Max-Flow Problem of Strang's Type

Takeshi MUKU, Maretsugu YAMASAKI
Shimane Univ.

1. Introduction

The celebrated duality theorem called max-flow min-cut theorem on a finite network due to Ford and Fulkerson [1] has been generalized to many directions. Among them, we shall be interested in Strang's work [4]. Strang’s results were further generalized by Nozawa [3] in the continuous case. Strang gave a max-flow min-cut theorem on a finite network as a motivation of his theory. Here we shall be concerned with the Strang’s max-flow problem on an infinite network. Related to this max-flow problem, we shall discuss several mathematical programming problems as in [5].

More precisely, let \( X \) be the countable set of nodes, \( Y \) be the countable set of arcs and \( K \) be the node-arc incidence matrix. We always assume that the graph \( G = \{X, Y, K\} \) is connected and locally finite and has no self-loop. For a strictly positive real function \( r \) on \( Y \), the pair \( N = \{G, r\} \) is called an infinite (discrete) network in this paper. In case \( r = 1 \), we can identify \( G \) with \( N = \{G, 1\} \), and we may call \( G \) an infinite network.

Denote by \( L(X) \) the set of real valued functions on \( X \). For \( u \in L(X) \), let \( Su \) be its support, i.e.,

\[
Su = \{x \in X; u(x) \neq 0\},
\]

and let \( L_0(X) \) be the set of \( u \in L(X) \) such that \( Su \) is empty or a finite set. For notation and terminology, we mainly follow [5] and [6].

For a given \( f \in L(X) \), we call \( w \in L(Y) \) a \( f \)-flow if there exists a number \( t \) which satisfies the condition

\[
\sum_{y \in Y} K(x, y)w(y) = tf(x) \quad \text{on} \quad X.
\]

Denote by \( F(f) \) the set of all \( f \)-flows. In case \( f \neq 0 \), the number \( t \) in the above definition is uniquely determined by \( w \), so we call it the strength of \( w \) and denote it by \( I(w) \).

Given a non-negative real function \( C \) on \( Y \) which is called a capacity, we consider the following max-flow problem which was studied by Strang in the case where \( G \) is a finite network:

(1.1) Find \( M(F(f); C) = \sup \{I(w); w \in F(f), |w(y)| \leq C(y) \quad \text{on} \quad Y\} \).

For a subset \( A \) of \( X \), denote by \( \varphi_A \) the characteristic function of \( A \), i.e., \( \varphi_A(x) = 1 \) for \( x \in A \) and \( \varphi_A(x) = 0 \) for \( x \in X - A \). Let \( a, b \) two distinct nodes and consider the special case where \( f = \varphi_{\{b\}} - \varphi_{\{a\}} \). Then \( w \in F(f) \) implies

\[
\sum_{y \in Y} K(x, y)w(y) = 0 \quad \text{on} \quad X - \{a, b\},
\]
\[ I(w) = -\sum_{y \in Y} K(a, y)w(y) = \sum_{y \in Y} K(b, y)w(y). \]

Namely every \( f \)-flow is a usual flow from the source \( a \) to the sink \( b \) and Problem (1.1) is the usual max-flow problem.

To state a dual problem of Problem (1.1), let us recall the definition of a cut. For mutually disjoint nonempty subsets \( A \) and \( B \) of \( X \), denote by \( A \ominus B \) the set of all arcs which connect directly \( A \) with \( B \). A subset \( Q \) of \( Y \) is a cut if there exists a nonempty proper subset \( A \) of \( X \) such that \( Q = A \ominus (X - A) \).

Let us define a quasi-norm \( \|u\|_C \) of \( u \in L(X) \) by
\[
\|u\|_C = \sum_{y \in Y} C(y) \left| \sum_{x \in X} K(x, y)u(x) \right|.
\]
For \( Q = A \ominus (X - A) \), we have
\[
\|\varphi_A\|_C = \|1 - \varphi_A\|_C = \sum_{y \in Q} C(y).
\]
Let us define an inner product \( \langle u, v \rangle \) of \( u, v \in L(X) \) by
\[
\langle u, v \rangle = \sum_{x \in X} u(x)v(x)
\]
whenever the sum is well-defined.

