

On Nilpotent Blocks of Finite Groups

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Let G be a finite group, F be an algebraically closed field of prime characteristic p and B be a p -block of G . If Q is a p -subgroup of G and b is a p -block of $C_G(Q)$ associated with B , we call (Q, b) a B -Brauer pair. When $Q = \langle \pi \rangle$, we call (π, b) a B -Brauer element. In Alperin-Broué [1], the method of the p -local theory for finite groups is generalized to a local theory for p -blocks. In the local theory for p -blocks the Brauer pairs correspond to the p -subgroups, and the B -Brauer elements correspond to the p -elements. Between Brauer pairs, "inclusion" is defined, and if for B -Brauer pair (Q, b_Q) and B -Brauer element (π, b_π) , $(\langle \pi \rangle, b_\pi) \subset (Q, b_Q)$ then we write $(\pi, b_\pi) \in (Q, b_Q)$. G naturally acts on the B -Brauer pairs and on the B -Brauer elements, respectively. Maximal B -Brauer pairs are mutually conjugate and they behave like Sylow p -subgroups of G .

On the other hand, Puig introduced so called pointed group theory in [6]. For any subgroup Q of G , put $(FG)^Q = \{a \in FG \mid ax = xa \text{ for all } x \in Q\}$. A pointed group over FG is any pair Q_δ where Q is a subgroup of G and δ is a equivalence class of primitive idempotents of $(FG)^Q$, and we say that δ is a point of Q over FG . If $Q = \langle u \rangle$, we say that the pair u_δ is a pointed element. For pointed groups Q_δ and P_ε , we write $Q_\delta \subset P_\varepsilon$ and say that Q_δ is contained in P_ε if $Q \subset P$ and

for any $i \in \varepsilon$ there exists $j \in \delta$ such that $ij = j = ji$. If $Q = \langle u \rangle$, then we write $u_\delta \in P_\varepsilon$ whenever $Q_\delta \subset P_\varepsilon$. We say that a pointed group Q_δ is local or that δ is a local point of Q over FG , if Q is a p -subgroup and for $i \in \delta$, the image of i by the Brauer homomorphism from $(FG)^Q$ to $FC_G(Q)$ does not vanish. Thus there is a bijection between the set of local points of Q over FG and the set of isomorphism classes of irreducible $FC_G(Q)$ -modules. G acts on the pointed groups over FG and the maximal local pointed groups contained in $G_{(B)}$ are mutually conjugate. (We must add that pointed groups are defined in more general situation).

Now nilpotent blocks were introduced in Broue-Puig [2] : We say that the block B is nilpotent if for any B -Brauer pair (Q, b_Q) , $N_G(Q, b_Q)/C_G(Q)$ is a p -group. The structure of nilpotent blocks is described in [2] and Puig [7], and in Külshammer [4] which is recently published. Let (D, b) be a maximal B -Brauer pair and D_γ be a maximal local pointed group over FG contained in $G_{(B)}$. If B is nilpotent, then FGB is isomorphic to a full matrix algebra over FD from the main theorem in [7]. Moreover, if B is nilpotent, then we have the following.

(I) The number of conjugacy classes of B -Brauer elements is equal to the number of conjugacy classes of D , that is, for B -Brauer elements $(\pi, b_\pi), (\sigma, b_\sigma) \in (D, b)$, if (π, b_π) and (σ, b_σ) are conjugate, we have $(\pi, b_\pi) = (\sigma, b_\sigma)^d$ where $d \in D$.

(II) For any B -Brauer pair (Q, b_Q) , $l(b_Q) = 1$.

(III) For any local pointed element $u_\varepsilon \in D_\gamma$ and any $x \in G$ such that $(u_\varepsilon)^x \in D_\gamma$ we have $x = zd$ where $z \in C_G(u)$ and $d \in D$.

Here $l(B)$ denote the number of isomorphism classes of irreducible

FG-modules in B . In this report we consider the converse of (I), (II) and (III), respectively.

I

The converse of (I) is true :

Theorem 1. Let B be a p -block of G and (D, b) be a maximal B -Brauer pair. Suppose that for any B -Brauer elements (π, b_π) and $(\sigma, b_\sigma) \in (D, b)$, if (π, b_π) and (σ, b_σ) are conjugate, then we have $(\pi, b_\pi) = (\sigma, b_\sigma)^d$ where $d \in D$. Then B is nilpotent.

Let B_0 be the principal p -block of G . By the Frobenius criterion for p -nilpotent groups and the third main theorem on p -blocks, B_0 is nilpotent if and only if G is p -nilpotent. Let P be a Sylow p -subgroup of G and suppose that for any $\pi, \sigma \in P$, if π and σ are conjugate, then we have $\pi = \sigma^d$ where $d \in P$. Then G is p -nilpotent. So Theorem 1 is a kind of generalization of this well known result to p -blocks.

