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ON QF-3 ALGEBRAS OF FINITE REPRESENTATION TYPE

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1. Introduction and Preliminaries

The purpose of this report is to give brief outline of the classification, the structure and the construction of QF-3 algebras finite-represention type. Throughout this report, an algebra means a finite dimensional algebra over algebraically closed field K.

Further we assume an algebra is representation-finite. Of course, we may assume an algebra is basic and connected. We sometimes have to consider a special kind of over-algebra, which is not an algebra of usual sense, when we take a covering of an algebra. The overalgebra is characterized as a locally bounded or locally representation finite K-category [3]. Here we distinguish algebras and K-category. The method in which we use these notions is called *a covering technique*. So we review the definitions.

Let $\overline{Q}$ be a locally finite quiver (i.e. quiver in which finite many arrows start or end at each vertex) with a relation $\overline{I}$. Here $\overline{I}$ is a ideal of $K\overline{Q}$ and $\overline{I} \in K\overline{Q}^2$. When $\overline{Q}$ is a finite quiver, we write Q and I instead of $\overline{Q}$ and $\overline{I}$ respectively.

Definition 1.1. (Locally bounded [3])

Let $\overline{Q}$ be a locally finite quiver and $\overline{I}$ be a set of relations in the path algebra $K\overline{Q}$

whose length is equal or more than 2 .

K-category $\overline{R} = K\overline{Q}/\overline{I}$ is called a locally bounded K-category if the following conditions are satisfied for any idempotent e corresponding to the vertices of $\overline{Q}$:

(1) $\overline{R}$ is basic (i.e., every e is non-isomorphic).

(2) $e\overline{R}e$ is a local ring.

(3) $\dim_K \overline{R}e, \dim_K e\overline{R}$ is finite.

We denote $(\overline{Q}, \overline{I})$ the a locally bounded K-category $K\overline{Q}/\overline{I}$.

Let $\mathrm{Aut}(\overline{Q}, \overline{I})$ be the quiver automorphism group of $\overline{Q}$ which induces K-linear automorphism of $\overline{I}$. Clearly this induces K-automorphism of K-category $K\overline{Q}/\overline{I}$. For $G < \mathrm{Aut}(\overline{Q}, \overline{I})$, we denote $(\overline{Q}, \overline{I}, G)$ the K-category $K\overline{Q}/\overline{I}$ with an K-automorphism group G. Next we define the notion of Galois covering of an algebra.

Definition 1.2. (Galois covering [3])

A Galois covering of an algebra $R = (Q, I)$ with a galois group G is an locally bounded K-category $\overline{R} = (\overline{Q}, \overline{I})$ with a K-linear functor F: $(\overline{Q}, \overline{I}) \to (Q, I)$ and $G < \mathrm{Aut}(\overline{Q}, \overline{I})$ satisfying

(1) $F = Fg$ for any $g \in G$.

(2) The orbit $(\overline{Q}, \overline{I})/G = (Q, I)$.

(3) G acts freely on $(\overline{Q}, \overline{I})$.

(4) For each vertex $x \in \overline{Q}_0$, $\quad a \in Q_0$

$$\sum_{F(y)=a} y\overline{R}x = aR \cdot F(x), \qquad \sum_{F(y)=a} x\overline{R}y = F(x) \cdot Ry.$$

The advantage we consider Galois covering is that this has simple structure and reflects the structure of the original algebra. The simply connected algebra is the typical one that we can find all the indecomposable modules and irreducible maps. This notion is originally defined in [3], but we adopt the result due to [1] as the definition.

Definition 1.3. (Simply connected [1])

$\overline{R} = K\overline{Q}/\overline{I}$ is called simply connected if $\overline{R}$ has separated radical.

(i.e.,) Let e be a local idempotent and $\mathrm{rad}(Re) = \sum \oplus T_i$ a direct decomposition. Then each pair of different direct summond T_i and T_j has no composition factors whose corresponding vertices have a common predecessor in Q.

It is a nice algebra that the universal Galois covering is simply connected. We remark that this is the same algebra as the standard algebra [3, 4].

Definition 1.4. (Standard algebra)

An algebra $R = (Q, I)$ is called a standard algebra if its universal Galois covering $(\overline{Q}, \overline{I})$ is simply connected.

In [2], it has been proved that non-standard algebra R happens only when $\mathrm{ch}\, K = 2$ and $a \cdot \mathrm{rad}\, R \cdot b = K \cdot ab$ for some $a, b \in \mathrm{rad}\, R$ but $a, b \notin \mathrm{rad}^2 R$. In our observation, non-standard algebras are treated in the same way as the standard case by remaining the subquiver including arrows corresponding to the above a and b. So we consider only the standard cases.

