)} C(y) \geq (V + \varepsilon) |< \varphi_A, f>|.$$ 

Let $u \in \mathbf{V}(f)$ and $u(X) = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ with $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_n \leq 1$ and put $A_k = \{x \in X; u(x) \geq \alpha_k\}$. Multiplying both sides of the above inequality (with $A = A_k$) by $\alpha_k - \alpha_{k-1}$ and summing both sides over $k$, we have by Lemmas 2.1 and 2.2

$$\|u\|_C = \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{y \in A_k \ominus (X - A_k)} C(y)$$

$$\geq \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1})(V + \varepsilon) |< \varphi_{A_k}, f>|$$

$$\geq (V + \varepsilon) \left| \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{x \in A_k} f(x) \right|$$

$$= (V + \varepsilon) |< u, f >|.$$ 

Namely we have $V + \varepsilon \leq \|u\|_C / |< u, f >|$ for all $u \in \mathbf{V}(f)$, and hence $V + \varepsilon \leq V$. This is a contradiction. Thus $V = M^*$. Since $\mathbf{U}(f)$ contains only a finite number of elements, there exists $\varphi \in \mathbf{U}(f)$ such that $M^* = \|\varphi\|_C / |< \varphi, f >|$.

Summing up (2.2), (2.3) and Lemma 2.3, we complete the proof of Theorem 1.1.

3. Max-flow min-cut theorems on an infinite network

In order to study a max-flow problem on an infinite network, we consider the subset $\mathbf{F}_0(f) = \mathbf{F}(f) \cap L_0(Y)$ of the set of $f$-flows. In this section, we always assume the following condition:

(3.1) $f \in L_0(X), f \neq 0$ and $< f, 1 > = 0$.

Let $\{G_n\} G_n = < X_n, Y_n >$ be an exhaustion of $G$, i.e., each $G_n$ is a finite subnetwork of $G$ and $\{G_n\}$ approximates $G$ increasingly. For simplicity, we assume that $Sf \subset X_1$. Define $C_n \in L^+(Y)$ by $C_n(y) = C(y)$ for $y \in Y_n$ and $C_n(y) = 0$ for $y \in Y - Y_n$ and consider the following extremum problems:

(3.2) Find $M_n = M(\mathbf{F}(f); C_n)$;

(3.3) Find $M^*_n = M^*(\mathbf{U}(f); C_n)$.

We shall be concerned with the limits of $\{M_n\}$ and $\{M^*_n\}$.

LEMMA 3.1. $\lim_{n \to \infty} M(\mathbf{F}(f); C_n) = M(\mathbf{F}_0(f); C)$.
PROOF. If \( w \) is a feasible solution of Problem (3.2), then \( w \in L_0(Y) \) by the condition \( |w(y)| \leq C_n(y) \) on \( Y \), and hence \( M_n \leq M_{n+1} \leq M(\mathbf{F}_0(f); C) \). For any \( \varepsilon > 0 \), there exists \( w \in \mathbf{F}_0(f) \) such that

\[
M(\mathbf{F}_0(f); C) - \varepsilon < I(w), \quad |w(y)| \leq C(y) \text{ on } Y.
\]

There exists \( n_0 \) such that \( Sw \subset Y_n \) for all \( n \geq n_0 \). Then \( w \) is a feasible solution of Problem (3.2) for \( n \geq n_0 \), and hence \( M_n \leq M_{n+1} \leq M(\mathbf{F}_0(f); C) \).

We see easily the following:

REMARK 3.2. The value of Problem (3.2) is equal to the value of the following max-flow problem on \( G_n \):

(3.4) Maximize \( t \) subject to \( w \in L(Y_n), |w(y)| \leq C_n(y) \) on \( Y \) and

\[
\sum_{y \in Y_n} K(x, y)w(y) = tf(x) \text{ on } X_n.
\]

Related to Problem (3.3), consider the following min-cut problem on \( G_n \):

(3.5) Find \( M^*(U(f; X_n); C_n) = \inf \{ \sum_{y \in Y_n} C_n(y) | \sum_{x \in X_n} K(x, y)\varphi(x) ; \varphi \in U(f; X_n) \} \),

where \( U(f; X_n) \) is the set of all \( \varphi \in L(X_n) \) such that \( \varphi(X_n) = \{0, 1\} \) and \( \sum_{x \in X_n} \varphi(x)f(x) \neq 0 \).

LEMMA 3.3. \( M^*_n = M^*(U(f; X_n); C_n) \) holds and there exists \( \varphi \in U(f) \) such that \( M^*_n = ||\varphi||_{C_n}/|\langle \varphi, f \rangle| \).

PROOF. The equality follows from our construction. Problem (3.5) has an optimal solution \( \varphi' \in U(f; X_n) \) by Theorem 1.1 and the extension \( \varphi \) of \( \varphi' \) to \( X - X_n \) by 0 belongs to \( U(f) \) and satisfies our requirement.

