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Kyoto University
Perfect Isometries for Blocks with Abelian Defect

Groups and Klein Four Inertial Quotients

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1. Alperin's weight conjecture for the case of abelian defect groups

Let $p$ be a prime number, $k$ an algebraically closed field of characteristic $p$, $O$ a complete discrete valuation ring with residue field $k$ and quotient field $K$ of characteristic zero, $G$ a finite group, $b$ a $p$-block of $G$ (i.e. a primitive idempotent of $Z(kG)$), $P$ a defect group of $b$, $e$ a root of $b$ in $C_G(P)$ and $E$ the inertial quotient $N_G(P,e)/P C_G(P)$. We assume that $K$ is large enough.

Alperin's weight conjecture states that the number $l(b)$ of isomorphism classes of simple $kG_b$-modules can be calculated by the function of its local structure. When $P$ is abelian, this is equivalent to the following one. ([1])

Conjecture 1. If $P$ is abelian, then $l(b)$ is the number of isomorphism classes of simple $kN_G(P,e)e$-modules.
This is known to be true if \(|E| \leq 3\) by the results of Brauer (cf. [3], Proposition (6G)) and Usami [13] (except the case \(|E| = 3\) and \(p = 2\)). Here we introduce the result which proves it in the case \(E\) is a Klein four group (and in the case \(|E| = 3\) and \(p = 2\)).

2. Reformed conjecture

First we want to reform Conjecture 1 in terms of a suitable \(k^*\)-central extension of \(E\). Setting \(\overline{N}_G(P,e) = N_G(P,e)/P\), \(\overline{C}_G(P) = C_G(P)/P\) and denoting by \(\overline{e}\) the image of \(e\) in \(k\overline{C}_G(P)\), it is well known from Brauer that \(k\overline{C}_G(P)\overline{e}\) is a simple \(k\)-algebra (i.e. a full matrix algebra over \(k\)) and, in particular, we have \(Z(k\overline{C}_G(P)\overline{e}) \cong k\); hence, by Skolem-Noether’s theorem, we have an exact sequence

\[
1 \rightarrow k^* \rightarrow (k\overline{C}_G(P)\overline{e})^* \xrightarrow{\pi} \text{Aut}(k\overline{C}_G(P)\overline{e}) \rightarrow 1
\]

so that \((k\overline{C}_G(P)\overline{e})^*\) can be seen as a \(k^*\)-central extension. Since \(N_G(P,e)\) acts on \(k\overline{C}_G(P)\overline{e}\), we have a group homomorphism \(\hat{\phi}: \overline{N}_G(P,e) \rightarrow \text{Aut}(k\overline{C}_G(P)\overline{e})\) and then \(\hat{\overline{N}}_G(P,e)\) is the \(k^*\)-central extension of \(\overline{N}_G(P,e)\) induced by \((k\overline{C}_G(P)\overline{e})^*\): that is to say, \(\hat{\overline{N}}_G(P,e)\) is the subgroup of

\[(\overline{a}, \overline{n}) \in (k\overline{C}_G(P)\overline{e})^* \times \overline{N}_G(P,e)\]

such that \(\pi(\overline{a}) = \hat{\phi}(\overline{n})\) and we get a commutative and exact diagram

\[
\begin{array}{ccc}
1 & \rightarrow & k^* & \rightarrow & (k\overline{C}_G(P)\overline{e})^* & \xrightarrow{\pi} & \text{Aut}(k\overline{C}_G(P)\overline{e}) & \rightarrow & 1 \\
\uparrow{id} & & \uparrow{\hat{\phi}} & & \uparrow{\pi} & & \uparrow{\hat{\phi}} & & \uparrow{\pi} \\
1 & \rightarrow & k^* & \rightarrow & \hat{\overline{N}}_G(P,e) & \rightarrow & \overline{N}_G(P,e) & \rightarrow & 1
\end{array}
\]

