A secretary problem with uncertain employment and restricted offering chances

Author(s)
Ano, Katsunori; Tamaki, Mitsushi

Citation
数理解析研究所講究録 (1992), 798: 61-67

Issue Date
1992-08

URL
http://hdl.handle.net/2433/82805

Type
Departmental Bulletin Paper

Textversion
publisher
Kyoto University
A secretary problem with uncertain employment and restricted offering chances

Katsunori Ano * and Mitsushi Tamaki †

June 9, 1991

Abstract

A version of the secretary problem with no recall, in which an offer of acceptance is refused by the applicant with a fixed known probability $1-p$, $(0 \leq p \leq 1)$ and the offering chances, until the decision maker gets one applicant, are at most $M$ times is treated. This problem is an extension of Smith (1975). The optimal strategy of this problem is obtained in Section 2. In Section 3, as example, the optimal offering strategies and the maximum probabilities of selecting the best applicant are given for the problems with $M = 1, 2, 3$ respectively.

DYNAMIC PROGRAMMING; OPTIMAL STOPPING; SECRETARY PROBLEM; UNCERTAIN EMPLOYMENT

1. Introduction

We consider a variation of the sequential observation and selection problem, often referred to as the secretary problem and studied extensively by Gilbert and Mosteller (1960). The basic framework of the classical secretary problem can be described as follows. $N$ applicants appear one by one in random order with all $N!$ orderings being equally likely. We are able, at any time, to rank the applicants that have so far appeared according to some order of preference. As each applicant appears, we must decide whether or not to make an offer to that applicant with the objective of maximizing the probability of choosing the best applicant. It is assumed that each applicant accepts an offer of employment with certainty and that an applicant to whom an offer is not made cannot be recalled later.

There are many interesting modifications of this problem, for an excellent review of the published work to date, see Freeman (1983) or Ferguson (1989). Smith (1975) is the first to consider the problem with uncertain employment where each applicant has the right to decline an offer of employment with a known fixed probability, $1 - p (= q, 0 \leq p \leq 1)$, independent of his/her rank and the arrangement of the other applicants. In Smith’s problem, we can make as many offers as we wish. The problem we consider here puts restriction on the number of offers and allows us to make offer at most $M$ times, where $M \leq N$ is the predetermined number. We call our problem $m$-problem if we are allowed to give $m$ more offers in the future. As easily seen, to solve the $M$-problem completely, we must also solve the $(M - 1)$-,$(M - 2)$-,$\cdots$,1-problems. The event that we can employ the overall best is called success and our objective is to find a strategy of maximizing the probability of success. Another modification of Smith’s problem was considered in Tamaki (1991). We derive the optimal strategy of the problem in Section 2 and investigate in detail, the 1, 2, 3-problems in Section 3.

2. The optimal strategy of the problem

Let $X_j$ denote the relative rank of the $j$th applicant among the first $j$ applicants (rank 1 being relative best). Then, since the applicants appear in random order, it is easy to see that

(i) the $X_j$ are independent random variables, and
(ii) $P(X_j = i) = 1/j$ for $i = 1, 2, \cdots, j$, for $j = 1, 2, \cdots, N$.

The $n$-th applicant is sometimes called a candidate if he/she is relative best, that is, $X_n = 1$.

Define the state of the process as $(n, m)$, $1 \leq n \leq N, 0 \leq m \leq M$, when we confront the $m$-problem and observe that the $n$th applicant is a candidate. In state $(n, m)$, we must decide either to give an offer or not to the current candidate. Let $w(n, m)$ be the probability of success starting from state $(n, m)$.

