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Max-Flow Problem of Strang's Type

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1. Introduction

The celebrated duality theorem called max-flow min-cut theorem on a finite network due to Ford and Fulkerson [1] has been generalized to many directions. Among them, we shall be interested in Strang's work [4]. Strang's results were further generalized by Nozawa [3] in the continuous case. Strang gave a max-flow min-cut theorem on a finite network as a motivation of his theory. Here we shall be concerned with the Strang's max-flow problem on an infinite network. Related to this max-flow problem, we shall discuss several mathematical programming problems as in [5].

More precisely, let $X$ be the countable set of nodes, $Y$ be the countable set of arcs and $K$ be the node-arc incidence matrix. We always assume that the graph $G = \{X, Y, K\}$ is connected and locally finite and has no self-loop. For a strictly positive real function $r$ on $Y$, the pair $N = \{G, r\}$ is called an infinite (discrete) network in this paper. In case $r = 1$, we can identify $G$ with $N = \{G, 1\}$, and we may call $G$ an infinite network.

Denote by $L(X)$ the set of real valued functions on $X$. For $u \in L(X)$, let $Su$ be its support, i.e.,

$$Su = \{x \in X; u(x) \neq 0\},$$

and let $L_0(X)$ be the set of $u \in L(X)$ such that $Su$ is empty or a finite set. For notation and terminology, we mainly follow [5] and [6].

For a given $f \in L(X)$, we call $w \in L(Y)$ a $f$-flow if there exists a number $t$ which satisfies the condition

$$\sum_{y \in Y} K(x, y)w(y) = tf(x) \text{ on } X.$$

Denote by $F(f)$ the set of all $f$-flows. In case $f \neq 0$, the number $t$ in the above definition is uniquely determined by $w$, so we call it the strength of $w$ and denote it by $I(w)$.

Given a non-negative real function $C$ on $Y$ which is called a capacity, we consider the following max-flow problem which was studied by Strang in the case where $G$ is a finite network:

(1.1) Find $M(F(f); C) = \sup\{I(w); w \in F(f), |w(y)| \leq C(y) \text{ on } Y\}.$

For a subset $A$ of $X$, denote by $\varphi_A$ the characteristic function of $A$, i.e., $\varphi_A(x) = 1$ for $x \in A$ and $\varphi_A(x) = 0$ for $x \in X - A$. Let $a, b$ two distinct nodes and consider the special case where $f = \varphi_{\{b\}} - \varphi_{\{a\}}$. Then $w \in F(f)$ implies

$$\sum_{y \in Y} K(x, y)w(y) = 0 \text{ on } X - \{a, b\},$$

and let $L_0(X)$ be the set of $u \in L(X)$ such that $Su$ is empty or a finite set. For notation and terminology, we mainly follow [5] and [6].
\[ I(w) = - \sum_{y \in Y} K(a, y)w(y) = \sum_{y \in Y} K(b, y)w(y). \]

Namely every \( f \)-flow is a usual flow from the source \( a \) to the sink \( b \) and Problem (1.1) is the usual max-flow problem.

To state a dual problem of Problem (1.1), let us recall the definition of a cut. For mutually disjoint nonempty subsets \( A \) and \( B \) of \( X \), denote by \( A \ominus B \) the set of all arcs which connect directly \( A \) with \( B \). A subset \( Q \) of \( Y \) is a cut if there exists a nonempty proper subset \( A \) of \( X \) such that \( Q = A \ominus (X - A) \).

Let us define a quasi-norm \( \| u \|_C \) of \( u \in L(X) \) by
\[
\| u \|_C = \sum_{y \in Y} C(y) \left| \sum_{x \in X} K(x, y)u(x) \right|.
\]

For \( Q = A \ominus (X - A) \), we have
\[
\| \varphi_A \|_C = \| 1 - \varphi_A \|_C = \sum_{y \in Q} C(y).
\]

Let us define an inner product \( \langle u, v \rangle \) of \( u, v \in L(X) \) by
\[
\langle u, v \rangle = \sum_{x \in X} u(x)v(x)
\]
whenever the sum is well-defined.

Let \( U(X) \) be the set of all functions \( u \in L(X) \) taking values only 0 and 1, i.e., the range \( u(X) \) of \( u \) is equal to \( \{0, 1\} \). Notice that for every cut \( Q = A \ominus (X - A) \), both \( \varphi_A \) and \( 1 - \varphi_A \) belong to \( U(X) \).