Now, the twisted algebra \(k^*\hat{\overline{N}}_G(P,e)\) is the quotient of the full group algebra by the ideal generated by the elements \(\lambda(\overline{a}, \overline{n}) - (\lambda \overline{a}, \overline{n})\) where \(\lambda\) runs over \(k^*\) and \((\overline{a}, \overline{n})\) over \(\hat{\overline{N}}_G(P,e)\). (We can
define $0_* \hat{N}_G(P,e)$, since there is a unique section $k^* \rightarrow 0^*$ of the canonical homomorphism $0^* \rightarrow k^*$. Moreover, we have an injective group homomorphism

$$\hat{C}_G(P) \rightarrow \hat{N}_G(P,e)$$

mapping $\tilde{z} \in \hat{C}_G(P)$ on $(\tilde{z}e, \tilde{z}) \in \hat{N}_G(P,e)$ and its image is a normal subgroup of $\hat{N}_G(P,e)$ intersecting trivially the image of $k^*$, so the corresponding quotient is a $k^*$-central extension of $E$. We denote by $\hat{E}$ the opposite one; that is to say, denoting by $\hat{N}_G(P,e)^*$ the set $\hat{N}_G(P,e)$ endowed with opposite product, we have the exact sequence

$$1 \rightarrow \hat{C}_G(P) \rightarrow \hat{N}_G(P,e)^* \rightarrow \hat{E} \rightarrow 1$$

where $\tilde{z} \in \hat{C}_G(P)$ maps on $(\tilde{z}e, \tilde{z})^{-1}$. The following more or less known lemma explains the role of $\hat{E}$ (see also [9], Proposition 14.6 in [11], Proposition 2.1 in [10] and Lemma 2.5 in [12]).

**Lemma 1.** With the notation above, there is an algebra isomorphism

$$k\hat{N}_G(P,e)e \cong k\hat{C}_G(P)e \otimes_{k^*} \hat{E}$$

mapping $\tilde{ne}$ on $\hat{f}(\tilde{n}) \otimes_{\hat{E}} (\tilde{n})^{-1}$, where $\tilde{n} \in \hat{N}_G(P,e)$ and $\tilde{n}$ is an element of $\hat{N}_G(P,e)$ lifting $\tilde{n}$.

Let $\hat{L}$ be the semidirect product of $\hat{E}$ and $P$. Since the number of isomorphism classes of simple $k_\hat{L}$-modules is equal to the number of isomorphism classes of simple $k_*\hat{E}$-modules, we can reform Conjecture 1 by Lemma 1 as follows.

**Conjecture 2.** If $P$ is abelian, then $1(b)$ is the number of
isomorphism classes of simple $k_\hat{L}$-modules.

Hence we must study the relation between $OG\hat{b}$ and $O_\hat{L}$, where $\hat{b}$ denotes the unique primitive idempotent lifting $b$ to $Z(OG)$. We denote respectively by $L_\hat{K}(\hat{L})$ and $L_\hat{K}(G,b)$ the Grothendieck groups of the categories of $K_\hat{L}$-modules and ordinary $K$-representations of $G$ in $b$. We expect that there exists a special kind of bijective isometry between $L_\hat{K}(\hat{L})$ and $L_\hat{K}(G,b)$.

3. Preliminaries and the main theorem

Following [2] and [6], we consider Brauer morphism $Br_\mathbb{Q}$ for a $p$-subgroup $Q$ of $G$ and $(b,G)$-Brauer pairs. Note that $(P,e)$ is a maximal $(b,G)$-Brauer pair and for a $p$-subgroup $Q$ of $P$, $(Q,e^{C_G(Q)})$ is a $(b,G)$-Brauer pair contained in $(P,e)$. One of the typical properties of blocks with abelian defect groups is the following one.

Lemma 2. (Proposition 4.21 in [2]) Assume that $P$ is abelian. If $(Q,f)$ is a $(b,G)$-Brauer pair such that $(Q,f) \subseteq (P,e)$ and $x$ an element of $G$ such that $(Q,f)^x \subseteq (P,e)$, then there are $z \in C_G(Q)$ and $n \in N_G(P,e)$ such that $x = zn$. In particular, if $U$ is a set of representatives for the orbits of $E$ in $P$, then
\[
\{ (u, e^{C_G(u)}) \}_{u \in U}
\]
is a set of representatives for the conjugacy classes of $(b,G)$-Brauer elements.

