ON AUSLANDER-REITEN QUIVERS
OF FINITE GROUPS

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1. Introduction

Let $G$ be a finite group and $k$ a field of characteristic $p > 0$. Let $\Gamma_s(kG)$ be the stable Auslander-Reiten quiver of the group algebra $kG$. By Webb's Theorem, the tree class of a connected component $\Delta$ of $\Gamma_s(kG)$ is restricted. We summarize results from [W, O1, Bt1, E-S] on the graph structure of connected components of $\Gamma_s(kG)$.

Theorem 1.1([W], [O1], [Bt1], [E-S]). Let $\Delta$ be a connected component of $\Gamma_s(kG)$. Then the tree class of $\Delta$ is $A_n$, $\tilde{A}_{1,2}$, $\tilde{B}_3$, $A_\infty$, $B_\infty$, $C_\infty$, $D_\infty$ or $A_\infty^\infty$. If $k$ is algebraically closed, then the tree class is not $B_\infty$ or $C_\infty$. Moreover if the tree class or the reduced graph of $\Delta$ is Euclidean, then the modules in $\Delta$ lie in a block whose defect group is a Klein four group $C_2 \times C_2$.

Moreover if $\Delta$ contains the trivial $kG$-module $k$, then the graph structure of $\Delta$ has been investigated [W, L, O1, E2].
Theorem 1.2([W], [L], [O1], [E2]). Let $\Delta_0$ be the connected component containing the trivial $kG$-module $k$ and $T$ the tree class of $\Delta_0$. Let $P$ be a Sylow $p$-subgroup of $G$. Then:

1. If $P$ is cyclic, then $T = A_n$ for some $n$.
2. If $P = C_2 \times C_2$ and $N_G(P) = C_G(P)$, then $T = \tilde{A}_{1,2}$.
3. If $P = C_2 \times C_2$ and $N_G(P) \neq C_G(P)$ but $k$ does not contain a primitive cube root of unity, then $T = \tilde{B}_3$.
4. If $P$ is a dihedral 2-group and neither (2) nor (3) holds, then $T = A_\infty$. Moreover if $P$ is dihedral of order at least 8, then $\Delta_0 \cong ZA_\infty$.
5. If $P$ is a semidihedral 2-group, then $T = D_\infty$ and $\Delta_0 \cong ZD_\infty$.
6. If $P$ is a generalized quaternion 2-group, then $T = A_\infty$ and $\Delta_0$ is a 2-tube.
7. $T = A_\infty$ and $\Delta_0 \cong ZA_\infty$ otherwise.

Here we study a connected component of $\Gamma_s(kG)$ containing an indecomposable $kG$-module whose $k$-dimension is not divided by $p$. Suppose that $M$ is an indecomposable $kG$-module and $p \nmid \dim_k M$. In Section 2, we will show that $M$ lies in a connected component isomorphic to $ZA_\infty$ if $k$ is an algebraically closed field of odd characteristic and a Sylow $p$-subgroup of $G$ is not cyclic. In Sections 3 and 4 we consider the situation where $p = 2$ and a Sylow 2-subgroup of $G$ is dihedral of order at least 8 or semidihedral. In Section 5 we make some remarks on tensoring the component containing the trivial $kG$-module $k$ with $M$.

The notation is almost standard. For an indecomposable non-projective $kG$-module $W$, we write $A(W)$ to denote the Auslander-Reiten sequence (AR-sequence) $0 \rightarrow \Omega^2 W \rightarrow m(W) \rightarrow W \rightarrow 0$.
terminating at $W$, where $\Omega$ is the Heller operator. The symbol $\otimes$ denotes tensor product over the coefficient field $k$. For an exact sequence of $kG$-modules $S : 0 \to A \to B \to C \to 0$ and a $kG$-module $W$, we write $S \otimes W$ to denote the tensor sequence $0 \to A \otimes W \to B \otimes W \to C \otimes W \to 0$. For tensoring the AR-sequence with an indecomposable $kG$-module, see [A-C, B-C]. If an exact sequence of $kG$-modules $S$ is of the form $0 \to \Omega^2 W \oplus U' \to m(W) \oplus U \oplus U' \to W \oplus U \to 0$, where $W$ is an indecomposable non-projective $kG$-module, and $U$ and $U'$ are projective or 0, we say that $S$ is the AR-sequence $A(W)$ \textit{modulo projectives}. Concerning some basic facts and terminologies used here, we refer to [Bn], [F] and [G].

2. $ZA_{\infty}$--Component

Throughout this section, we assume that

\(#2\) $k$ is algebraically closed and a Sylow $p$-subgroup $P$ of $G$ is not cyclic, dihedral, semidihedral or generalized quaternion.