Let \( U(X) \) be the set of all functions \( u \in L(X) \) taking values only 0 and 1, i.e., the range \( u(X) \) of \( u \) is equal to \( \{0, 1\} \). Notice that for every cut \( Q = A \ominus (X - A) \), both \( \varphi_A \) and \( 1 - \varphi_A \) belong to \( U(X) \).

Now we consider the general case where \( f \) satisfies the condition
\[
(1.2) \quad f \neq 0, \quad <|f|, 1> < \infty \quad \text{and} \quad <f, 1> = 0.
\]
This condition holds if \( G \) is a finite network and \( F(f) \) contains \( w \) such that \( I(w) \neq 0 \).

Strang introduced the following min-cut problem:
\[
(1.3) \quad \text{Find } M^*(U(f); C) = \inf \{ \|\varphi\|_C / \langle \varphi, f \rangle; \varphi \in U(f) \},
\]
where \( U(f) = \{ \varphi \in U(X); \langle \varphi, f \rangle \neq 0 \} \).

In the special case where \( f = \varphi_{\{b\}} - \varphi_{\{a\}} \) as above, it is easily seen that Problem (1.3) is reduced to the usual min-cut problem.

Strang stated the following duality theorem [4; p.128]:

**THEOREM 1.1.** Let \( G \) be a finite network. Then \( M(F(f); C) = M^*(U(f); C) \) holds and both Problems (1.1) and (1.3) have optimal solutions.

In the next section, we shall begin with proving this theorem which was roughly stated in [4]. We shall study whether this theorem is valid or not on an infinite network. Related
to the $f$-flows, we shall consider an extremum problem which is analogous to the extremal width of $a$ and $b$ (cf. [5]).

2. Max-flow min-cut theorem on a finite network

In this section, we always assume that $G$ is a finite network, i.e., $X$ and $Y$ are finite sets. To apply the duality theory in [2], we shall formulate Problem (1.1) as a usual linear programming problem on paired spaces.

Let us take

$$
\begin{align*}
X &= Y = L(Y) \times R, \quad Z = W = L(X) \times L(Y) \times L(Y), \\
\mathcal{P} &= L(Y) \times R, \quad \mathcal{Q} = \{0\} \times L^+(Y) \times L^+(Y), \\
Tx &= T(w, t) = (\Sigma_{y \in Y} K(\cdot, y)w(y) - tf, w, -w), \\
y_0 &= (0, -1), \quad z_0 = (0, -C, -C).
\end{align*}
$$

Define bilinear functionals:

$$(x, y)_1 = ((w, t), (w', t'))_1 = \sum_{y \in Y} w(y)w'(y) + tt'$$

for $x = (w, t), y = (w', t') \in L(Y) \times R$;

$$(z, w)_2 = ((u, v, w), (u', v', w'))_2 = \sum_{y \in Y} v(y)v'(y) + \sum_{y \in Y} w(y)w'(y)$$

for $z = (u, v, w), w = (u', v', w') \in L(X) \times L(Y) \times L(Y)$. Then $\mathcal{X}$ and $\mathcal{Y}$ (resp. $Z$ and $W$) are paired linear spaces with respect to $(\cdot, \cdot)_1$ (resp. $(\cdot, \cdot)_2$). We see that the quintuple $\{T, \mathcal{P}, Q, y_0, z_0\}$ is a linear program and

$$-M(\mathcal{F}(f); C) = \inf\{(x, y_0)_1; x \in \mathcal{P}, Tx - z_0 \in \mathcal{Q}\}.$$