It is not difficult to handle $O_\hat{L}$, since there are a finite
subgroup $L'$ of $\hat{L}$ and a $p$-block $b'$ of $L'$ such that the inclusion $L' \subset L$ induces a bijective isometry $L_K(\hat{L}) = L_K(L', b')$ and an algebra isomorphism $O_\hat{L} \cong OL' \hat{b}'$ (see Remark 5 in section 1 in [8], Lemma 5.5 and Proposition 5.15 in [11]). Furthermore P is also a defect group of $b'$ and E is also the inertial quotient of $b'$, since P is the normal Sylow $p$-subgroup of $L'$ and $(P, Br_p(b'))$ is the unique maximal $(b', L')$-Brauer pair. We remark that (3.1) $(Q, Br_Q(b'))$ is the unique $(b', L')$-Brauer pair for a fixed $p$-subgroup $Q$ of $P$.

From now on we introduce some general notation and results without any hypothesis until we state Theorem 1 (i.e. $E$ is arbitrary and we do not assume that $P$ is abelian for the moment). We denote by $CF_K(G)$ and $CF_0(G)$ the sets of $K$- and $O$-valued central functions over $G$, so that $CF_0(G) \subset CF_K(G)$, and we identify $CF_K(G)$ with $K \otimes O CF_0(G)$ and with the set of central $K$-linear forms over $KG$ (or $OG$). We denote respectively by $L_K(G)$ and $L_k(G)$ the Grothendieck groups of the categories of $KG$- and $kG$-modules (of finite dimension) and we identify $L_K(G)$ with its image in $CF_0(G)$. We also identify any element of $L_k(G)$ with its Brauer character; that is to say, denoting respectively by $BCF_K(G)$ and $BCF_0(G)$ (BCF for "Brauer central function") the sets of $K$- and $O$-valued $G$-central functions over the set $G_p$ of elements of $G$ of order prime to $p$, we also identify $L_k(G)$ with its image in $BCF_0(G)$ and $BCF_K(G)$ with $K \otimes O BCF_0(G)$. Recall that the inclusion $L_k(G) \subset BCF_0(G)$ induces an isomorphism

(3.2) $0 \otimes L_k(G) \cong BCF_0(G)$.

Following Brauer, we denote by
\[ d_G : \text{CF}_K(G) \rightarrow \text{BCF}_K(G) \]

the restriction map, which fulfills

(3.3) \[ d_G(L_K(G)) = L_K(G). \]

Moreover we denote by \( \text{CF}_K^0(G) \) the kernel of \( d_G \) and set \( L_K^0(G) = \text{CF}_K^0(G) \cap L_K(G) \). It is clear that \( d_G \) induces a bijection between the orthogonal subspace of \( \text{CF}_K^0(G) \) and \( \text{BCF}_K(G) \), and then the inverse map determines a section of \( d_G \)

\[ e_G : \text{BCF}_K(G) \rightarrow \text{CF}_K(G) \]

and induces an scalar product on \( \text{BCF}_K(G) \); thus, \( d_G \) and \( e_G \) become adjoint maps.

More generally, following Broué [4], for any \( p \)-element \( u \) of \( G \), we consider the "twisted" restriction

\[ d_G^u : \text{CF}_K(G) \rightarrow \text{BCF}_K(C_G(u)) \]

mapping \( \chi \in \text{CF}_K(G) \) on the \( C_G(u) \)-central function over \( C_G(u)_p \), which maps \( s \in C_G(u)_p \) on \( \chi(us) \), and denote by

\[ e_G^u : \text{BCF}_K(C_G(u)) \rightarrow \text{CF}_K(G) \]

the adjoint \( K \)-linear map, which is a section of \( d_G^u \).

It is well-known that any idempotent of \( Z(kG) \) determines a selfadjoint projector over \( \text{CF}_K(G) \) which stabilizes \( \text{CF}_0(G) \) and \( L_K(G) \), and commutes with \( e_G d_G \), so that it determines a self-adjoint projector over \( \text{BCF}_K(G) \) stabilizing \( \text{BCF}_0(G) \) and \( L_K(G) \).