---

*Postal address: Department of Information Systems and Quantitative Sciences, Nanzan University, Nagoya 466, Japan. Research supported in part by Center for Management Studies, Nanzan University
†Postal address: Department of Business Administration, Aichi University, Nagoya 466, Japan. Research supported by Aichi University Grant C-19
Let $w_n^{(m)}(u_n^{(m)})$ be the corresponding probability when we make an offer (when we decline to make an offer) to the current candidate in state $(n, m)$ and proceed optimally thereafter. Then from the principle of optimality

$$w_n^{(m)} = \max\{u_n^{(m)}, v_n^{(m)}\}, \quad m \geq 1, 1 \leq n \leq N,$$

where

$$u_n^{(m)} = \frac{n}{N} + qw_n^{(m-1)},$$

$$v_n^{(m)} = \frac{1}{n+1}w_{n+1}^{(m)} + (1 - \frac{1}{n+1})v_{n+1}^{(m)}.$$

The boundary conditions are $w_n^{(0)} \equiv 0, v_n^{(m)} \equiv 0, u_n^{(m)} \equiv p$ for $m \geq 1$. Eq.(2) follows since the availability of the applicant can be ascertained by giving an offer and the $m$-problem enters into the $(m-1)$-problem once an offer is declined. Eq.(1),(2) and (3) can be solved recursively to yield the optimal strategy and the probability of success $v^{(m)} \equiv u_1^{(m)}$. The following lemma gives the monotonicity property of $v_n^{(m)}$.

**Lemma 1.** (i) $v_n^{(m)}$ is non-increasing in $n$.

(ii) $v_n^{(m)}$ is non-decreasing in $m$.

**Proof.** (i) is evident from (3). (ii) shall be shown by backward induction on $n$. The assertion holds for $n = N$ and all $m$ because $v_N^{(m+1)} - v_N^{(m)} = 0$ by definition. Assume now that the assertion holds for $n = k (\leq N)$ and for all $m$. We have from (3)

$$v_{k-1}^{(m+1)} - v_{k-1}^{(m)} = \frac{1}{k} \max\{p\frac{k}{N} + qv_k^{(m)} - v_k^{(m+1)}\}$$

$$- \max\{q\frac{k}{N} + qv_k^{(m-1)} - v_k^{(m)}\}$$

$$+ (1 - \frac{1}{k})(v_k^{(m+1)} - v_k^{(m)}).$$

Applying the fact that $\max(a_1, b_1) - \max(a_2, b_2) \geq \min(a_1 - a_2, b_1 - b_2)$ to the right-hand side, we obtain

$$v_{k-1}^{(m+1)} - v_{k-1}^{(m)} \geq \frac{1}{k} \min\{q(v_k^{(m)} - v_k^{(m-1)}), v_k^{(m+1)} - v_k^{(m)}\}$$

$$+ (1 - \frac{1}{k})(v_k^{(m+1)} - v_k^{(m)}).$$

The right-hand side in the above inequality is nonnegative from the induction hypothesis and the proof is complete.

Repeated use of (3) yields

$$v_n^{(m)} = \sum_{j=n+1}^{N} \frac{n}{j(j-1)}u_j^{(m)}.$$

Throughout this paper, the vacuous sum is assumed to be zero. Now let

$$v_n^{(m)} = \sum_{j=n+1}^{N} \frac{n}{j(j-1)}u_j^{(m)}.$$

Then $v_n^{(m)}$ represents the probability of success attainable by giving an offer to the first candidate that appears after leaving state $(n, m)$ and preceding optimally thereafter. Let $B_m$ be the one-stage look-ahead (OLA) stopping region for the $m$-problem, that is, $B_m$ is the set of state $(n, m)$ for which giving an offer immediately is at least as good as waiting for the first candidate to appear to whom an offer is given. Thus

$$B_m = \{(n, m) : u_n^{(m)} \geq v_n^{(m)}\}.$$

From (2) and (5) we have

$$u_n^{(m)} - v_n^{(m)} = p\frac{n}{N}(1 - \psi_n) + q\{v_n^{(m-1)} - \sum_{j=n+1}^{N} \frac{n}{j(j-1)}u_j^{(m-1)}\},$$

where

$$\psi_n = \sum_{j=n+1}^{N} \frac{n}{j(j-1)} = \frac{N}{n+1}.$$
where \( \psi_n \equiv \sum_{j=n+1}^{N} \frac{1}{j(j-1)}. \) Define \( A_n^{(m)} \) as \( (u_n^{(m)} - \tilde{v}_n^{(m)})/n \), that is,

\[
A_n^{(m)} = \frac{p}{N}(1 - \psi_n) + \frac{q}{n} \sum_{j=n+1}^{N} \frac{1}{j(j-1)} (w_j^{(m)} - v_j^{(m-1)}).
\]