Now we consider the general case where \( f \) satisfies the condition
\[(1.2) \quad f \neq 0, \quad <|f|, 1> < \infty \quad \text{and} \quad <f, 1> = 0. \]

This condition holds if \( G \) is a finite network and \( F(f) \) contains \( w \) such that \( I(w) \neq 0 \).

Strang introduced the following min-cut problem:
\[(1.3) \quad \text{Find } M^*(U(f); C) = \inf\{\| \varphi \|_C / <f, \varphi>; \varphi \in U(f)\}, \]
where \( U(f) = \{\varphi \in U(X); <\varphi, f> \neq 0\} \).

In the special case where \( f = \varphi_{\{b\}} - \varphi_{\{a\}} \) as above, it is easily seen that Problem (1.3) is reduced to the usual min-cut problem.

Strang stated the following duality theorem [4; p.128]:

**THEOREM 1.1.** Let \( G \) be a finite network. Then \( M(F(f); C) = M^*(U(f); C) \) holds and both Problems (1.1) and (1.3) have optimal solutions.

In the next section, we shall begin with proving this theorem which was roughly stated in [4]. We shall study whether this theorem is valid or not on an infinite network. Related
to the \( f \)-flows, we shall consider an extremum problem which is analogous to the extremal width of \( a \) and \( b \) (cf. [5]).

2. Max-flow min-cut theorem on a finite network

In this section, we always assume that \( G \) is a finite network, i.e., \( X \) and \( Y \) are finite sets. To apply the duality theory in [2], we shall formulate Problem (1.1) as a usual linear programming problem on paired spaces.

Let us take

\[
\begin{align*}
\mathcal{X} = \mathcal{Y} &= L(Y) \times R, \\
\mathcal{Z} = \mathcal{W} &= L(X) \times L(Y) \times L(Y), \\
\mathcal{P} &= L(Y) \times R, \\
\mathcal{Q} &= \{0\} \times L^+(Y) \times L^+(Y), \\
Tx = T(w, t) &= (\sum_{y \in Y} K(\cdot, y)w(y) - tf, w, -w), \\
y_0 &= (0, -1), \\
z_0 &= (0, -C, -C).
\end{align*}
\]

Define bilinear functionals:

\[
(x, y)_1 = ((w, t), (w', t'))_1 = \sum_{y \in Y} w(y)w'(y) + tt'
\]

for \( x = (w, t), y = (w', t') \in L(Y) \times R; \)

\[
(z, w)_2 = ((u, v, w), (u', v', w'))_2 = u, u' + \sum_{y \in Y} v(y)v'(y) + \sum_{y \in Y} w(y)w'(y)
\]

for \( z = (u, v, w), w = (u', v', w') \in L(X) \times L(Y) \times L(Y). \) Then \( \mathcal{X} \) and \( \mathcal{Y} \) (resp. \( \mathcal{Z} \) and \( \mathcal{W} \)) are paired linear spaces with respect to \((\cdot, \cdot)_1\) (resp. \((\cdot, \cdot)_2\)). We see that the quintuple \( \{T, P, Q, y_0, z_0\} \) is a linear program and

\[
-M(\mathcal{F}(f); C) = \inf\{(x, y_0)_1; x \in P, Tx - z_0 \in Q\}.
\]

Denote by \( T^* \) the adjoint of \( T \). Then

\[
T^*(u, w_1, w_2) = (\sum_{x \in X} K(x, \cdot)u(x) + w_1 - w_2, -< u, f >).
\]

The dual problem is to find the value

\[
\tilde{M}^* = \sup\{(z_0, w)_2; w \in Q^+, y_0 - T^*w \in P^+\},
\]

where \( P^+ \) and \( Q^+ \) are dual cones of \( P \) and \( Q \) respectively and given by

\[
P^+ = \{0\} \times \{0\}, \\
Q^+ = L(X) \times L^+(Y) \times L^+(Y).
\]