In particular, for any element \( \chi \) of \( \text{CF}_K(G) \) or \( \text{BCF}_K(G) \), we denote by \( b.\chi \) the image of \( \chi \) by the projector determined by \( b \) and set

\[ b.\text{CF}_K(G) = \text{CF}_K(G,b) \quad \text{and} \quad b.\text{BCF}_K(G) = \text{BCF}_K(G,b). \]

Moreover, for any \( p \)-element \( u \) of \( G \), we have (cf. [4] Appendixes)

(3.4) \[ d_G^u(b.\chi) = Br_u^b(d_G^u(\chi)) \quad \text{and} \quad e_G^u(Br_u(b.\varphi)) = b.e_G^u(\varphi) \]

for any \( \chi \in \text{CF}_K(G) \) and any \( \varphi \in \text{BCF}_K(C_G(u)) \) ( where \( Br_u = Br_{\langle u \rangle} \)).
Consequently, for any \( \chi \in CF_K(G, b) \) and any \((b, G)\)-Brauer element \((u, g)\) we consider the central function
\[
\chi^{(u, g)} = e^u_g \cdot d^u_g(\chi)
\]
which still belongs to \( CF_K(G, b) \). Notice that we have
\[
\chi^{(u, g)}(u) = \chi(u\widehat{g}),
\]
where \( \widehat{g} \) is the unique primitive idempotent of \( Z(O\mathcal{C}_G(u)) \) lifting \( g \). We remark that
\[
(3.5) \quad \chi = \sum_{(u, g)} \chi^{(u, g)}
\]
and for any \( \chi, \chi' \in CF_K(G, b) \) we get
\[
(3.6) \quad (\chi, \chi')_G = \sum_{(u, g)} (\chi^{(u, g)}, \chi'^{(u, g)})_G
\]
where \((u, g)\) runs over a set of representatives for the conjugacy classes of \((b, G)\)-Brauer elements.

Following [6], a central function \( \lambda \) over \( P \) is called \((G, e)\)-stable if, for any \((b, G)\)-Brauer element \((u, g)\) such that \((\langle u \rangle, g) \subset (P, e)\) and any \( x \in G \) such that \((\langle u^x \rangle, g^x) \subset (P, e)\), we have \( \lambda(u^x) = \lambda(u) \). In that case, for any \( \chi \in CF_K(G, b) \), we consider the new central function
\[
\lambda \ast \chi = \sum_{(u, g)} \lambda(u) \chi^{(u, g)}
\]
where \((u, g)\) runs over a set of representatives such that \((\langle u \rangle, g) \subset (P, e)\) for the conjugacy classes of \((b, G)\)-Brauer elements, which still belongs to \( CF_K(G, b) \) and does not depend on the choice of the set of representatives. We remark that
\[
g \cdot d^u_g(\lambda \ast \chi) = \lambda(u) (g \cdot d^u_g(\chi)).
\]
Then, by the main result in [6], if \( \lambda \) and \( \chi \) are generalized characters, so is \( \lambda \ast \chi \). Notice that, by Lemma 2, if \( P \) is
abelian, a central function over $P$ is $(G,e)$-stable if and only if it is $E$-stable. We denote by $\text{CF}_0(P)^E$ the $0$-module of $E$-stable 0-valued central functions over $P$.

We are ready to state our main theorem (Theorem 1.5 in [12]).

Theorem 1. With the notation above, assume that $P$ is abelian and $E$ is a Klein four group. Then there is a bijective isometry

$$\Delta : \text{CF}_0(\hat{L}) \rightarrow \text{CF}_0(G,b)$$

such that

$$\Delta (L_K(\hat{L})) = L_K(G,b)$$

and

$$(3.7) \quad \Delta (\lambda \ast \eta) = \lambda \ast \Delta (\eta)$$

for any $\lambda \in \text{CF}_0(P)^E$ and any $\eta \in \text{CF}_0(\hat{L})$.

$(3.7)$ implies that $\Delta$ fulfills Definition 4.3 in [5] (i.e. (3.7) guarantees the existence of a local system in Broué's terms) and therefore, by Lemma 4.5 in [5], $\Delta$ is a perfect isometry in Broué's terms. (We discuss perfect isometries in section 5.) By Proposition 1.3 and Theorem 1.5 in [5] we have a following corollary.

Corollary 1. If $P$ is abelian and $E$ is a Klein four group, then the following hold with the notation of Theorem 1.