Denote by $T^*$ the adjoint of $T$. Then

$$T^*(u, w_1, w_2) = (\sum_{x \in X} K(x, \cdot)u(x) + w_1 - w_2, -<u, f>).$$

The dual problem is to find the value

$$\tilde{M}^* = \sup\{(z_0, w)_2; w \in \mathcal{Q}^+, y_0 - T^*w \in \mathcal{P}^+\},$$

where $\mathcal{P}^+$ and $\mathcal{Q}^+$ are dual cones of $\mathcal{P}$ and $\mathcal{Q}$ respectively and given by

$$\mathcal{P}^+ = \{0\} \times \{0\}, \quad \mathcal{Q}^+ = L(X) \times L^+(Y) \times L^+(Y).$$

Rewriting the right hand side of $\tilde{M}^*$, we see that $-\tilde{M}^*$ is equal to the value of the following extremum problem: Minimize the objective function

$$\sum_{y \in Y} C(y)[w_1(y) + w_2(y)].$$
subject to $w_1, w_2 \in L^+(Y), <u, f> = 1$ and

\[ \sum_{x \in X} K(x, y)u(x) + w_1(y) - w_2(y) = 0 \text{ on } Y. \]

Therefore we have

\[ -\tilde{M}^* = V := \inf\{||u||_C; u \in L(X), <u, f> = 1\}. \]

Since $X$ and $Z$ are finite dimensional and $P$ and $Q$ are polyhedral cones, there is no duality gap (cf. [2]), i.e., $M(F(f); C) = \tilde{M}^*$. It follows that $M(F(f); C) = V$. By an easy calculation, we obtain

\[ (2.1) \quad V = \min\{||u||_C/|<u, f>|; u \in L(X), <u, f> \neq 0\}, \]

and hence

\[ (2.2) \quad V = \min\{||u||_C/|<u, f>|; u \in V(f)\}, \]

where $V(f) = \{u \in L(X); 0 \leq u(x) \leq 1 \text{ on } X, <u, f> \neq 0\}$.

Our next step is to show that $V(f)$ can be replaced by $U(f)$ in (2.2). To do this, we need a discrete analogue to the coarea formula.

**LEMMA 2.1.** Let $u \in L^+(X)$ and $u(X) = \{\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n\}$ with $\alpha_0 = 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$ and put $A_k = \{x \in X; u(x) \geq \alpha_k\}$. Then

\[ \sum_{x \in X} u(x)f(x) = \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{x \in A_k} f(x). \]

**PROOF.** Put $\beta_k = \sum_{x \in A_k} f(x)$ for $0 \leq k \leq n$ and let $A_{n+1} = \emptyset$ and $\beta_{n+1} = 0$. By the relation

\[ B_k := A_k - A_{k+1} = \{x \in X; u(x) = \alpha_k\}, \]

we see that

\[ \sum_{x \in X} u(x)f(x) = \sum_{k=1}^{n+1} \sum_{x \in B_{k-1}} u(x)f(x) = \sum_{k=1}^{n+1} \alpha_{k-1}(\beta_k - \beta_{k-1}). \]

Changing the order of summation, we obtain the desired relation.

**LEMMA 2.2.** Let $u, \{\alpha_k\}$ and $A_k$ be the same as above and put $Q_k = A_k \ominus (X - A_k)$ for $k = 1, \cdots, n$. Then