Rewriting the right hand side of \( \tilde{M}^* \), we see that \( -\tilde{M}^* \) is equal to the value of the following extremum problem: Minimize the objective function

\[
\sum_{y \in Y} C(y)[w_1(y) + w_2(y)]
\]
subject to $w_1, w_2 \in L^+(Y), <u, f> = 1$ and
\[
\sum_{x \in X} K(x, y)u(x) + w_1(y) - w_2(y) = 0 \quad \text{on } Y.
\]
Therefore we have
\[-\tilde{M}^* = V := \inf\{||u||_C; u \in L(X), <u, f> = 1\}.
\]
Since $\mathcal{X}$ and $\mathcal{Z}$ are finite dimensional and $\mathcal{P}$ and $\mathcal{Q}$ are polyhedral cones, there is no duality gap (cf. [2]), i.e., $M(\mathbf{F}(f); C) = \tilde{M}^*$. It follows that $M(\mathbf{F}(f); C) = V$. By an easy calculation, we obtain
\[
(2.1) \quad V = \min\{||u||_C/|<u, f>|; u \in L(X), <u, f> \neq 0\},
\]
and hence
\[
(2.2) \quad V = \min\{||u||_C/|<u, f>|; u \in \mathcal{V}(f)\},
\]
where $\mathcal{V}(f) = \{u \in L(X); 0 \leq u(x) \leq 1 \text{ on } X, <u, f> \neq 0\}$.

Our next step is to show that $\mathcal{V}(f)$ can be replaced by $\mathcal{U}(f)$ in (2.2). To do this, we need a discrete analogue to the coarea formula.

**Lemma 2.1.** Let $u \in L^+(X)$ and $u(X) = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ with $\alpha_0 = 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$ and put $A_k = \{x \in X; u(x) \geq \alpha_k\}$. Then
\[
\sum_{x \in X} u(x)f(x) = \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{x \in A_k} f(x).
\]

**Proof.** Put $\beta_k = \sum_{x \in A_k} f(x)$ for $0 \leq k \leq n$ and let $A_{n+1} = \emptyset$ and $\beta_{n+1} = 0$. By the relation
\[
B_k := A_k - A_{k+1} = \{x \in X; u(x) = \alpha_k\},
\]
we see that
\[
\sum_{x \in X} u(x)f(x) = \sum_{k=1}^{n+1} \sum_{x \in B_k} u(x)f(x)
= \sum_{k=1}^{n+1} \alpha_k (\beta_{k-1} - \beta_k).
\]
Changing the order of summation, we obtain the desired relation.

**Lemma 2.2.** Let $u, \{\alpha_k\}$ and $A_k$ be the same as above and put $Q_k = A_k \ominus (X - A_k)$ for $k = 1, \cdots, n$. Then
\[
\sum_{y \in Y} C(y) \mid \sum_{x \in X} K(x, y)u(x) \mid = \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{y \in Q_k} C(y).
\]
Note that $B_j \cap B_k = \emptyset$ if $j \neq k$ and

$$| \sum_{x \in X} K(x, y) u(x) | = \alpha_k - \alpha_j$$

if $y \in B_j \ominus B_k$ and $j < k$. Note that if the endpoints of $y$ belong to $B_j$, i.e., $\{x \in X; K(x, y) \neq 0\} \subset B_j$, then

$$| \sum_{x \in X} K(x, y) u(x) | = 0.$$

Put

$$\mu_{jk} = \sum_{y \in B_j \ominus B_k} C(y) \quad \nu_j = \sum_{y \in Q_j} C(y)$$

with $\nu_{n+1} = 0$. Then it is easily seen that

$$\sum_{k=0}^{J} \mu_{kj} = \sum_{y \in A_j \ominus (X-A_j)} C(y) = \sum_{y \in Q_j} C(y) = \nu_j$$

and similarly

$$\sum_{k=j+1}^{n} \mu_{jk} = \sum_{y \in Q_{j+1}} C(y) = \nu_{j+1}.$$

By the above observation, we have

$$\sum_{y \in Y} C(y) | \sum_{x \in X} K(x, y) u(x) | = \sum_{j=0}^{n} \sum_{k=j+1}^{n} \mu_{jk} (\alpha_k - \alpha_j)$$

$$= \sum_{j=1}^{n} \alpha_j \sum_{k=0}^{j} \mu_{kj} - \sum_{j=0}^{n} \alpha_j \sum_{k=j+1}^{n} \mu_{jk}$$

$$= \sum_{j=1}^{n} \alpha_j \nu_j - \sum_{j=0}^{n} \alpha_j \nu_{j+1}.$$
holds for all $\varphi \in U(f)$. Since (2.3) holds trivially for $\varphi \in U(X) - U(f)$, (2.3) holds for all $\varphi \in U(X)$. For any proper subset $A$ of $X$, we have $\varphi_A \in U(X)$ and by (2.3)

$$\sum_{y \in A \ominus (X - A)} C(y) \geq (V + \varepsilon) |< \varphi_A, f>| .$$