First of all, we show

Theorem 2.1. Suppose that $\Theta$ is a connected component of $\Gamma_s(kG)$ containing an indecomposable $kG$-module whose $k$-dimension is not divided by $p$. Then

(1) $\Theta$ is isomorphic to $ZA_{\infty}$ or $ZD_{\infty}$.

(2) If $p$ is odd, then $\Theta$ is isomorphic to $ZA_{\infty}$. 
(3) All modules in \( \Theta \) have the same vertex \( P \).

Remark. The above (3) follows from [U, Theorem 4.3].

Let \( M \) be an indecomposable \( kG \)-module with a Sylow \( p \)-subgroup \( P \) of \( G \) as vertex, and let \( S \) be a \( P \)-source of \( M \). Then \( p \nmid \dim_k M \) if and only if \( p \nmid \dim_k S \) from [B-C, Proposition 2.4].

Proposition 2.2. Let \( M \) be an indecomposable \( kG \)-module such that \( p \nmid \dim_k M \), and let \( S \) be a \( P \)-source of \( M \). Let \( \Theta \) be the connected component of \( \Gamma_S(kG) \) containing \( M \), and let \( \Xi \) be the connected component of \( \Gamma_S(kP) \) containing \( S \). Then

1. \( \Theta \) is isomorphic to \( ZA_\infty \) if and only if \( \Xi \) is isomorphic to \( ZA_\infty \).
2. \( M \) lies at the end of \( ZA_\infty \)-component if and only if \( S \) lies at the end of \( ZA_\infty \)-component.
3. Suppose that \( \Theta \) is isomorphic to \( ZA_\infty \) and \( M \) lies at the end of \( \Theta \). Let \( M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_n \rightarrow \cdots \) is a maximal tree of \( \Theta \) with an irreducible map \( M_{n+1} \rightarrow M_n \) (\( n \geq 1 \)). Then there is a \( P \)-source \( S_n \) of \( M_n \) (\( n \geq 2 \)) such that \( S \rightarrow S_2 \rightarrow \cdots \rightarrow S_n \rightarrow \cdots \) is a maximal tree of \( \Xi \) with an irreducible map \( S_{n+1} \rightarrow S_n \) (\( n \geq 1 \)).

Now we give examples of indecomposable \( kG \)-modules lying at the ends of \( ZA_\infty \)-components.

Proposition 2.3. Let \( M \) be an indecomposable \( kG \)-module whose \( k \)-dimension is not divided by \( p \). Let \( Q \) be a proper subgroup of \( P \). Suppose that \( M \) satisfies the following conditions (with respect to \( Q \));
(1) The trivial \( kQ \)-module \( k \) is a direct summand of \( (M \otimes M) \downarrow Q \) with multiplicity one;

(2) If \( Q \) is generalized quaternion, then \( \Omega^2 k \nmid (M \otimes M) \downarrow Q \).

Then \( M \) lies at the end of \( ZA_{\infty\text{-component}} \).

Remark. The above condition (1) is equivalent to the following condition: \((1')\) We have an indecomposable direct sum decomposition \( N \oplus (\oplus_i W_i) \) of \( M \downarrow Q \), where \( p \nmid \text{dim}_k N \) and \( p \mid \text{dim}_k W_i \) for all \( i \).

From Proposition 2.3, we have following

Example 2.4. (1) Suppose that \( p \) is odd. Let \( M \) be an indecomposable \( kG \)-module with vertex \( P \) and \( S \) a \( P \)-source of \( M \). Suppose that \( \text{dim}_k S = 2 \). Then \( M \) lies at the end of \( ZA_{\infty\text{-component}} \).

(2) Suppose that \( p \neq 3 \). Let \( M \) be an indecomposable \( kG \)-module with vertex \( P \), and \( S \) a \( P \)-source of \( M \). Suppose that \( \text{dim}_k S = 3 \). Then \( M \) lies at the end of \( ZA_{\infty\text{-component}} \).

Proof. There exists an element \( x \) of \( P \) such that \( x \) does not act on \( S \) trivially. Let \( Q = \langle x \rangle \). Then \( S \) satisfies the conditions (with respect to \( Q \)) in Proposition 2.3.

Remark. In [E3], Erdmann proved that if \( k \) is algebraically closed and a \( p \)-group \( P \) is not cyclic, dihedral, semidihedral or generalized quaternion, then there are infinitely many \( kP \)-modules of dimension 2 or 3 lying at the ends of \( ZA_{\infty\text{-components}} \) ([E3], Propositions 4.2 and 4.4.). Using this result, she consequently showed that for a block \( B \) over an algebraically closed field, the stable Auslander-Reiten quiver \( \Gamma_s(B) \) has infinitely many components of the
form $\mathbb{Z}A_\infty$ if a defect group of $B$ is not cyclic, dihedral, semidihedral or generalized quaternion.