The following is used to prove Theorem 1.

Lemma 2. Let Z be a central p -subgroup of G and \bar{B} be the p -block of \bar{G} corresponding to B , where $\bar{G} = G/Z$. Then B is nilpotent if and only if \bar{B} is nilpotent.

Sketch of proof of Theorem 1. We prove the theorem by induction

on the order $|G|$ of G . Put $Z = Z(D)$ and $b_Z = b^{C_G(Z)}$. We can show by routine argument that b_Z satisfies the assumption of the theorem. Therefore if $C_G(Z) \neq G$, then b_Z is nilpotent.

In the case $C_G(Z) \neq G$. Let (R, b_R) be a B-Brauer pair such that $(R, b_R) \subset (D, b)$ and (R, b_R) is extremal in (D, b) and that R is a defect group of b_R as a p -block of $RC_G(R)$. Then by [1, Proposition 4.4], $R \supset Z$ and $b_R^{C_G(Z)} = b_Z$. So (R, b_R) can be seen as a b_Z -Brauer pair. Furthermore for $\sigma \in Z$ and $t \in N_G(R, b_R)$, $(\sigma, b_\sigma) \in (R, b_R)$ and $(\sigma, b_\sigma)^t \in (R, b_R) \subset (D, b)$. So by the assumption $N_G(R, b_R) \subset C_G(Z)$, and hence $N_G(R, b_R)/C_G(R)$ is a p -group, because b_Z is nilpotent. Next let (Q, b_Q) be an arbitrary B-Brauer pair and S a defect group of b_Q regarded as a p -block of $QC_G(Q)$ and (S, b_S) be a maximal b_Q -Brauer pair. By replacing (Q, b_Q) by its conjugate if necessary, we may assume $(S, b_S) \subset (D, b)$ and (S, b_S) is extremal in (D, b) . As is shown in just above, $N_G(S, b_S)/C_G(S)$ is a p -group. Since $N_G(Q, b_Q) \subset N_G(S, b_S)C_G(Q)$, $N_G(Q, b_Q)/C_G(Q)$ is a p -group.

In the case $C_G(Z) = G$. Put $\bar{G} = G/Z$ and \bar{B} be the p -block of \bar{G} corresponding to B . We can show that \bar{B} satisfies the assumption of the theorem. Since we may assume that $D \neq 1$, \bar{B} is nilpotent by the induction hypothesis. Therefore by Lemma 2, B is nilpotent. Thus the proof is complete.

II

In Puig [7, 1.9] he says that the converse of (II) is probably true. A partial answer for his question is given.

Proposition 3. Let G be a p -solvable group and B be a p -block of G . If $l(b_Q) = 1$ for any B -Brauer pair (Q, b_Q) , then B is nilpotent.

We can show that if Alperin's weight conjecture is true, then the converse of (II) is true. For a p -block B and a p -subgroup Q of G , denote by $\text{Bl}(N_G(Q), B)$ the set of p -blocks of $N_G(Q)$ associated with B and denote by $l_B(Q)$ the number of isomorphism classes of irreducible FG -modules in B with vertex Q . By Okuyama [5, Theorem], Alperin's conjecture is true for p -solvable groups as follows.

Lemma 4. Let G be a p -solvable group and B be a p -block of G . Then the following hold.

(i) We have $l(B) = \sum_Q \sum_{b \in \text{Bl}(N_G(Q), B)} l_b(Q)$, where Q ranges over a complete set of representatives for the conjugacy classes of p -subgroups of G .

(ii) Let P be a normal p -subgroup of G . then we have $l(B) = \sum_Q \sum_{b \in \text{Bl}(N_G(Q), B)} l_b(Q)$, where Q ranges over a complete set of representatives for the conjugacy classes of p -subgroups of G containing P .

III

The converse of (III) is true :

Theorem 5. Let B be a p -block of G and D_γ be a maximal local

pointed group over FG contained in $G_{\{B\}}$. Suppose that for any local pointed element $u_{\varepsilon} \in D_{\gamma}$ and any element $x \in G$ such that $(u_{\varepsilon})^x \in D_{\gamma}$ we have $x = zd$ where $z \in C_G(u)$ and $d \in D$. Then B is nilpotent.

The above result is due to L. Puig. He proves it by the same argument as in the proof of Theorem 1. But in this time, ε is no longer uniquely determined by γ and u , and hence its proof is more complicated than Brauer element case. In particular it is not easy to show that \bar{B} satisfies the assumption. Section 3 in Külshammer-Puig [3] satisfactorily meets the requirement.

The detail version of this report will probably be published elsewhere.

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