Let $u \in V(f)$ and $u(X) = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ with $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_n \leq 1$ and put $A_k = \{x \in X; u(x) \geq \alpha_k\}$. Multiplying both sides of the above inequality (with $A = A_k$) by $\alpha_k - \alpha_{k-1}$ and summing both sides over $k$, we have by Lemmas 2.1 and 2.2

$$||u||_C = \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{y \in A_k \ominus (X - A_k)} C(y) \geq \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1})(V + \varepsilon) |< \varphi_{A_k}, f>| .$$

Namely we have $V + \varepsilon \leq ||u||_C / |< u, f >|$ for all $u \in V(f)$, and hence $V + \varepsilon \leq V$. This is a contradiction. Thus $V = M^*$. Since $U(f)$ contains only a finite number of elements, there exists $\varphi \in U(f)$ such that $M^* = ||\varphi||_C / |< \varphi, f >|$.

Summing up (2.2), (2.3) and Lemma 2.3, we complete the proof of Theorem 1.1.

3. Max-flow min-cut theorems on an infinite network

In order to study a max-flow problem on an infinite network, we consider the subset $F_0(f) = F(f) \cap L_0(Y)$ of the set of $f$-flows. In this section, we always assume the following condition:

(3.1) $f \in L_0(X), f \neq 0$ and $< f, 1 > = 0$.

Let $\{G_n\}(G_n =< X_n, Y_n >)$ be an exhaustion of $G$, i.e., each $G_n$ is a finite subnetwork of $G$ and $\{G_n\}$ approximates $G$ increasingly. For simplicity, we assume that $Sf \subset X_1$. Define $C_n \in L^+(Y)$ by $C_n(y) = C(y)$ for $y \in Y_n$ and $C_n(y) = 0$ for $y \in Y - Y_n$ and consider the following extremum problems:

(3.2) Find $M_n = M(F(f); C_n)$;

(3.3) Find $M_n^* = M^*(U(f); C_n)$.

We shall be concerned with the limits of $\{M_n\}$ and $\{M_n^*\}$.

**Lemma 3.1.** $\lim_{n \to \infty} M(F(f); C_n) = M(F_0(f); C)$. 


PROOF. If $w$ is a feasible solution of Problem (3.2), then $w \in L_0(Y)$ by the condition $|w(y)| \leq C_n(y)$ on $Y$, and hence $M_n \leq M_{n+1} \leq M(F_0(f); C)$. For any $\varepsilon > 0$, there exists $w \in F_0(f)$ such that

$$M(F_0(f); C) - \varepsilon < I(w), \quad |w(y)| \leq C(y) \text{ on } Y.$$ 

There exists $n_0$ such that $S_w \subset Y_n$ for all $n \geq n_0$. Then $w$ is a feasible solution of Problem (3.2) for $n \geq n_0$, and hence $M(F_0(f); C) - \varepsilon < I(w) \leq M_n$ for all $n \geq n_0$.

We see easily the following:

REMARK 3.2. The value of Problem (3.2) is equal to the value of the following max-flow problem on $G_n$:

$$\text{(3.4) Maximize } t \text{ subject to } w \in L(Y_n), |w(y)| \leq C_n(y) \text{ on } Y \text{ and }$$

$$\sum_{y \in Y_n} K(x, y)w(y) = tf(x) \text{ on } X_n.$$ 

Related to Problem (3.3), consider the following min-cut problem on $G_n$:

$$\text{(3.5) Find } M^*(U(f; X_n); C_n) = \inf\{\sum_{y \in Y_n} C_n(y) | \sum_{x \in X_n} K(x, y)\varphi(x) ; \varphi \in U(f; X_n)\},$$

where $U(f; X_n)$ is the set of all $\varphi \in L(X_n)$ such that $\varphi(X_n) = \{0, 1\}$ and $\sum_{x \in X_n} \varphi(x)f(x) \neq 0$.

LEMMA 3.3. $M_n^* = M^*(U(f; X_n); C_n)$ holds and there exists $\varphi \in U(f)$ such that $M_n^* = ||\varphi||_{C_n} / \langle \varphi, f \rangle$.

PROOF. The equality follows from our construction. Problem (3.5) has an optimal solution $\varphi' \in U(f; X_n)$ by Theorem 1.1 and the extension $\varphi$ of $\varphi'$ to $X - X_n$ by 0 belongs to $U(f)$ and satisfies our requirement.