Applying (4) to \( \psi_n^{(m-1)} \) of the above equation, we have

\[
A_n^{(m)} = \frac{p}{N}(1 - \psi_n) + \frac{q}{n} \sum_{j=n+1}^{N} \frac{1}{j(j-1)} (w_j^{(m-1)} - v_j^{(m-1)}).
\]

Then \( B_m \) can be written as

\[
B_m = \{(n, m) : A_n^{(m)} \geq 0\}.
\]

It is well known that if \( B_m \) is closed, i.e., \( B_m = \{(n, m) : n \geq s^*_m\} \) for some specified value \( s^*_m \), then the optimal strategy in state \((n, m)\) is to give an offer as soon as state enters \( B_m \) (see, e.g., Ross or Chow et al). The following theorem is the main result of this paper.

**Theorem 1.** Let \( s^*_m \) be specified as \( s^*_m = \min\{n : A_n^{(m)} \geq 0\} \). Then \( B_m \) is closed and gives an optimal offering region for the m-problem. Moreover \( s^*_m \) is non-increasing in \( m \).

**Proof.** It suffices to show that for \( k \geq 1 \), (H-1) \( A_n^{(k)} \) is non-decreasing in \( n \) and (H-2) \( A_n^{(k+1)} \geq A_n^{(k)} \) for \( n = 1, \ldots, N \). We show these by induction on \( k \). The assertion for \( k = 1 \) is immediate since we have from (7)

\[
A_n^{(1)} = \frac{p}{N}(1 - \psi_n),
\]

and

\[
A_n^{(2)} - A_n^{(1)} = \frac{q}{n} \sum_{j=n+1}^{N} \frac{n}{j(j-1)} (w_j^{(1)} - v_j^{(1)}). \]

Assume both (H-1) and (H-2) hold for \( k = m - 1 \), that is, assume \( A_n^{(m-1)} \) is non-decreasing in \( n \), \( A_n^{(m)} \geq A_n^{(m-1)} \) and define \( s^*_{m-1} = \min\{n : A_n^{(m-1)} \geq 0\} \). Then, from the induction hypothesis and (2),

\[
w_j^{(m-1)} - v_j^{(m-1)} = \begin{cases} 0 & j \leq s^*_{m-1} - 1 \\ u_j^{(m-1)} - v_j^{(m-1)} & j \geq s^*_{m-1}. \end{cases}
\]

Substituting (8) into (7), we obtain

\[
A_n^{(m)} = \begin{cases} \frac{p}{N}(1 - \psi_n) & n \leq s^*_{m-1} \\ \frac{p}{N}(1 - \psi_n) + \frac{q}{n} \sum_{j=n+1}^{N} \frac{1}{j(j-1)} (w_j^{(m-1)} - v_j^{(m-1)}) & n + 1 \geq s^*_{m-1}. \end{cases}
\]

When \( n + 1 \leq s^*_{m-1} \), \( A_n^{(m)} \) is clearly non-decreasing in \( n \). When \( n + 1 \geq s^*_{m-1} - 1 \), \( A_n^{(m)} \) is also non-decreasing in \( n \) from Lemma 1 (ii). Thus (H-1) for \( k = m \) is established.

From (7) we have

\[
A_n^{(m+1)} - A_n^{(m)} = \frac{q}{n} \sum_{j=n+1}^{N} \frac{1}{j(j-1)} ((w_j^{(m)} - v_j^{(m)} - w_j^{(m-1)} - v_j^{(m-1)})).
\]

As \( A_n^{(m)} \) is non-decreasing in \( n \), we can define \( s^*_m \) as \( s^*_m = \min\{n : A_n^{(m)} \geq 0\} \) such that

\[
w_j^{(m)} - v_j^{(m)} = \begin{cases} 0 & j \leq s^*_m - 1 \\ v_j^{(m)} - v_j^{(m)} & j \geq s^*_m. \end{cases}
\]
Considering that $v_j^{(m)} = \tilde{v}_j^{(m)}$ for $j \geq s_m^{*}$, we have for $j \geq s_m^{*}$

$$u_j^{(m)} - v_j^{(m)} = u_j^{(m)} - \sum_{i=j+1}^{N} \frac{j}{i(i-1)} u_i^{(m)}$$