(i) $\Delta$ is a perfect isometry from $L_K(\hat{L})$ onto $L_K(G,b)$.

(ii) $\Delta$ induces a bijective isometry from $L_K(\hat{L})$ onto $L_K(G,b)$ and hence Alperin's weight conjecture (Conjecture 2) holds in this case.
(iii) The algebra isomorphism $\Delta^*$ from $Z(K^\wedge_{\mathbb{Q}})$ onto $Z(KG^\wedge)$ determined by the isometry $\Delta$ maps $Z(O^\wedge_{\mathbb{Q}})$ onto $Z(OG^\wedge)$.

(iv) $\Delta$ preserves the height of irreducible ordinary characters. In particular, all the irreducible ordinary characters of $G$ in $b$ have height zero and Alperin-McKay conjecture holds in this case.

(v) $\Delta$ preserves the elementary divisors of the Cartan matrices.

4. Local systems for blocks with abelian defect groups

Before we introduce Theorem 2 in this section, we assume only that $P$ is abelian (and $E$ is arbitrary).

By Lemma 2, $E$ controls the fusion of $(b,G)$-Brauer pairs (resp. $(b',L')$-Brauer pairs). Then in the summation of (3.5) we have only to make $u$ run over $U$.

Applying inductive method, we hope to construct a bijective isometry $\Delta_Q$ from $\text{CF}_0(L)^Q$ onto $\text{CF}_0(G)^Q$, for each $p$-subgroup $Q$ of $P$ where $f = e^Q_{C_G(Q)}$. (That is to say, first we construct it for $Q = P$ and we hope to construct it for $Q = 1$ finally.) We note that $(P,e)$ is also a maximal $(f,C_G(Q))$-Brauer pair and $(u,g)$ is a $(f,C_G(Q))$-Brauer element contained in it for any element $u$ of $P$, where $g = e^Q_{C_G(Q)}$. By Lemma 2, $C_E(Q)$ controls the fusion of $(f,C_G(Q))$-Brauer pairs, and by (3.5) for any $\chi \in \text{CF}_K(C_G(Q),f)$

\begin{equation}
\chi = \sum_{u \in U_Q} e^u_{C_G(Q)} (g.d^u_{C_G(Q)}(\chi))
\end{equation}
where $U_Q$ is a set of representatives for the orbits of $C_E(Q)$ in $P$. Similarly we note that $(P, Br_P(b'))$ is the maximal $(Br_Q(b'), C_L(Q))$-Brauer pair and $(u, Br_Q(u))(b')$ is a $(Br_Q(b'), C_L(Q))$-Brauer element contained in it for any element $u$ of $P$ by (3.1).

By Lemma 2 and (3.5) for any $\eta \in CF_K(C_L(Q))$

$$\eta = \sum_{u \in U_Q} e_{C_L(Q)}^u (d_{C_L(Q)}^u(\eta)),$$

since we can omit $Br_Q(u)(b')$ by (3.1) and (3.4).

Let $X$ be an $E$-stable non-empty set of subgroups of $P$ and assume that $X$ contains any subgroup of $P$ containing an element of $X$. We call any map $\gamma$ $(G,b)$-local system over $X$, if $\gamma$ is defined over $X$ and sends $Q \in X$ to a bijective isometry

$$\gamma_Q : BCF_K(C_L(Q)) \cong BCF_K(C_G(Q), f)$$

where $f = e_{C_G(Q)}$, and fulfills the following two conditions:

(4.3) For any $Q \in X$, any $\eta \in BCF_K(C_L(Q))$ and any $s \in E$, we have

$$\gamma_Q^s(\eta) = \gamma_Q^s(\eta).$$

(4.4) For any $Q \in X$ and any $\eta \in L_K(C_L(Q))$, the sum

$$\sum_{u \in U_Q} e_{C_G(Q)}^u (\gamma_Q^u(d_{C_L(Q)}^u(\eta)))$$

is a generalized character of $C_G(Q)$.

We examine these conditions to give more explicit expression.