\[ \sum_{y \in Y} C(y) \sum_{x \in X} K(x, y)u(x) = \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{y \in Q_k} C(y). \]
PROOF. Note that $B_j \cap B_k = \emptyset$ if $j \neq k$ and
\[ | \sum_{x \in X} K(x, y) u(x) | = \alpha_k - \alpha_j \]
if $y \in B_j \ominus B_k$ and $j < k$. Note that if the endpoints of $y$ belong to $B_j$, i.e., \( \{ x \in X; K(x, y) \neq 0 \} \subset B_j \), then
\[ | \sum_{x \in X} K(x, y) u(x) | = 0. \]
Put
\[ \mu_{jk} = \sum_{y \in B_j \ominus B_k} C(y), \quad \nu_j = \sum_{y \in Q_j} C(y), \]
with $\nu_{n+1} = 0$. Then it is easily seen that
\[ \sum_{k=0}^{j} \mu_{kj} = \sum_{y \in A_j \ominus (X - A_j)} C(y) = \sum_{y \in Q_j} C(y) = \nu_j \]
and similarly
\[ \sum_{k=j+1}^{n} \mu_{jk} = \sum_{y \in Q_{j+1}} C(y) = \nu_{j+1}. \]
By the above observation, we have
\[ \sum_{y \in Y} C(y) | \sum_{x \in X} K(x, y) u(x) | = \sum_{j=0}^{n} \sum_{k=j+1}^{n} \mu_{jk} (\alpha_k - \alpha_j) \]
\[ = \sum_{j=1}^{n} \alpha_j \sum_{k=0}^{j} \mu_{kj} - \sum_{j=0}^{n} \alpha_j \sum_{k=j+1}^{n} \mu_{jk} \]
\[ = \sum_{j=1}^{n} \alpha_j \nu_j - \sum_{j=0}^{n} \alpha_j \nu_{j+1}. \]

Now we shall prove a fundamental lemma.

**Lemma 2.3.** The relation $V = M^*(U(f); C)$ holds and there exists $\varphi \in U(f)$ such that
\[ M^*(U(f); C) = \| \varphi \|_C / | \langle \varphi, f \rangle |. \]

**Proof.** Let us put $M^* = M^*(U(f); C)$. Clearly, $V \leq M^*$. Suppose that $V < M^*$, i.e., there exists $\varepsilon > 0$ such that $M^* \geq V + \varepsilon$. Then
\[ (2.3) \quad \| \varphi \|_C \geq (V + \varepsilon) | \langle \varphi, f \rangle | \]
holds for all \( \varphi \in U(f) \). Since (2.3) holds trivially for \( \varphi \in U(X) - U(f) \), (2.3) holds for all \( \varphi \in U(X) \). For any proper subset \( A \) of \( X \), we have \( \varphi_A \in U(X) \) and by (2.3)

\[
\sum_{y \in A \ominus (X - A)} C(y) \geq (V + \varepsilon) \left| <\varphi_A, f> \right|
\]

Let \( u \in V(f) \) and \( u(X) = \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \) with \( \alpha_0 = 0 < \alpha_1 < \cdots \alpha_n \leq 1 \) and put \( A_k = \{x \in X; u(x) \geq \alpha_k\} \). Multiplying both sides of the above inequality (with \( A = A_k \)) by \( \alpha_k - \alpha_{k-1} \) and summing both sides over \( k \), we have by Lemmas 2.1 and 2.2

\[
\|u\|_C = \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{y \in A_k \ominus (X - A_k)} C(y) \\
\geq \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1})(V + \varepsilon) \left| <\varphi_{A_k}, f> \right| \\
\geq (V + \varepsilon) \left| \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{x \in A_k} f(x) \right| \\
= (V + \varepsilon) \left| <u, f> \right|
\]

Namely we have \( V + \varepsilon \leq \|u\|_C / \left| <u, f> \right| \) for all \( u \in V(f) \), and hence \( V + \varepsilon \leq V \). This is a contradiction. Thus \( V = M^* \). Since \( U(f) \) contains only a finite number of elements, there exists \( \varphi \in U(f) \) such that \( M^* = \|\varphi\|_C / \left| <\varphi, f> \right| \).

Summing up (2.2), (2.3) and Lemma 2.3, we complete the proof of Theorem 1.1.

3. Max-flow min-cut theorems on an infinite network

In order to study a max-flow problem on an infinite network, we consider the subset \( F_0(f) = F(f) \cap L_0(Y) \) of the set of \( f \)-flows. In this section, we always assume the following condition:

(3.1) \( f \in L_0(X), f \neq 0 \) and \( <f, 1> = 0 \).