LEMMA 3.4. $\lim_{n \to \infty} M_n^* = M^*(U(f); C)$ and there exists $\varphi \in U(f)$ such that $M_n^* = ||\varphi||_C / \langle \varphi, f \rangle$.

PROOF. By definition, $M_n^* \leq M_{n+1}^* \leq M^*(U(f); C)$ is clear. There exists $\varphi_n \in U(f)$ such that $M_n^* = ||\varphi_n||_{C_n} / \langle \varphi_n, f \rangle$. Since $f \in L_0(X)$, it should be noted that the set $\{|\varphi, f \rangle ; \varphi \in U(f)\}$ contains only a finite number of real numbers which are apart from 0, so that there exists $\alpha > 0$ such that

$$\text{(3.6) } |\varphi, f \rangle \geq \alpha > 0 \text{ for all } \varphi \in U(f).$$

Since $\varphi_n(X) = \{0, 1\}$, we may assume that $\{\varphi_n\}$ converges pointwise to $\tilde{\varphi} \in L(X)$ by choosing subsequences if necessary. We see by (3.6) that $\tilde{\varphi} \in U(f)$. Since $f \in L_0(X)$,
< \varphi_n, f > \rightarrow < \tilde{\varphi}, f > as n \rightarrow \infty. It follows that

\[ \liminf_{n \rightarrow \infty} M_n^* \geq \sum_{y \in Y} \liminf_{n \rightarrow \infty} C_n(y) | \sum_{x \in X} K(x, y) \varphi_n(x) | / | < \varphi_n, f > | \]

\[ \geq \sum_{y \in Y} C(y) | \sum_{x \in X} K(x, y) \tilde{\varphi}(x) | / | < \tilde{\varphi}, f > | \]

\[ \geq M^*(U(f); C). \]

This completes the proof.

By Theorem 1.1 and Lemmas 3.1, 3.3 and 3.4 and Remark 3.2, we obtain the following:

**THEOREM 3.5.** \( M(F_0(f); C) = M^*(U(f); C) \) holds and there exists an optimal solution of the min-cut problem.

In the special case where \( f = \varphi_{\{b\}} - \varphi_{\{a\}} \), this theorem was proved in [5].

4. Extremal width of a network

Denote by \( Q(f) \) the set of all cuts generated by \( \varphi \in U(f) \), i.e.,

\[ Q(f) = \{ S \varphi \ominus (X - S \varphi); \varphi \in U(f) \} \]

and consider the following extremum problem of minimizing

\[ H(W) := \sum_{y \in Y} r(y) W(y)^2 \]

subject to \( W \in L^+(Y) \) and

\[ \sum_{y \in Q} W(y) / | < \varphi, f > | \geq 1 \text{ for all } Q = S \varphi \ominus (X - S \varphi) \in Q(f). \]

Let \( \mu(Q(f))^{-1} \) be the value of this problem. In the case where \( f = \varphi_{\{b\}} - \varphi_{\{a\}} \), this value is called the extremal width between \( \{a\} \) and \( \{b\} \) of \( N \) in [5].

Denote by \( E^*(Q(f)) \) the set of all feasible solutions of this problem, i.e.,

\[ E^*(Q(f)) = \{ W \in L^+(Y); M^*(U(f); W) \geq 1 \}. \]

Then we have

\[ \mu^*(Q(f))^{-1} = \inf\{ H(W); W \in E^*(Q(f)) \}. \]

We shall consider the extremum problem of finding the following value related to \( f \)-flows:

\[ d^*(F_0(f)) = \inf\{ H(w); w \in F_0(f), I(w) = 1 \}. \]

We shall prove
THEOREM 4.1. Assume Condition (3.1). Then $d^*(F_0(f)) = \mu^*(Q(f))^{-1}$.