For any $Q \in X$ and any $\eta \in CF_K(C_L(Q))$, the sum

$$\Delta_Q^s(\eta) = \sum_{u \in U_Q} e_{C_G(Q)}^u (\gamma_Q^u(d_{C_L(Q)}^u(\eta)))$$

is certainly an element of $CF_K(C_G(Q), f)$ (cf.(3.4), (4.1) and (4.2)), and we have, setting $g = e_{C_G(Q)}(u)$,
(4.6) \[ \Delta_Q(\eta)^{(u,g)} = e^u_{C_G(Q)}(\Gamma_Q \omega) d^u_{C^\wedge_L(Q)}(\eta) \]

and therefore, for any \( \eta' \in CF_K(C^\wedge_L(Q)) \) we get (cf.(3.6))

\[ (\Delta_Q(\eta), \Delta_Q(\eta'))_{C_G(Q)} \]
\[ = \sum_{u \in U_Q} (d^u_{C^\wedge_L(Q)}(\eta), d^u_{C^\wedge_L(Q)}(\eta'))_{C^\wedge_L(Q)}(\omega_u) \]
\[ = (\eta, \eta')_{C^\wedge_L(Q)} \]

(recall that \( e^u_{C_G(Q)} \) and \( e^u_{C^\wedge_L(Q)} \) are isometries!). Hence for any \( Q \in X \) we get a bijective isometry

(4.7) \[ \Delta_Q = \sum_{u \in U_Q} e^u_{C_G(Q)} \Gamma_Q \omega_u d^u_{C^\wedge_L(Q)} \]

from \( CF_K(C^\wedge_L(Q)) \) onto \( CF_K(C_G(Q), f) \) and condition (4.3) insures that \( \Delta_Q \) does not depend on the choice of \( U_Q \) whereas condition (4.4) demands that \( L_K(C^\wedge_L(Q), f) \) contains \( \Delta_Q(L_K(C^\wedge_L(Q))) \) which actually implies the equality

(4.8) \[ \Delta_Q(L_K(C^\wedge_L(Q))) = L_K(C_G(Q), f) \]

since both members have orthonormal basis of the same cardinal (cf.(4.7)). Moreover, notice that \( d_{C_G(Q)} \circ \Delta_Q = \Gamma_Q \omega d_{C^\wedge_L(Q)} \) (cf. (4.5)) and therefore we get (cf.(3.3) and (4.8))

\[ \Gamma_Q(L_K(C^\wedge_L(Q))) = L_K(C_G(Q), f) \]

which then implies (cf.(3.2))

(4.9) \[ \Gamma_Q(BCF_0(C^\wedge_L(Q))) = BCF_0(C_G(Q), f). \]

Consequently, since (4.9) is true for any \( R \in X \) and the maps \( d^u_{C^\wedge_R(Q)} \) and \( e^u_{C_G(Q)} \) send \( 0 \)-valued functions to \( 0 \)-valued functions, we have for any \( Q \in X \)

(4.10) \[ \Delta_Q(CF_0(C^\wedge_L(Q))) = CF_0(C_G(Q), f). \]
An immediate consequence of the definition of \( \Delta_Q \), which does not depend on conditions (4.3) and (4.4), is that for any \( Q \in X \), any \( \lambda \in CF_K(P)_{C_E(Q)} \) and any \( \eta \in CF_K(C_L(Q)) \) we have

\[
\Delta_Q(\lambda \ast \eta) = \lambda \ast \Delta_Q(\eta).
\]

These (4.8), (4.10) and (4.11) show that for any \( Q \in X \), \( \Delta_Q \) (for \( C_L(Q) \) and \( C_G(Q), f \)) fulfills the similar conditions to \( \Delta \) (for \( \widehat{L} \) and \( (G,b) \)) in Theorem 1. (Hence, if \( 1 \in X \), then \( \Delta = \Delta_1 \) is a required one in Theorem 1.) Since \( C_L(P) \cong k^* \times P \), we can easily prove that there are exactly two \((G,b)\)-local systems defined over \( \{P\} \) (cf. (4.11)). (Notice that, up to sign, there is just one possibility for the isometry \( \Gamma_P \).