3. Dihedral 2-group

In this section we consider the following situation:

(#3) $k$ is an algebraically closed field of characteristic 2 and a Sylow 2-subgroup $P$ of $G$ is dihedral of order at least 8.

Let $\Delta_0$ be the connected component containing the trivial $kG$-module $k$. Then $\Delta_0$ is isomorphic to $\mathbb{Z}A_\infty^\infty$ by Theorem 1.2. It is known that all modules in $\Delta_0$ are endotrivial $kG$-modules (see, e.g., [Bt2]). Hence the following holds.

Proposition 3.1. Assume (#3). Let $M$ be an odd dimensional indecomposable $kG$-module. Let $\Theta$ be the connected component of $\Gamma_s(kG)$ containing $M$ and $\Delta_0$ the connected component containing $k$. Then $\Theta$ is isomorphic to $\mathbb{Z}A_\infty^\infty$ and tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$. Moreover all modules in $\Theta$ have the same vertex $P$.

4. Semidihedral 2-group

Throughout this section, we assume that
(#4) \( k \) is an algebraically closed field of characteristic 2 and a Sylow 2-subgroup \( P \) of \( G \) is semidihedral.

Let \( A_0 \) be the connected component of \( \Gamma_S(kP) \) containing the trivial \( kP \)-module \( k \). Then \( A_0 \) is isomorphic to \( ZD_{\infty} \) (see [E2, p.76, II. 10.7 Remark]). Thus a part of \( A_0 \) is as follows for some indecomposable \( kG \)-modules \( H_2, H_3 \) and \( I \).

\[
\begin{array}{c}
\Omega^2k \\
\downarrow \\
\Omega^2H_2 \\
\downarrow \\
H_3 \\
\end{array}
\xrightarrow{\quad} \begin{array}{c}
\Omega^2I \\
\downarrow \\
H_2 \\
\downarrow \\
\Omega^2H_3 \\
\end{array}
\xrightarrow{\quad} \begin{array}{c}
k \\
\downarrow \\
I \\
\downarrow \\
\Omega^2H_2 \\
\end{array}
\]

Let \( P = \langle x, y \mid x^2 = y^{2^n-1} = 1, \; y^x = y^{-1+2^n-2} \rangle \) and \( \mathcal{K} = \{<x>\} \). Then an \( \mathcal{K} \)-projective cover resolution of \( k \) is \( 0 \to \Omega_{\mathcal{K}}k \to (k_{\downarrow<\mathcal{K}})^\uparrow P \to k \to 0 \), where \( (k_{\downarrow<\mathcal{K}})^\uparrow P \to k \) is a canonical epimorphism and \( \Omega_{\mathcal{K}}k \) is its kernel. Concerning some basic facts on relative projective cover, we refer to [Kn, T, O2].

In [O2], Okuyama showed the following

Theorem 4.1[O2]. With the same assumption and notations as above,

(1) \( I \equiv \Omega(\Omega_{\mathcal{K}}k) \) and \( I \) is an endotrivial \( kP \)-module.

(2) \( I \) is self-dual and odd dimensional.

(3) If \( I' \) is self-dual, odd dimensional and indecomposable, then
$I' \cong k$ or $I$.

Applying Theorem 4.1, we have

Lemma 4.2. Let $S$ be an odd dimensional indecomposable $kP$-module. Then $S \not\cong S \otimes I$.

If $S$ is an odd dimensional indecomposable $kP$-module, then the projective-free part $S'$ of $S \otimes I$ is odd dimensional indecomposable and $S \not\cong S'$ by Theorem 4.1 and Lemma 4.2. Moreover it follows that the projective-free part of $S \otimes H_2$ is indecomposable. Therefore the following holds.

Proposition 4.3. Let $S$ be an odd dimensional indecomposable $kP$-module and $\Xi$ the connected component of $\Gamma_s(kP)$ containing $S$. Then

1. $\Xi$ is isomorphic to $\mathbb{Z}D_\infty$.
2. All indecomposable $kP$-modules in $\Xi$ have the same vertex $P$.

Remark. The above (2) follows from [E1, Theorem A].

Let $k \rightarrow H_2 \rightarrow H_3 \rightarrow \cdots \rightarrow H_n \rightarrow \cdots$ be a maximal tree of $\Lambda_0$.