$$= \frac{j}{N} (1 - \psi_j) + q (v_j^{(m-1)} - \sum_{i=j+1}^{N} \frac{j}{i(i-1)} v_i^{(m-1)})$$

$$= j A_j^{(m)}.$$

Hence

$$u_j^{(m)} - v_j^{(m)} = \begin{cases} 0, & j \leq s_m^{*} - 1 \\ j A_j^{(m)}, & j \geq s_m^{*} \end{cases}$$

and similarly we have

$$w_j^{(m-1)} - v_j^{(m-1)} = \begin{cases} 0, & j \leq s_{m-1}^{*} - 1 \\ j A_j^{(m-1)}, & j \geq s_{m-1}^{*} \end{cases}$$

Since $s_m^{*} \leq s_{m-1}^{*}$, (11) and (12) yield

$$\{u_j^{(m)} - v_j^{(m)}\} - \{w_j^{(m-1)} - v_j^{(m-1)}\} = \begin{cases} 0, & j \leq s_m^{*} - 1 \\ j A_j^{(m)}, & j \geq s_m^{*} \end{cases}$$

where the last inequality follows from the definition of $s_m^{*}$ and the induction hypothesis. Applying (13) to (10) immediately yields

$$A_n^{(m+1)} - A_n^{(m)} \geq 0, \quad 1 \leq n \leq N,$$

which proves (H-2) for $k = m$.

From Theorem 1, the optimal strategy of the $M$-problem can be summarized as follows: We pass over the first $s_M^{*} - 1$ applicants and give an offer to the first candidate that appears thereafter. If the $M - m$ offers are all declined, the next offer is only given to the candidate that appears on or after $s_m^{*}$, $m = 1, 2, \ldots, M - 1$.

Tables 1, 2, 3, and 4 give the values of $s_1^{*}, s_2^{*}, s_3^{*}$, and the maximum probabilities of success for various values of $p$ and $N$, the values for $p = 1.0$ being taken from Table 2 in Gilbert and Mosteller (1966). Moreover, Tables 5 and 6 give the values of $s_4^{*}, s_5^{*}, s_6^{*}$, and $s_7^{*}$ and the probabilities of success.

Let $s_m^{*}$ be the unique root between 0 and 1 of the equation $\lim_{N,narrow\to\infty} A_n^{(m)} = 0$ for $m \geq 1$ and writing $v_n^{(m)}$ as $v^{(m)}(N)$ when $N, n \to \infty$, with $\frac{N}{n} \to \tilde{z}$, then substituting (4) into (9) we have from $s_m^{*} \leq s_{m-1}^{*}$

$$\lim_{N,narrow\to\infty} A_n^{(m)} = p(1 + log x) + q \left( \frac{v^{(m-1)}(s_{m-1}^{*})}{s_{m-1}^{*}} - \int_{s_{m-1}^{*}}^{1} \frac{v^{(m-1)}(y)}{y^2} dy \right).$$

Thus $s_m^{*}$ and $v^{(m)}(x)$ are easily found to be the forms in the following corollary.

**Corollary 1.** For $m \geq 1$,

$$s_m^{*} = \exp\left\{-\left(1 + \frac{q}{p} \left( \frac{v^{(m-1)}(s_{m-1}^{*})}{s_{m-1}^{*}} - \int_{s_{m-1}^{*}}^{1} \frac{v^{(m-1)}(y)}{y^2} dy \right) \right)\right\},$$

$$v^{(m)}(x) = \begin{cases} v^{(m)}(s_m^{*}), & 0 < x \leq s_m^{*} \\ -pxlogx + q \int_{x}^{1} y v^{(m-1)}(y) dy, & s_m^{*} \leq x < 1. \end{cases}$$

and the limiting value of the maximum probability of success is given by $v^{(m)}(0+) = v^{(m)}(s_m^{*})$. 