LEMMA 3.4. \( \lim_{n \to \infty} M^*_n = M^*(U(f); C) \) and there exists \( \varphi \in U(f) \) such that \( M^*(U(f); C) = ||\varphi||_{C}/|\langle \varphi, f \rangle| \).

PROOF. By definition, \( M^*_n \leq M^*_{n+1} \leq M^*(U(f); C) \) is clear. There exists \( \varphi_n \in U(f) \) such that \( M^*_n = ||\varphi_n||_{C}/|\langle \varphi_n, f \rangle| \). Since \( f \in L_0(X) \), it should be noted that the set \( \{ |\langle \varphi, f \rangle| ; \varphi \in U(f) \} \) contains only a finite number of real numbers which are apart from 0, so that there exists \( \alpha > 0 \) such that

(3.6) \( |\langle \varphi, f \rangle| \geq \alpha > 0 \) for all \( \varphi \in U(f) \).

Since \( \varphi_n(X) = \{0, 1\} \), we may assume that \( \{\varphi_n\} \) converges pointwise to \( \tilde{\varphi} \in L(X) \) by choosing subsequences if necessary. We see by (3.6) that \( \tilde{\varphi} \in U(f) \). Since \( f \in L_0(X) \),
<\varphi_n, f > \rightarrow < \tilde{\varphi}, f > \text{ as } n \rightarrow \infty. \text{ It follows that }

\liminf_{n \rightarrow \infty} M^*_n \geq \sum_{y \in Y} \liminf_{n \rightarrow \infty} C_n(y) \left| \sum_{x \in X} K(x, y) \varphi_n(x) \right| / \left| \varphi_n, f \right|

\geq \sum_{y \in Y} C(y) \left| \sum_{x \in X} K(x, y) \tilde{\varphi}(x) \right| / \left| \tilde{\varphi}, f \right|

\geq M^*(U(f); C).

This completes the proof.

By Theorem 1.1 and Lemmas 3.1, 3.3 and 3.4 and Remark 3.2, we obtain the following:

**THEOREM 3.5.** \( M(F_0(f); C) = M^*(U(f); C) \) holds and there exists an optimal solution of the min-cut problem.

In the special case where \( f = \varphi_{\{b\}} - \varphi_{\{a\}} \), this theorem was proved in [5].

4. **Extremal width of a network**

Denote by \( Q(f) \) the set of all cuts generated by \( \varphi \in U(f) \), i.e.,

\[ Q(f) = \{ S\varphi \ominus (X - S\varphi); \varphi \in U(f) \} \]

and consider the following extremum problem of minimizing

\[ H(W) := \sum_{y \in Y} r(y)W(y)^2 \]

subject to \( W \in L^+(Y) \) and

\[ \sum_{y \in Q} W(y) / \left| \varphi, f \right| \geq 1 \text{ for all } Q = S\varphi \ominus (X - S\varphi) \in Q(f). \]

Let \( \mu(Q(f))^{-1} \) be the value of this problem. In the case where \( f = \varphi_{\{b\}} - \varphi_{\{a\}} \), this value is called the extremal width between \( \{a\} \) and \( \{b\} \) of \( N \) in [5].

Denote by \( E^*(Q(f)) \) the set of all feasible solutions of this problem, i.e.,

\[ E^*(Q(f)) = \{ W \in L^+(Y); M^*(U(f); W) \geq 1 \}. \]

Then we have

\[ \mu^*(Q(f))^{-1} = \inf \{ H(W); W \in E^*(Q(f)) \}. \]

We shall consider the extremum problem of finding the following value related to \( f \)-flows:

\[ d^*(F_0(f)) = \inf \{ H(w); w \in F_0(f), I(w) = 1 \}. \]

We shall prove
THEOREM 4.1. Assume Condition (3.1). Then \( d^{*}(F_{0}(f)) = \mu^{*}(Q(f))^{-1} \).

PROOF. Let \( w \in F_{0}(f), I(w) = 1 \) and put \( W(y) = |w(y)| \). For any \( \varphi \in U(f) \),

\[
|<\varphi, f>| = \left| \sum_{y \in Y} w(y) \sum_{x \in X} K(x, y)\varphi(x) \right|
\leq \sum_{y \in Y} W(y) \left| \sum_{x \in X} K(x, y)\varphi(x) \right|,
\]

so that \( W \in E^{*}(Q(f)) \). Thus \( \mu^{*}(Q(f))^{-1} \leq H(W) = H(w) \), and hence \( \mu^{*}(Q(f))^{-1} \leq d^{*}(F_{0}(f)) \). On the other hand, let \( W \in L^{+}(Y) \) satisfy \( M^{*}(U(f); W) \geq 1 \). Then by Theorem 3.5,

\[ M(F_{0}(f); W) = M^{*}(U(f); W) \geq 1. \]