We want to extend \( X \) and \( \Gamma \) step by step. Assume that \( X \) does not contain all the subgroups of \( P \) and let \( Q \) be a subgroup of \( P \) which is maximal such that \( Q \not\in X \). We will discuss now a necessary and sufficient condition to extend \( \Gamma \) to a \((G,b)\)-local system \( \Gamma' \) over the union \( X' \) of \( X \) and the \( E \)-orbit of \( Q \). Since any subgroup \( R \) of \( P \) properly containing \( Q \) belongs to \( X \), for any \( u \in P - Q \) we still have the map (as in (4.6))

\[
e^u_{C_G(Q)} \ast \Gamma_Q \ast d^u_{C_L(Q)} : CF_K(C_L(Q)) \longrightarrow CF_K(C_G(Q), f)
\]

where \( f = e \). We consider the sum

\[
\Delta_Q^\circ = \sum_{u \in U_Q - Q} e^u_{C_G(Q)} \ast \Gamma_Q \ast d^u_{C_L(Q)}
\]

where, as above, \( U_Q \) is a set of representatives for the orbits of \( C_E(Q) \) in \( P \) and by condition (4.3) again, \( \Delta_Q^\circ \) does not depend on the choice of \( U_Q \).

Denote by \( \bar{f} \) the image of \( f \) in \( kC_G(Q) \), where we set \( \bar{C}_G(Q) = \)...
\( \mathcal{C}_G(Q)/Q \). We also set \( \mathcal{C}_L(Q) = \mathcal{C}_L(Q)/Q \). In [12] we proved following propositions ( Proposition 3.7 and Proposition 3.11 ).

Proposition 1. With the notation and the hypothesis above, \( \Delta^*_Q \) induces a bijective isometry

\[
\Delta^*_Q : \text{CF}_K(\mathcal{C}_L(Q)) \cong \text{CF}_K(\mathcal{C}_G(Q), \mathcal{f})
\]
such that \( \Delta^*_Q(L_K(\mathcal{C}_L(Q))) \subset L_K(\mathcal{C}_G(Q), \mathcal{f}) \).

Proposition 2. With the notation and the hypothesis above, the \((G,b)\)-local system \( \Gamma \) over \( X \) can be extended to a \((G,b)\)-local system \( \Gamma' \) over \( X' \) if and only if the bijective isometry \( \Delta^*_Q \) can be extended to a \( N_E(Q) \)-stable bijective isometry

\[
\Delta_Q : \text{CF}_K(\mathcal{C}_L(Q)) \cong \text{CF}_K(\mathcal{C}_G(Q), \mathcal{f})
\]
such that \( \Delta_Q(L_K(\mathcal{C}_L(Q))) = L_K(\mathcal{C}_G(Q), \mathcal{f}) \).

Now we try to extend \( \Delta^*_Q \) to a \( N_E(Q) \)-stable bijective isometry \( \Delta_Q \). When \( E \) is a Klein four group, we obtain the following slightly stronger theorem (Theorem 4.2 in [12]) than Theorem 1. Unfortunately we do not succeed when \( E \) is cyclic of order 4.

Theorem 2. If \( P \) is abelian and \( E \) is a Klein four group, then there is a \((G,b)\)-local system over the set of all the subgroups of \( P \).

5. Perfect isometry
In this section we introduce some Broué's terms. Let \((H, f)\) (resp. \((H', f')\)) be a pair of a finite group \(H\) and its block \(f\) (resp. a finite group \(H'\) and its block \(f'\)).

Definition 1 (Definition 1.4 and Proposition 4.1 in [5]). A bijective isometry \(I\) from \(L_K(H, f)\) onto \(L_K(H', f')\) is called a perfect isometry if it induces a bijection from \(CF_0(H, f)\) onto \(CF_0(H', f')\) and a bijection from \(BCF_K(H, f)\) onto \(BCF_K(H', f')\). (We can extend \(I\) \(K\)-linearly.)

Such special kind of bijective isometry has various properties as follows.

Proposition 3 (Proposition 1.3 and Theorem 1.5 in [5]). If \(I\) is a perfect isometry from \(L_K(H, f)\) onto \(L_K(H', f')\), then the following hold.

(i) \(I\) induces a bijective isometry from \(L_K(H, f)\) onto \(L_K(H', f')\) and then \(l(f) = l(f')\).

(ii) \(I\) induces a bijective isometry from the \(Z\)-module generated by the characters of projective \(OH\hat{f}\)-modules onto the \(Z\)-module generated by the characters of projective \(OH'\hat{f}'\)-modules.