The asymptotic examples for $m = 1, 2, 3$ are as follows:

$$
\tilde{s}^*(1) = e^{-1}, \tilde{s}^*(2) = e^{-(1+\frac{1}{2})}, \tilde{s}^*(3) = e^{-(1+\frac{1}{2}+\frac{1}{3})}. 
$$

$$
v^1(0+) = p\tilde{s}^* + pq\tilde{s}^*_{1}, v^2(0+) = p\tilde{s}^* + pq\tilde{s}^*_{1}, v^3(0+) = p\tilde{s}^* + pq\tilde{s}^*_{1} + pq^2\tilde{s}^*.
$$

$$
v^1(x) = \begin{cases} 
  p\tilde{s}^*_{1}, & 0 < x \leq \tilde{s}^*_{1} \\
  -pz\log x, & \tilde{s}^*_{1} \leq x < 1.
\end{cases}
$$

$$
v^2(x) = \begin{cases} 
  p\tilde{s}^*_{1} + pq\tilde{s}^*_{1}, & 0 < x \leq \tilde{s}^*_{2} \\
  -pz\log x - \frac{1}{2}pqz + pq\tilde{s}^*_{1}, & \tilde{s}^*_{2} \leq x \leq \tilde{s}^*_{1} \\
  -pz\log x + \frac{1}{2}pq\log^2 x, & \tilde{s}^*_{1} \leq x < 1.
\end{cases}
$$

$$
v^3(x) = \begin{cases} 
  p\tilde{s}^*_{1} + pq\tilde{s}^*_{1} + pq^2\tilde{s}^*_{1}, & 0 < x \leq \tilde{s}^*_{2} \\
  -pz\log x - \left(\frac{1}{2} + \frac{1}{2}q + \frac{1}{2}q^2\right)pqz + pq\tilde{s}^*_{1} + pq^2\tilde{s}^*_{1}, & \tilde{s}^*_{2} \leq x \leq \tilde{s}^*_{1} \\
  -pz\log x + \frac{1}{2}pq\log^2 (x + \log x) - \frac{1}{2}pq^2 x + pq^2\tilde{s}^*_{1}, & \tilde{s}^*_{1} \leq x \leq \tilde{s}^*_{2} \\
  -pz\log x + \frac{1}{2}pq\log^2 x - \frac{1}{2}pq^2\log^3 x, & \tilde{s}^*_{2} \leq x < 1.
\end{cases}
$$

From the above asymptotic values we conjecture $v^m(0) = p\tilde{s}^*_m + pq\tilde{s}^*_{m-1} + pq^2\tilde{s}^*_{m-2} + \cdots + pq^{m-1}\tilde{s}^*$ and $\tilde{s}^*_m \to p^\frac{1}{m} = \exp\left(\left(-\log(1-q)\right)\right) = \exp\left(\left(-1 + \frac{1}{2} + \frac{1}{2}q^2 + \cdots\right)\right)$ as $m \to \infty$.

3. **Concluding remarks**

1. The proof of Theorem 1 is made by induction but it is complicated. If it is shown that $v^m_n - v^{(m-1)}_n$ is nonincreasing in $n$, we can more easily show, from the optimality equation (9), that the cutoff points $s^*_m$ are unique for each $m$ and $B_m$ is closed.

2. It would be reasonable to allow the probability of refusing, $1 - p$, to be a decreasing function of the absolute rank of the applicant and to depend on the number of offer, and also reasonable to allow the additional offering cost. Uncertainty of employment and restricted offering chances could be extended to the class of problems considered by Gusein-Zade (1966).

3. We can consider the problem in which there is not uncertainty of employment and we can select at most $m$ applicants, i.e., multiple choice is allowed to be made. If we define the state of process as $(n, m)$, when we observe that the $nth$ applicant is a candidate and we can select more $m$ applicants in future.

Let $w^m_n, w^m_{n+1}$ and $v^m_n$ be the same definition of $m$-problem, although the definitions of the state are different. Then the dynamic programming equation of this multiple choice problem can be derived by putting conveniently $p = q = 1$ in the dynamic programming equations (1)-(3) of the $m$-problem. Thus we can apply the same analysis of the $m$-problem to the multiple choice problem, noting that the relation $p + q = 1$ can not be apply. Hence we can easily derive the optimal strategy of the multiple choice problem.