2. QF-3 ALGEBRAS

There are many investigation about QF-3 algebras, for example [6, 9].

Here we define the notion of QF-3 algebras applicable to K-categories.

Definition 2.1. (QF-3 K-category)

A locally bounded K-category $\overline{R} = K\overline{Q}/\overline{I}$ with an automorphism group $G < \mathrm{Aut}(\overline{Q}, \overline{I})$ is called a QF-3 K-category if there exist projective injective ideals $\overline{R}e_1, \ldots, \overline{R}e_t$ (i.e., $\overline{R}e_i = D(f_i\overline{R})$ for some f_j), satisfying that for any non-zero $a \in \overline{R}$ there are some $g_1, \ldots, g_n \in G$ such that

$$a\{\overline{R} \cdot g_1(e_1) \oplus \cdots \oplus \overline{R} \cdot g_n(e_n)\} \neq 0.$$

We call

$$(\overline{R}e_1 \oplus \cdots \oplus \overline{R}e_n)$$

a minimal faithful module.

This definition is the same as the original one when R is an algebra. The important theorem is the following.

Theorem 2.1. *Let R be a representation-finite algebra. Then R is QF-3 if and only if a universal Galois covering $\overline{R}$ of R is QF-3.*

From this theorem, we can use techniques that we can treat a QF-3 algebra in its universal Galois covering. Since a universal covering is simply connected, we can apply the discussion in [8]. In fact all the simply connected QF-3 algebras are determined in [8]. These are constructed from 59 many elementary QF-3 quivers (see the list in [8]) and the important thing is that the relations are uniquely determined by the way of interlacing.

From [8], we consider the following condition.

THE CONDITIONS (SQF-3):

Let $\overline{R} = (\overline{Q}, \overline{I}, G)$ be a locally bounded QF-3 K-category with a automorphism group G.

(1) There are finite number of elementary QF-3 quivers $Q_1, \ldots, Q_n$ and their embedding f into Q.

(2) $f(Q_1)^G \cup \cdots \cup f(Q_n)^G = Q$ and Q has no oriented cycles.

(3) All the maximal vertices (resp. minimal vertices) are mapped to different vertices each other.

(4) For any $g, h \in G$ and any pair of quivers Q_i and Q_j, $f(Q_i)^g \cap f(Q_j)^h$ is empty or some interval $[a, b]$, which satisfies the property (*);

(*) b is maximal in $f(Q_j)^h$ iff a is minimal in $f(Q_i)^g$.

(5) The generator of $\overline{I}$ are as following;

(a) The commutative relations of rectangles in Q_i^g for any i and $g \in G$.

(b) The minimal zero relations not to make all the rectangles in Q_i^g zero for any

i and $g \in G$.

(6) Assume $f(Q_i)^g \cap f(Q_j)^h = [c,d]$, then $\overline{R}c$ has a separated radical.

(7) $\overline{R}$ is locally representation-finite.

Hence we get the following proposition.

Proposition 2.2. *Let $(\overline{Q}, \overline{I}, G)$ be a simply connected QF-3 K-category with the indecomposable minimal faithful module. Then $\overline{Q} = Q^G$ for some elementary QF-3 quiver Q listed in* [8] *and $\overline{I}$ is generated by the rule of the condition (SQF-3) (5).*

Example 1. Consider an elementary QF-3 quiver

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 3 \arrow[r] \arrow[d] & 5 \arrow[d] \\ 2 \arrow[r] & 4 \arrow[r] & 6 \end{tikzcd}$$

and an automorphism group $G = < g >, g(i) = i + 2$ for any integer i. Then we get a quiver

$$\begin{tikzcd} \cdots \arrow[r, "\alpha"] & 1 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & 3 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & 5 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & \cdots \\ \cdots \arrow[r, "\beta"] & 2 \arrow[r, "\beta"] & 4 \arrow[r, "\beta"] & 6 \arrow[r, "\beta"] & \cdots. \end{tikzcd}$$

By the rule of giving relations in the condition (SQF-3) to make it QF-3, we have the the relation

$$\gamma\alpha = \beta\gamma, \quad \alpha^3 = \beta^3 = 0.$$

We put $\overline{R}$ the locally representation-finite QF-3 K-category defined the above quiver and relations. Then $R = \overline{R}/G$ is a QF-3 algebra defined the following quiver and relations;

$$\begin{tikzcd} 1 \arrow[r, "\gamma"] \arrow[loop below, "\alpha"] & 2 \arrow[loop below, "\alpha"] \end{tikzcd}$$

$$\gamma\alpha = \beta\gamma, \quad \alpha^3 = \beta^3 = 0.$$

We get the following theorem by summarize the above explanation.