Let \( \{G_n\} \) (\( G_n =< X_n, Y_n > \)) be an exhaustion of \( G \), i.e., each \( G_n \) is a finite subnetwork of \( G \) and \( \{G_n\} \) approximates \( G \) increasingly. For simplicity, we assume that \( Sf \subseteq X_1 \). Define \( C_n \in L^+(Y) \) by \( C_n(y) = C(y) \) for \( y \in Y_n \) and \( C_n(y) = 0 \) for \( y \in Y - Y_n \) and consider the following extremum problems:

(3.2) Find \( M_n = M(F(f); C_n) \);
(3.3) Find \( M^*_n = M^*(U(f); C_n) \).

We shall be concerned with the limits of \( \{M_n\} \) and \( \{M^*_n\} \).

**Lemma 3.1.** \( \lim_{n \to \infty} M(F(f); C_n) = M(F_0(f); C) \).
PROOF. If $w$ is a feasible solution of Problem (3.2), then $w \in L_0(Y)$ by the condition $|w(y)| \leq C_n(y)$ on $Y$, and hence $M_n \leq M_{n+1} \leq M(F_0(f);C)$. For any $\varepsilon > 0$, there exists $w \in F_0(f)$ such that

$$M(F_0(f);C) - \varepsilon < I(w), \quad |w(y)| \leq C(y) \text{ on } Y.$$ 

There exists $n_0$ such that $Sw \subset Y_n$ for all $n \geq n_0$. Then $w$ is a feasible solution of Problem (3.2) for $n \geq n_0$, and hence $M(F_0(f);C) - \varepsilon < I(w) \leq M_n$ for all $n \geq n_0$.

We see easily the following:

REMARK 3.2. The value of Problem (3.2) is equal to the value of the following max-flow problem on $G_n$:

(3.4) Maximize $t$ subject to $w \in L(Y_n), |w(y)| \leq C_n(y)$ on $Y$ and

$$\sum_{y \in Y_n} K(x, y)w(y) = tf(x) \text{ on } X_n.$$ 

Related to Problem (3.3), consider the following min-cut problem on $G_n$:

(3.5) Find $M^*(U(f;X_n);C_n) = \inf\{\sum_{y \in Y_n} C_n(y) | \sum_{x \in X_n} K(x, y)\varphi(x) ; \varphi \in U(f;X_n)\}$, where $U(f;X_n)$ is the set of all $\varphi \in L(X_n)$ such that $\varphi(X_n) = \{0, 1\}$ and $\sum_{x \in X_n} \varphi(x)f(x) \neq 0$.

LEMMA 3.3. $M_n^* = M^*(U(f;X_n);C_n)$ holds and there exists $\varphi \in U(f)$ such that $M_n^* = \|\varphi\|_{C_n}/|<\varphi,f>|$.

PROOF. The equality follows from our construction. Problem (3.5) has an optimal solution $\varphi' \in U(f;X_n)$ by Theorem 1.1 and the extension $\varphi$ of $\varphi'$ to $X - X_n$ by 0 belongs to $U(f)$ and satisfies our requirement.

LEMMA 3.4. $\lim_{n \to \infty} M_n^* = M^*(U(f);C)$ and there exists $\varphi \in U(f)$ such that $M^*(U(f);C) = \|\varphi\|_{C}/|<\varphi,f>|$.

PROOF. By definition, $M_n^* \leq M_{n+1}^* \leq M^*(U(f);C)$ is clear. There exists $\varphi_n \in U(f)$ such that $M_n^* = \|\varphi_n\|_{C}/|<\varphi_n,f>|$. Since $f \in L_0(X)$, it should be noted that the set $\{|<\varphi,f>| ; \varphi \in U(f)\}$ contains only a finite number of real numbers which are apart from 0, so that there exists $\alpha > 0$ such that

(3.6) $|<\varphi,f>| \geq \alpha > 0$ for all $\varphi \in U(f)$.

