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Representation theory for finite groups
in computer system "CAYLEY"

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Recently, computational methods are useful for the representation theory, and have been executed by the CAYLEY system by Cannon[1]. In this paper, we will show a usage and some applications of the CAYLEY in the representation theory.

1. Representation in CAYLEY

Let $G$ be a finite group with a set of generators $\{g_1, \ldots, g_l\}$ and $F$ a splitting field for $G$ such that the characteristic of $F$ divides the group order $|G|$.

In this paper we treat the action of an element $g$ of $G$ on the $F$-vector space $V$ as the product of a vector by a matrix $V(g)$ on the right. So we can consider the vector space $V$ as a right $FG$-module for the group algebra $FG$. In the CAYLEY system, we treat a set $\{M(g_1), \ldots, M(g_l)\}$ as a representation of $FG$-module $V$. A series of submodules of $V$

$$0 = V_0 < V_1 < \cdots < V_n = V$$

where $V_i/V_{i-1}$ is simple

is called a composition series for an $FG$-module $V$.

2. The socle

Let $\text{Soc}(V)$ denote the socle of $V$, namely the sum of all simple $FG$-submodules of $V$.

**Lemma 1.** Let $V$ be an $FG$-module and $U$ an $FG$-submodule of $V$ such that $V/U$ is isomorphic to a simple $FG$-module $W$. Then the following statements are equivalent.

(i) There is an $FG$-submodule $T$ which is isomorphic to $W$ and $\text{Soc}(V) = \text{Soc}(U) \oplus T$.

(ii) $V$ is isomorphic to $U \oplus W$.

**Proof:** (i) $\Rightarrow$ (ii). Since $U \cap T = \text{Soc}(U) \cap T = 0$, $U \oplus T$ is an $FG$-submodule of $V$. But the dimension of $V$ is equal to this submodule. So $V = U \oplus T$. 
(ii) ⇒ (i). Immediate from the definition of the socle.

There is the standard function *composition factor* which is written by Schneider[3] in the CAYLEY system. From Lemma 1, we can get the socle of the FG-module V by the following algorithm.

**Algorithm SOC:**
1. Let get a composition series \( \{V_i\}_{(i=1,\ldots,n)} \) of \( V \) and \( \text{socsq} \) be empty.
2. For each \( i \), see whether \( V_i \) is isomorphic to \( V_{i-1} \oplus V_i/V_{i-1} \) or not. If \( V_i \) can split then append \( V_i/V_{i-1} \) to \( \text{socsq} \).
3. Print \( \text{socsq} \) as the socle of the FG-module \( V \).

The main part of this algorithm is investigating that \( V_i \) can split or not. Let \( V \) be an FG-module and \( U \) an FG-submodule of \( V \) such that \( V/U \) is isomorphic to a simple FG-module \( W \). The dimension of the module \( U \) and the module \( W \) are \( u \) and \( w \), respectively. In a good basis of \( V \), \( V(g) \) is a following matrix

\[
\begin{pmatrix}
    U(g) & 0 \\
    D(g) & W(g)
\end{pmatrix}
\]

for each element \( g \) of \( G \)

where \( D(g) \) is a \( w \times u \)-matrix. Since \( V \) is an FG-module, \( D \) is satisfies a following equation.

\[ (*) \quad D(gg') = D(g)U(g') + W(g)D(g') \quad \text{for any } g, g' \text{ in } G \]

The module \( V \) is isomorphic to \( U \oplus W \) if and only if there are some regular matrices \( P \) and

\[ (1) \quad PV(g)P^{-1} = \begin{pmatrix} U(g) & 0 \\ 0 & W(g) \end{pmatrix} \]

for all elements \( g \) of \( G \). What made it difficult is the number of unknowns which have to be processed to find the matrix \( P \). Thus we prove the next lemma to reduce the number of unknowns.
LEMMA 2. Using the above conditions, the following statements are equivalent.

(i) There is such a matrix $P$.

(ii) There is a $w \times u$ matrix $Q$ such that $D(g) = W(g)Q - QU(g)$ for any $g$ in $G$.

By Lemma 2, it suffices to find the matrix $Q$ instead of the matrix $P$. So we can reduce the number of unknowns from $(m + n)^2$ to $nm$ and see it as the problem of basic linear algebra.

PROOF: (i) $\Rightarrow$ (ii)

Let $P = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}$ where $p_1 : u \times u$ matrix, $p_2 : u \times w$ matrix, $p_3 : w \times u$ matrix, $p_4 : w \times w$ matrix

Then from (1), we get the following equations for all elements $g$ of $G$.

(2) $p_1 U(g) + p_2 D(g) = U(g)p_1$

(3) $p_2 W(g) = U(g)p_2$

(4) $p_3 U(g) + p_4 D(g) = W(g)p_3$

(5) $p_4 W(g) = W(g)p_4$

If matrix $p_4$ is regular then let $Q$ be $p_4^{-1}p_3$. The matrix $Q$ satisfies the condition (ii) from (4) and (5).

Since $W$ is the simple module and we can see that $p_4$ is an endomorphism of $FG$-module $W$ from (5).