(iii) The bijection between primitive idempotents of \(ZKH\hat{f}\) and \(ZKH'\hat{f}'\) defined by \(I\) induces an algebra isomorphism between \(ZOH\hat{f}\) and \(ZOH'\hat{f}'\).

(iv) \(I\) preserves the height of irreducible ordinary characters and the elementary divisors of the Cartan matrices.
Let \((P, f_P)\) be a maximal \((f, H)\)-Brauer pair and for any \(p\)-subgroup \(Q\) of \(P\), let \((Q, f_Q)\) be a \((f, H)\)-Brauer pair contained in it.

**Definition 2.** Let \(\text{Br}_f(H)\) be the category whose objects are \((f, H)\)-Brauer pairs and whose morphisms from \((Q, f_Q)\) to \((R, f_R)\) are the homomorphisms from \(Q\) to \(R\) induced by the inner automorphisms of \(G\) which send \((Q, f_Q)\) onto a pair contained in \((R, f_R)\). This is called the Brauer category of \(H\) for \(f\).

**Hypothesis for pairs \((H, f)\) and \((H', f')\) (Hypothesis 4.2 in [5]).** We suppose that \(P\) is a defect group of \(f\) and \(f'\). We also suppose that the inclusions of \(P\) in \(H\) and \(H'\) induce a equivalence of Brauer categories \(\text{Br}_f(H)\) and \(\text{Br}_{f'}(H')\).

**Definition 3 (Definition 4.3 in [5]).** With the above Hypothesis, a linear map \(I\) from \(\text{CF}_K(H, f)\) to \(\text{CF}_K(H', f')\) is called compatible with the fusion, if for every cyclic subgroup \(<u>\) of \(P\), there exists a linear map \(I_{p'}^{<u>}\) from \(\text{BCF}_K(C_H(u), f'_{<u>})\) onto \(\text{BCF}_K(C_{H'}(u), f'_{<u>})\) such that

\[
(f'_{<u>} \cdot d^u_{H'}) \circ I = I_{p'}^{<u>} \circ (f'_{<u>} \cdot d^u_{H'}).
\]

Here the family \(\{ I_{p'}^{<u>} \mid <u> \subseteq P \}\) is called the local system of \(I\).

**Definition 4 (Definition 4.6 and "good definition" in its
Remark in [5]). We say that the pair \((H,f)\) and \((H',f')\) are the same type (in "good definition"), if the following conditions are satisfied.

(i) The Brauer categories \(\text{Br}_f(H)\) and \(\text{Br}_{f'}(H')\) are equivalent.

(ii) There exists a family of perfect isometries

\[
\{I^Q : L_K(C_H(Q),f_Q) \longrightarrow L_K(C_{H'}(Q),f'_Q) \mid Q \subseteq P\}
\]

such that if for any \(Q\) we denote by

\[
I^Q_p' : BCF_K(C_H(Q),f_Q) \longrightarrow BCF_K(C_{H'}(Q),f'_Q)
\]

the map induced by \(I^Q\), then \(I^Q\) is compatible with the fusion and its local system is

\[
\left\{I^Q_p' \mid \langle u \rangle \subseteq C_p(Q)\right\}.
\]

Broué conjectured that if \(b\) has an abelian defect group \(P\), and \((P,e)\) is a maximal \((b,G)\)-Brauer pair, then \((G,b)\) and \((N_G(P,e),e)\) are the same block type (Conjecture 6.1 in [5]).

By Lemma 2 \(\text{Br}_b(G)\) and \(\text{Br}_e(N_G(P,e))\) are equivalent. Notice that by (4.5) for any \(p\)-subgroup \(Q\) of \(P\) and any \(u \in U_Q\) we have

\[
g^{-d_{C_G(Q)}} \Delta_Q = \Gamma_{Q,\langle u \rangle} \circ d_{C_L(Q)}
\]

and in particular, \(\Gamma_Q\) is the restriction of \(\Delta_Q\) to \(BCF_K(C_L(Q))\).

Then by (4.8), (4.10) and Theorem 2, this conjecture holds when \(E\) is a Klein four group (and it also holds when \(|E| \leq 3\)).
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