Since $\varphi_n(X) = \{0,1\}$, we may assume that $\{\varphi_n\}$ converges pointwise to $\tilde{\varphi} \in L(X)$ by choosing subsequences if necessary. We see by (3.6) that $\tilde{\varphi} \in U(f)$. Since $f \in L_0(X)$,
<φₙ, f> → <φ̃, f> as n → ∞. It follows that
\[
\liminf_{n \to \infty} M_n^* \geq \sum_{y \in Y} \liminf_{n \to \infty} C_n(y) \mid \sum_{x \in X} K(x, y)φ_n(x) \mid / \mid <φ_n, f> |
\geq \sum_{y \in Y} C(y) \mid \sum_{x \in X} K(x, y)φ̃(x) \mid / \mid <φ̃, f> |
\geq \mu(U(f); C).
\]

This completes the proof.

By Theorem 1.1 and Lemmas 3.1, 3.3 and 3.4 and Remark 3.2, we obtain the following:

**THEOREM 3.5.** \(M(F_0(f); C) = M^*(U(f); C)\) holds and there exists an optimal solution of the min-cut problem.

In the special case where \(f = φ_{\{b\}} - φ_{\{a\}}\), this theorem was proved in [5].

4. **Extremal width of a network**

Denote by \(Q(f)\) the set of all cuts generated by \(φ \in U(f)\), i.e.,
\[
Q(f) = \{Sφ \ominus (X - Sφ); φ \in U(f)\}
\]
and consider the following extremum problem of minimizing
\[
H(W) := \sum_{y \in Y} r(y)W(y)^2
\]
subject to \(W \in L^+(Y)\) and
\[
\sum_{y \in Q} W(y)/ \mid <φ, f> \mid \geq 1 \text{ for all } Q = Sφ \ominus (X - Sφ) \in Q(f).
\]

Let \(\mu(Q(f))^{-1}\) be the value of this problem. In the case where \(f = φ_{\{b\}} - φ_{\{a\}}\), this value is called the extremal width between \{a\} and \{b\} of \(N\) in [5].

Denote by \(E^*(Q(f))\) the set of all feasible solutions of this problem, i.e.,
\[
E^*(Q(f)) = \{W \in L^+(Y); M^*(U(f); W) \geq 1\}.
\]
Then we have
\[
\mu^*(Q(f))^{-1} = \inf\{H(W); W \in E^*(Q(f))\}.
\]
We shall consider the extremum problem of finding the following value related to \(f\)-flows:
\[
d^*(F_0(f)) = \inf\{H(w); w \in F_0(f), I(w) = 1\}.
\]
We shall prove
THEOREM 4.1. Assume Condition (3.1). Then $d^*(F_0(f)) = \mu^*(Q(f))^{-1}$.

PROOF. Let $w \in F_0(f), I(w) = 1$ and put $W(y) = |w(y)|$. For any $\varphi \in U(f)$,

$$|<\varphi, f>| = \left| \sum_{y \in Y} w(y) \sum_{x \in X} K(x, y) \varphi(x) \right|$$

$$\leq \left( \sum_{y \in Y} W(y) \left| \sum_{x \in X} K(x, y) \varphi(x) \right| \right)$$

so that $W \in E^*(Q(f))$. Thus $\mu^*(Q(f))^{-1} \leq H(W) = H(w)$, and hence $\mu^*(Q(f))^{-1} \leq d^*(F_0(f))$. On the other hand, let $W \in L^+(Y)$ satisfy $M^*(U(f); W) \geq 1$. Then by Theorem 3.5,

$$M(F_0(f); W) = M^*(U(f); W) \geq 1.$$  

For any positive number $t < 1$, there exists $w \in F_0(f)$ such that $|w(y)| \leq W(y)$ and $I(w) > t$. Clearly $w' := w/I(w) \in F_0(f)$ and $I(w') = 1$, so that

$$d^*(F_0(f)) \leq H(w/I(w)) < H(W)/t^2.$$ 

Letting $t \to 1$, we have $d^*(F_0(f)) \leq H(W)$, and hence $d^*(Q(f)) \leq \mu^*(Q(f))^{-1}$. This completes the proof.