Theorem 2.3. *A locally bounded K-category is a simply connected QF-3 K-category of locally finite representation type iff it satisfies the condition* $(SQF-3)$.

By the above theorem, QF-3 algebras are classified in terms of the elementary QF-3 quivers and the Galois groups. We remark that the relations are uniquely determined by the Galois group and the interlacing on their elementary QF-3 quivers $Q_1, \ldots, Q_n$.

The Galois group of a QF-3 algebra is very simple.

Theorem 2.4. *The Galois group of a QF-3 algebra is a cyclic group.*

QF-algebra (Quasi-Frobenius algebra or self-injective algebra) is very important algebra in ring theory. We can distinguish QF-algebras from QF-3 algebras.

Theorem 2.5. *Let $\overline{R} = (\overline{Q}, \overline{I})$ be a QF-3 K-category with the elementary QF-3 quiver $Q_1, \ldots, Q_n$. Then the algebra $R = \overline{R}/G$ is a QF-algebra iff any vertex in $\overline{Q}$ belongs to an orbit of minimal vertex of some $Q_i, i = 1, \ldots, n$.*

From the above theorem, we can construct easily construct non-QF but QF-3 algebras. Also we know the way to construct a QF-algebra from a QF-3 algebra.

Example 2. We consider the following quiver

$$\begin{tikzcd}
 & & 3 \arrow[dr, "\delta"] & \\
1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & 5. \\
 & & 4 \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

with the relations $\delta\beta = \epsilon\gamma$.

We consider a group $G =< g >, g(i) = i + 4$ for any integer i. Then we get a quiver

$$\begin{tikzcd}
 & & & 3 \arrow[dr, "\delta"] & & & & 7 \arrow[dr, "\delta"] & & \\
\cdots & 1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & 5 \arrow[r, "\alpha"] & 6 \arrow[urr, phantom, ""] & & & 9 & \cdots. \\
 & & & 4 \arrow[ur, "\epsilon"'] & & & & 8 \arrow[ur, "\epsilon"'] & &
\end{tikzcd}$$

By the rule of the condition (SQF-3) to make QF-3 K-category, we get the relations

$\delta\beta = \epsilon\gamma$, and $\alpha\delta = \alpha\epsilon = 0$.

Hence we get a QF-3 non-QF algebra with the following quiver and relations.

$$\begin{tikzcd}
 & 3 \arrow[dl, "\delta"'] \\
1 \arrow[r, "\alpha"] & 2 \arrow[u, "\beta"'] \arrow[d, "\gamma"] \\
 & 4 \arrow[ul, "\epsilon"]
\end{tikzcd}$$

$\delta\beta = \epsilon\gamma$, and $\alpha\delta = \alpha\epsilon = 0$.

In the above example, only the vertex corresponding to 1 is a orbit corresponding to minimal vertices. In the following way, we can make a QF-algebra from a QF-3 non-QF-algebra.

Example 3. Consider the elementary QF-3 quivers in addition to the above one

$$\begin{tikzcd}
3 \arrow[dr, "\delta"] & & & 7, \\
& 5 \arrow[r, "\alpha"] & 6 \arrow[ur, "\beta"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 5 \arrow[r, "\alpha"] & 6 \arrow[dr, "\gamma"] & \\
4 \arrow[ur, "\epsilon"] & & & 8,
\end{tikzcd}
\qquad
\begin{tikzcd}
& 3 \arrow[dr, "\delta"] & & \\
2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & 5 \arrow[r, "\alpha"] & 6. \\
& 4 \arrow[ur, "\epsilon"'] & &
\end{tikzcd}$$

Then we get the quiver

$$\begin{tikzcd}
& & & 3 \arrow[dr, "\delta"] & & & 7 \arrow[dr, "\delta"] & & \\
\cdots & 1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & 5 \arrow[r, "\alpha"] & 6 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & 9 & \cdots. \\
& & & 4 \arrow[ur, "\epsilon"'] & & & 8 \arrow[ur, "\epsilon"'] & &
\end{tikzcd}$$

with the relation $\delta\beta = \epsilon\gamma$, $0 = \alpha\epsilon\gamma\alpha = \beta\alpha\epsilon = \gamma\alpha\delta$.

Hence we get a QF-algebra defined by the following quiver and relations.

$$\begin{tikzcd}
& 3 \arrow[dl, "\delta"'] \\
1 \arrow[r, "\alpha"] & 2 \arrow[u, "\beta"'] \arrow[d, "\gamma"] \\
& 4 \arrow[ul, "\epsilon"]
\end{tikzcd}$$

$$\delta\beta = \epsilon\gamma, 0 = \alpha\epsilon\gamma\alpha = \beta\alpha\epsilon = \gamma\alpha\delta.$$

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