PROOF. Let $w \in F_0(f), I(w) = 1$ and put $W(y) = |w(y)|$. For any $\varphi \in U(f),

|<\varphi, f>| = |\sum_{y \in Y} w(y) \sum_{x \in X} K(x, y) \varphi(x)|

\leq \sum_{y \in Y} W(y) |\sum_{x \in X} K(x, y) \varphi(x)|,

so that $W \in E^*(Q(f))$. Thus $\mu^*(Q(f))^{-1} \leq H(W) = H(w)$, and hence $\mu^*(Q(f))^{-1} \leq d^*(F_0(f))$. On the other hand, let $W \in L^+(Y)$ satisfy $M^*(U(f); W) \geq 1$. Then by Theorem 3.5,

$M(F_0(f); W) = M^*(U(f); W) \geq 1.$

For any positive number $t < 1$, there exists $w \in F_0(f)$ such that $|w(y)| \leq W(y)$ and $I(w) > t$. Clearly $w' := w/I(w) \in F_0(f)$ and $I(w') = 1$, so that

$d^*(F_0(f)) \leq H(w/I(w)) < H(W)/t^2.$

Letting $t \to 1$, we have $d^*(F_0(f)) \leq H(W)$, and hence $d^*(Q(f)) \leq \mu^*(Q(f))^{-1}$. This completes the proof.

Related to the above flow problems, let us consider the following extremum problem of minimizing the Dirichlet sum:

(4.1) Find $d(f) = \inf \{D(u); u \in L(X) \text{ and } <u, f> = 1\}$,

where $D(u) := H(du)$ and

$du(y) = -r(y) \sum_{x \in X} K(x, y) u(x).$

We have the following reciprocal relation:

THEOREM 4.2. Assume Condition (3.1). Then $d(f)d^*(F_0(f)) = 1$.

PROOF. Let $w \in F_0(f), I(w) = 1$ and $u \in L(X), <u, f> = 1$. Then

$1 = <u, f> = \sum_{y \in Y} w(y) \sum_{x \in X} K(x, y) u(x)

\leq [H(w)]^{1/2}[D(u)]^{1/2},$

so that $1 \leq d(f)d^*(F_0(f))$. Denote by $F_2(f)$ the closure of $F_0(f)$ in the Hilbert space $L_2(Y; r) = \{w \in L(Y); H(w) < \infty\}$ with the inner product

$H(w, w') = \sum_{y \in Y} r(y) w(y) w'(y).$
Then we have $d^*(F_0(f)) = d^*(F_2(f))$. Let $\{w_n\}$ be a sequence in $F_0(f)$ such that $I(w_n) = 1$ and $H(w_n) \to d^*(F_0(f))$ as $n \to \infty$. Since $(w_n + w_m)/2 \in L_0(Y)$ is a $f$-flow of unit strength, we see by the standard method that $H(w_n - w_m) \to 0$ as $n, m \to \infty$. There exists $\tilde{w} \in L_2(Y; r)$ such that $I(\tilde{w}) = 1$ and $H(\tilde{w}) \to d^*(F_0(f))$ as $n \to \infty$. Clearly $\tilde{w} \in F_2(f)$ and $I(w_n) \to I(\tilde{w})$ as $n \to \infty$. It follows that $I(\tilde{w}) = 1$ and $d^*(F_2(f)) = H(\tilde{w})$. For any $w' \in F_0(f)$ (a finite cycle) and for any real number $t$, we have $\tilde{w} + tw' \in F_2(f)$, so that $H(\tilde{w}) \leq H(\tilde{w} + tw')$. By the usual variational method, we have $H(\tilde{w}, w') = 0$. We see by the same argument as in [7] that there exists $\tilde{u} \in D(N)$ such that $d\tilde{u}(y) = \tilde{w}(y)$ on $Y$. Here $D(N)$ is the set of all $u \in L(X)$ with finite Dirichlet sum. Notice that $H(\tilde{w}, w_n - w_m) = 0$ for all $n, m$ by the above observation, so that $H(\tilde{w}) = H(\tilde{w}, w_n)$. It follows that

$$<\tilde{u}, f> = \sum_{x \in X} \tilde{u}(x) \sum_{y \in Y} K(x, y)w_n(y)$$

$$= \sum_{y \in Y} w_n(y) \sum_{x \in X} K(x, y)\tilde{u}(x)$$

$$= H(w_n, \tilde{w}) = H(\tilde{w}) = D(\tilde{u}).$$

Therefore $<f, \tilde{u}/D(\tilde{u})> = 1$, and

$$d(f) \leq D(\tilde{u}/D(\tilde{u})) = D(\tilde{u})^{-1} = H(\tilde{w})^{-1} = d^*(F_0(f))^{-1}.$$ 

Thus $d(f)d^*(F_0(f)) \leq 1$. This completes the proof.

Theorems 4.1 and 4.2 were proved in [5] in the case where $f = \varphi_{\{b\}} - \varphi_{\{a\}}$.

References