So if the matrix $p_4$ is not regular then $p_4$ must be a zero-matrix by Schur's lemma. From the equations (3) and (4), $p_3p_2 W(g) = W(g)p_3p_2$. If $p_3p_2$ is not a zero-matrix then $p_3p_2$ is $\alpha I$ by Schur's lemma where $\alpha$ is a non-zero element of $F$ and $I$ is the unit matrix. The product of (2) and $\alpha^{-1}p_3$ on the left gives

$\alpha^{-1}p_3p_1 U(g) + D(g) = W(g)\alpha^{-1}p_3p_1$

by the equation (4). So $Q$ is $\alpha^{-1}p_3p_1$.

If $p_3p_2$ is a zero-matrix then there is a positive integer $k$ such that $p_3p_1^n p_2 = 0$ ($0 \leq n \leq k$) and $p_3p_1^{k+1} p_2 \neq 0$ and

(2') $p_1^{n+1} U(g) = U(g)p_1^{n+1} - \sum_{i=0}^{n} p_1^i p_2 D(g)p_1^{n-i}$ for the natural number $n$
by the easy calculation. When $n = k$, the product of $(2')$ and $p_3$ on the left gives $p_3p_1^{k+1}U(g) = W(g)p_3p_1^{k+1}$ by the equation (4) and $p_3p_1^{k+1}p_2W(g) = W(g)p_3p_1^{k+1}p_2$ by the equation (3). We can see that $p_3p_1^{k+1}p_2$ is $\alpha I$ by Schur's lemma where $\alpha$ is a non-zero element of $F$ and $I$ is the unit matrix. When $n = k + 1$, the product of $(2')$ and $\alpha^{-1}p_3$ on the left gives $\alpha^{-1}p_3p_1^{k+2}U(g) = W(g)\alpha^{-1}p_3p_1^{k+2} - D(g)$ by the equation (4). So $Q$ is $\alpha^{-1}p_3p_1^{k+2}$.

(ii) $\Rightarrow$ (i)

Let $P = \begin{pmatrix} I_m & 0 \\ Q & I_n \end{pmatrix}$ where $I_m$ and $I_n$ are the $m$ and $n$-dimensional unit matrix. Then the matrix $P$ satisfies the equation (1).

By the way, let think about a $w \times u$-matrix $D(g)$. Let $F^{w \times u}$ be a set of $w \times u$-matrices over $F$, $E(W, U)$ a set of map $D$ from $G$ to $F^{w \times u}$ which is satisfies $(\ast)$ and $e(W, U)$ a set of map $D_Q$ such that $D_Q(g) = W(g)Q - QU(g)$ where $Q$ is a $w \times u$-matrix. Then $E(W, U)$ is an $F$-space and $e(W, U)$ an $F$-subspace of $E(W, U)$. And $E(W, U)/e(W, U)$ is isomorphic to $\text{Ext}^1_{FG}(W, U)$ as $F$-space. So we can compute the dimension of $\text{Ext}^1_{FG}(W, U)$ from this equation. In particular, $E(W, U)$ and $e(W, U)$ are $Z^1(G, U)$ and $B^1(G, U)$ respectively if $W$ is the trivial module.

3. $\Omega^{-1}(M)$

Suppose $G$ is p-group. Using $E(W, U)$, we can construct the Heller module $\Omega^{-1}(M)$ of an $FG$-module $M$. Let $\tilde{E}(M)$ denote $E(\mathbf{F}, M)/e(\mathbf{F}, M)$ where $\mathbf{F}$ is the trivial $FG$-module and $\{d_i\} (1 \leq i \leq m_1)$ an $F$-basis of $\tilde{E}(M)$. Then we can make a following representation

$$M_1(g) = \begin{pmatrix} M(g) & 0 \\ d_1^i(g) & 1 \\ \vdots & \ddots \\ d_{m_1}^1(g) & 1 \end{pmatrix}$$

where the $FG$-module $M_1$ has $M$ as a submodule of $M_1$ and $M_1/M$ is isomorphic to $m_1$ copies of the trivial module $\mathbf{F}$. Moreover $\text{Soc}(M) \cong \text{Soc}(M_1)$. By the same process, we can make $FG$-module $M_2$ such that $M_2$ has $M_1$ as a submodule and $M_2/M_1$ is isomorphic
to $m_2(= \dim_F \bar{E}(M_1))$ copies of the trivial module $F$ and $soc(M_1) \simeq Soc(M_2)$. So if we continue this process, then we get

$$M_k(g) = \begin{pmatrix}
M(g) & 1 & 0 \\
\vdots & \ddots & \ddots \\
0 & 1 & 1 \\
\vdots & \ddots & \ddots \\
0 & 1 & 1 \\
\end{pmatrix}$$

as the injective hull of $M$. And it's easy to calculate $\Omega^{-1}(M) = M_k/M$.

4. Example

Let $G = \langle x, y, z | x^3 = y^3 = z^3 = (x, z) = (y, z) = 1, (x, y) = z \rangle$ and $F = GF(3)$ then $G$ is the extra-special 3-group, $|G| = 27$ and

<table>
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<tr>
<th>$M$</th>
<th>the dimension of $M$</th>
<th>the dimension of the socle series of $M$</th>
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<tr>
<td>$FG$</td>
<td>27</td>
<td>(1, 2, 4, 4, 5, 4, 4, 2, 1)</td>
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<tr>
<td>$\Omega^{-1}(F)$</td>
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<td>(2, 3, 3, 5, 4, 4, 2, 1)</td>
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<tr>
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<td>(6, 9, 14, 13, 14, 12, 8, 4)</td>
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<td>$\Omega^{-4}(F)$</td>
<td>82</td>
<td>(7, 10, 16, 12, 15, ...)</td>
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References