For any positive number \( t < 1 \), there exists \( w \in F_{0}(f) \) such that \( |w(y)| \leq W(y) \) and \( I(w) > t \). Clearly \( w' := w/I(w) \in F_{0}(f) \) and \( I(w') = 1 \), so that

\[ d^{*}(F_{0}(f)) \leq H(w/I(w)) < H(W)/t^{2}. \]

Letting \( t \to 1 \), we have \( d^{*}(F_{0}(f)) \leq H(W) \), and hence \( d^{*}(Q(f)) \leq \mu^{*}(Q(f))^{-1} \). This completes the proof.

Related to the above flow problems, let us consider the following extremum problem of minimizing the Dirichlet sum:

\[ (4.1) \quad \text{Find } \tilde{d}(f) = \inf \{ D(u); u \in L(X) \text{ and } <u, f> = 1 \}, \]

where \( D(u) := H(du) \) and

\[ du(y) = -r(y)^{-1} \sum_{x \in X} K(x, y)u(x). \]

We have the following reciprocal relation:

THEOREM 4.2. Assume Condition (3.1). Then \( d(f)d^{*}(F_{0}(f)) = 1 \).

PROOF. Let \( w \in F_{0}(f), I(w) = 1 \) and \( u \in L(X) \), \( <u, f> = 1 \). Then

\[ 1 = <u, f> = \sum_{y \in Y} w(y) \sum_{x \in X} K(x, y)u(x) \leq [H(w)]^{1/2}[D(u)]^{1/2}, \]

so that \( 1 \leq d(f)d^{*}(F_{0}(f)) \). Denote by \( F_{2}(f) \) the closure of \( F_{0}(f) \) in the Hilbert space \( L_{2}(Y; r) = \{w \in L(Y); H(w) < \infty\} \) with the inner product

\[ H(w, w') = \sum_{y \in Y} r(y)w(y)w'(y). \]
Then we have $d^*(F_0(f)) = d^*(F_2(f))$. Let $\{w_n\}$ be a sequence in $F_0(f)$ such that $I(w_n) = 1$ and $H(w_n) \to d^*(F_0(f))$ as $n \to \infty$. Since $(w_n + w_m)/2 \in L_0(Y)$ is a $f$-flow of unit strength, we see by the standard method that $H(w_n - w_m) \to 0$ as $n, m \to \infty$. There exists $\bar{w} \in L_2(Y; r)$ such that $H(w_n) = I(w_n)$ and $H(w_n - \bar{w}) \to 0$ as $n \to \infty$. Clearly $\bar{w} \in L_0(Y)$ is a $f$-flow of unit strength, we see by the standard method that $H(w_n - \bar{w}) \to 0$ as $n \to \infty$. There exists $\tilde{w} \in L_2(Y; r)$ such that $H(w_n - \tilde{w}) \to 0$ as $n \to \infty$. It follows that $I(\tilde{w}) = 1$ and $d^*(F_2(f)) = H(\tilde{w})$. For any $w' \in F_0(f)$ (a finite cycle) and for any real number $t$, we have $\hat{w} + tw' \in F_2(f)$, so that $H(\hat{w}) \leq H(\hat{w} + tw')$. By the usual variational method, we have $H(\hat{w}, w') = 0$. We see by the same argument as in [7] that there exists $\tilde{u} \in D(N)$ such that $d(\tilde{u}) = \tilde{w}(y)$ on $Y$. Here $D(N)$ is the set of all $u \in L(X)$ with finite Dirichlet sum. Notice that $H(\hat{w}, w_n - w_m) = 0$ for all $n, m$ by the above observation, so that $H(\tilde{w}) = H(\tilde{w}, w_n)$. It follows that

\[
<\tilde{u}, f> = \sum_{x \in X} \tilde{u}(x) \sum_{y \in Y} K(x, y)w_n(y)
\]

\[
= \sum_{y \in Y} w_n(y) \sum_{x \in X} K(x, y)\tilde{u}(x)
\]

\[
= H(w_n, \tilde{w}) = H(\tilde{w}) = D(\tilde{u}).
\]

Therefore $<f, \tilde{u}/D(\tilde{u}) > = 1$, and

\[
d(f) \leq D(\tilde{u}/D(\tilde{u})) = D(\tilde{u})^{-1} = H(\tilde{w})^{-1} = d^*(F_0(f))^{-1}.
\]

Thus $d(f)d^*(F_0(f)) \leq 1$. This completes the proof.

Theorems 4.1 and 4.2 were proved in [5] in the case where $f = \varphi_{\{b\}} - \varphi_{\{a\}}$.

References