**Acknowledgements**

The authors are grateful to Professor M. Sakaguchi for reading earlier draft and making a number of useful comments on the asymptotic values which led to improvements in the paper.
### Table 1: $p = 0.3$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$s_1^*$</th>
<th>$s_2^*$</th>
<th>$s_3^*$</th>
<th>$a_1^{(1)}$</th>
<th>$a_1^{(2)}$</th>
<th>$a_1^{(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
<td>0.1500000</td>
<td>0.2550000</td>
<td>0.2295000</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0.1500000</td>
<td>0.2050000</td>
<td>0.2295000</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0.1375000</td>
<td>0.1900000</td>
<td>0.2080000</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>0.1300000</td>
<td>0.1862500</td>
<td>0.1985000</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>0.1196071</td>
<td>0.1686506</td>
<td>0.1860433</td>
</tr>
<tr>
<td>25</td>
<td>10</td>
<td>6</td>
<td>5</td>
<td>0.1142749</td>
<td>0.1604136</td>
<td>0.1774775</td>
</tr>
<tr>
<td>50</td>
<td>19</td>
<td>13</td>
<td>10</td>
<td>0.1122825</td>
<td>0.1576602</td>
<td>0.1745811</td>
</tr>
<tr>
<td>100</td>
<td>38</td>
<td>26</td>
<td>21</td>
<td>0.1113128</td>
<td>0.1563364</td>
<td>0.1731780</td>
</tr>
<tr>
<td>1000</td>
<td>369</td>
<td>259</td>
<td>211</td>
<td>0.1104587</td>
<td>0.1551576</td>
<td>0.1719363</td>
</tr>
</tbody>
</table>
| $\infty$ | 0.36788N | 0.25924N | 0.21093N | 0.1103638 | 0.1550268 | 0.1717940  

### Table 2: $p = 0.5$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$s_1^*$</th>
<th>$s_2^*$</th>
<th>$s_3^*$</th>
<th>$a_1^{(1)}$</th>
<th>$a_1^{(2)}$</th>
<th>$a_1^{(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
<td>0.2500000</td>
<td>0.3750000</td>
<td>0.3125000</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0.2500000</td>
<td>0.2916667</td>
<td>0.2968750</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0.2291667</td>
<td>0.2916667</td>
<td>0.2968750</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0.2166667</td>
<td>0.2812500</td>
<td>0.2916667</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>0.1993452</td>
<td>0.2547098</td>
<td>0.2678205</td>
</tr>
<tr>
<td>25</td>
<td>10</td>
<td>7</td>
<td>7</td>
<td>0.1904582</td>
<td>0.2432968</td>
<td>0.2553796</td>
</tr>
<tr>
<td>50</td>
<td>19</td>
<td>14</td>
<td>13</td>
<td>0.1871375</td>
<td>0.2392014</td>
<td>0.2512869</td>
</tr>
<tr>
<td>100</td>
<td>38</td>
<td>26</td>
<td>26</td>
<td>0.1855214</td>
<td>0.2371828</td>
<td>0.2493212</td>
</tr>
<tr>
<td>1000</td>
<td>369</td>
<td>259</td>
<td>259</td>
<td>0.1840978</td>
<td>0.2354182</td>
<td>0.2475618</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.36788N</td>
<td>0.28650N</td>
<td>0.25951N</td>
<td>0.1839397</td>
<td>0.2352273</td>
<td>0.2473664</td>
</tr>
</tbody>
</table>

### References


### Table 3: $p = 0.9$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$s_1^*$</th>
<th>$s_2^*$</th>
<th>$s_3^*$</th>
<th>$\psi^{(1)}$</th>
<th>$\psi^{(2)}$</th>
<th>$\psi^{(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
<td>0.4500000</td>
<td>0.4950000</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0.4500000</td>
<td>0.4650000</td>
<td>0.4650000</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0.4125000</td>
<td>0.4350000</td>
<td>0.4337500</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0.3900000</td>
<td>0.4035000</td>
<td>0.4036500</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0.3588214</td>
<td>0.3787527</td>
<td>0.3793534</td>
</tr>
<tr>
<td>25</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0.3428247</td>
<td>0.3597193</td>
<td>0.3602341</td>
</tr>
<tr>
<td>50</td>
<td>19</td>
<td>18</td>
<td>10</td>
<td>0.3368475</td>
<td>0.3559776</td>
<td>0.3545874</td>
</tr>
<tr>
<td>100</td>
<td>38</td>
<td>35</td>
<td>35</td>
<td>0.3339385</td>
<td>0.3510086</td>
<td>0.3516011</td>
</tr>
<tr>
<td>1000</td>
<td>369</td>
<td>350</td>
<td>349</td>
<td>0.3313761</td>
<td>0.3489551</td>
<td>0.3489711</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.36788N</td>
<td>0.34994N</td>
<td>0.34873N</td>
<td>0.3310915</td>
<td>0.3480531</td>
<td>0.3486620</td>
</tr>
</tbody>
</table>