Related to the above flow problems, let us consider the following extremum problem of minimizing the Dirichlet sum:

(4.1) Find $d(f) = \inf \{D(u); u \in L(X) \text{ and } <u, f>=1\}$,

where $D(u) := H(du)$ and

$$du(y) = -r(y)^{-1} \sum_{x \in X} K(x, y) u(x).$$

We have the following reciprocal relation:

THEOREM 4.2. Assume Condition (3.1). Then $d(f)d^*(F_0(f)) = 1$.

PROOF. Let $w \in F_0(f), I(w) = 1$ and $u \in L(X), <u, f>=1$. Then

$$1 = <u, f> = \sum_{y \in Y} w(y) \sum_{x \in X} K(x, y) u(x)$$

$$\leq [H(w)]^{1/2}[D(u)]^{1/2},$$

so that $1 \leq d(f)d^*(F_0(f))$. Denote by $F_2(f)$ the closure of $F_0(f)$ in the Hilbert space $L_2(Y; r) = \{w \in L(Y); H(w) < \infty\}$ with the inner product

$$H(w, w') = \sum_{y \in Y} r(y) w(y) w'(y).$$
Then we have $d^*(F_0(f)) = d^*(F_2(f))$. Let $\{w_n\}$ be a sequence in $F_0(f)$ such that $I(w_n) = 1$ and $H(w_n) \to d^*(F_0(f))$ as $n \to \infty$. Since $(w_n + w_m)/2 \in L_0(Y)$ is a $f$-flow of unit strength, we see by the standard method that $H(w_n - w_m) \to 0$ as $n, m \to \infty$. There exists $\tilde{w} \in F_2(f)$ such that $I(\tilde{w}) = 1$. Since $L_{0}(Y)_{2}$ is a $f$-flow of unit strength, we see by the standard method that $H(w_n - \tilde{w}) \to 0$ as $n \to \infty$. There exists $\tilde{u} \in L_{2}(Y;r)$ such that $H(w_n - \tilde{w}) = 0$ as $n \to \infty$. It follows that $I(\tilde{u}) = 1$ and $d^*(F_2(f)) = H(\tilde{w})$. For any $w' \in F_0(0)$ (a finite cycle) and for any real number $t$, we have $\tilde{w} + tw' \in F_2(f)$, so that $H(\tilde{w}) \leq H(\tilde{w} + tw')$. By the usual variational method, we have $H(\tilde{w}, w') = 0$. We see by the same argument as in [7] that there exists $\tilde{u} \in D(N)$ such that $d(\tilde{u}) = \tilde{w}(y)$ on $Y$. Here $D(N)$ is the set of all $u \in L(X)$ with finite Dirichlet sum. Notice that $H(\tilde{w}, w_n - w_n) = 0$ for all $n, m$ by the above observation, so that $H(\tilde{w}) = H(\tilde{w}, w_n)$. It follows that
\[
< \tilde{u}, f > = \sum_{x \in X} \tilde{u}(x) \sum_{y \in Y} K(x, y)w_n(y)
= \sum_{y \in Y} w_n(y) \sum_{x \in X} K(x, y)\tilde{u}(x)
= H(w_n, \tilde{w}) = H(\tilde{w}) = D(\tilde{u}).
\]
Therefore $< f, \tilde{u}/D(\tilde{u}) > = 1$, and
\[
d(f) \leq D(\tilde{u}/D(\tilde{u})) = D(\tilde{u})^{-1} = H(\tilde{w})^{-1} = d^*(F_0(f))^{-1}.
\]
Thus $d(f)d^*(F_0(f)) \leq 1$. This completes the proof.

Theorems 4.1 and 4.2 were proved in [5] in the case where $f = \varphi_{\{b\}} - \varphi_{\{a\}}$.

References