### Table 4: $p = 1.0$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$s_1^*$</th>
<th>$s_2^*$</th>
<th>$s_3^*$</th>
<th>$\psi^{(1)}$</th>
<th>$\psi^{(2)}$</th>
<th>$\psi^{(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
<td>0.5000000</td>
<td>0.5000000</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0.5000000</td>
<td>0.5000000</td>
<td>0.5000000</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0.4583333</td>
<td>0.4583333</td>
<td>0.4583333</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0.4333333</td>
<td>0.4333333</td>
<td>0.4333333</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0.3986905</td>
<td>0.3986905</td>
<td>0.3986905</td>
</tr>
<tr>
<td>25</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0.3809164</td>
<td>0.3809164</td>
<td>0.3809164</td>
</tr>
<tr>
<td>50</td>
<td>19</td>
<td>19</td>
<td>19</td>
<td>0.3742750</td>
<td>0.3742750</td>
<td>0.3742750</td>
</tr>
<tr>
<td>100</td>
<td>38</td>
<td>38</td>
<td>38</td>
<td>0.3710428</td>
<td>0.3710428</td>
<td>0.3710428</td>
</tr>
<tr>
<td>1000</td>
<td>369</td>
<td>369</td>
<td>369</td>
<td>0.3681956</td>
<td>0.3681956</td>
<td>0.3681956</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$e^{-1}N$</td>
<td>$e^{-1}N$</td>
<td>$e^{-1}N$</td>
<td>$e^{-1}$</td>
<td>$e^{-1}$</td>
<td>$e^{-1}$</td>
</tr>
</tbody>
</table>

### Table 5: $p = 0.3$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$s_1^*$</th>
<th>$s_2^*$</th>
<th>$s_3^*$</th>
<th>$s_4^*$</th>
<th>$\psi^{(4)}$</th>
<th>$\psi^{(5)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td>0.1993575</td>
<td>0.1995502</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0.1896865</td>
<td>0.1909424</td>
</tr>
<tr>
<td>25</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>0.1825669</td>
<td>0.1838990</td>
</tr>
<tr>
<td>50</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0.1799344</td>
<td>0.1812443</td>
</tr>
<tr>
<td>100</td>
<td>20</td>
<td>19</td>
<td>19</td>
<td>19</td>
<td>0.1785684</td>
<td>0.1799735</td>
</tr>
<tr>
<td>1000</td>
<td>191</td>
<td>183</td>
<td>180</td>
<td>180</td>
<td>0.1773773</td>
<td>0.1788235</td>
</tr>
</tbody>
</table>

### Table 6: $p = 0.9$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$s_1^*$</th>
<th>$s_2^*$</th>
<th>$s_3^*$</th>
<th>$s_4^*$</th>
<th>$\psi^{(4)}$</th>
<th>$\psi^{(5)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
<td>0.4036500</td>
<td>0.4036500</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0.3793640</td>
<td>0.3793641</td>
</tr>
<tr>
<td>25</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0.3602450</td>
<td>0.3602451</td>
</tr>
<tr>
<td>50</td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>0.3546025</td>
<td>0.3546027</td>
</tr>
<tr>
<td>100</td>
<td>36</td>
<td>36</td>
<td>36</td>
<td>36</td>
<td>0.3516160</td>
<td>0.3516163</td>
</tr>
<tr>
<td>1000</td>
<td>349</td>
<td>349</td>
<td>349</td>
<td>349</td>
<td>0.3489711</td>
<td>0.3489715</td>
</tr>
</tbody>
</table>