If $S$ is an odd dimensional indecomposable $kG$-module, then the projective-free part $S_n$ of $H_n \otimes S$ is indecomposable and the tensor sequence $A(H_n) \otimes S$ is the AR-sequence $A(S_n)$ modulo projectives. Hence the following holds.
Lemma 4.4. Let $S$ be an odd dimensional indecomposable $kP$-module and $\Xi$ the connected component of $\Gamma_S(kP)$ containing $S$. Then tensoring with $S$ induces a graph isomorphism from $\Delta_0$ onto $\Xi$.

Using [Ka1, Theorem and Ka2, Theorem], we obtain

Proposition 4.5. Let $M$ be an odd dimensional indecomposable $kG$-module and $\Theta$ the connected component containing $M$. Let $\Delta_0$ be the connected component containing the trivial $kG$-module $k$. Then

1. $\Theta$ is isomorphic to $\mathbb{Z}D_\infty$ and tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$.

2. All indecomposable $kG$-modules in $\Theta$ have the same vertex $P$.

5. Remarks on tensoring with a certain module

Suppose that $M$ is an indecomposable $kG$-module and $p \nmid \dim_k M$. Let $\Theta$ be the connected component of $\Gamma_S(kG)$ containing $M$ and $\Delta_0$ the connected component containing the trivial $kG$-module $k$. If a Sylow $p$-subgroup $P$ of $G$ is dihedral of order at least 8 or semidihedral, then tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$ as we have seen in Propositions 3.1 and 4.5.

In this section we consider on tensoring modules in $\Delta_0$ with $M$ under the same hypothesis as in Section 2. Throughout this section, we assume that
(#2) \( k \) is algebraically closed and a Sylow \( p \)-subgroup \( P \) of \( G \) is not cyclic, dihedral, semidihedral or generalized quaternion.

Hence the connected component \( \Delta_0 \) of \( \Gamma_s(kG) \) containing the trivial kG-module \( k \) is of the form \( ZA_{\infty} \) by Theorem 1.2.

Proposition 5.1. Suppose that \( M \) is indecomposable kG-module and \( p \nmid \dim_k M \). Let \( \Theta \) be the connected component of \( \Gamma_s(kG) \) containing \( M \). Let \( S \) be a \( P \)-source of \( M \) and \( \Xi \) the connected component of \( \Gamma_s(kP) \) containing \( S \). Suppose that \( \Theta \) is isomorphic to \( ZA_{\infty} \) and \( M \) lies at the end of \( \Theta \). Then the following are equivalent.

1. Tensoring with \( M \) induces a graph isomorphism from \( \Delta_0 \) onto \( \Theta \).
2. Tensoring with \( S \) induces a graph isomorphism from the connected component of \( \Gamma_s(kP) \) containing the trivial kP-module \( k \) onto \( \Xi \).

Note that the hypothesis of Proposition 5.1 implies that \( \Xi \cong ZA_{\infty} \) and \( S \) lies at the end of \( \Xi \) by Proposition 2.2.

Example 5.2. Let \( M \) be a trivial source module with vertex \( P \). Let \( \Theta \) be the connected component of \( \Gamma_s(kG) \) containing \( M \). Then \( \Theta \) is isomorphic to \( ZA_{\infty} \) and \( M \) lies at the end of \( \Theta \). Moreover tensoring with \( M \) induces a graph isomorphism from \( \Delta_0 \) onto \( \Theta \).

We consider an indecomposable kG-module \( M \) lying at the end of its connected component \( \Theta \) isomorphic to \( ZA_{\infty} \). In the following,
we give conditions which imply that tensoring with \( M \) induces a graph isomorphism from \( \Delta_0 \) onto \( \Theta \).

**Proposition 5.3.** Let \( M \) be an indecomposable \( kG \)-module with \( p \nmid \dim_k M \), and let \( \Theta \) be the connected component of \( \Gamma_s(kG) \) containing \( M \). Suppose that \( M \) lies at the end of \( \Theta \) and \( M \otimes M^* \cong k \otimes (\otimes_i W_i) \), where each \( W_i \) is indecomposable and \( p \mid \dim_k W_i \). Then tensoring with \( M \) induces a graph isomorphism from \( \Delta_0 \) onto \( \Theta \).

**Example 5.4.** Suppose that \( M \) is an endotrivial \( kG \)-module. Let \( \Theta \) be the connected component containing \( M \). Then \( M \) satisfies the condition in Proposition 5.5. Hence tensoring with \( M \) induces a graph isomorphism from \( \Delta_0 \) onto \( \Theta \).

**Remark.** Without the assumption \((#2)\), if \( M \) is an endotrivial \( kG \)-module, then tensoring with \( M \) induces a graph isomorphism from the connected component containing the trivial \( kG \)-module onto the connected component containing \( M \) (Bt2, Theorem 2.3]). For related results on endotrivial modules, see also [Bt2].

**Proposition 5.5.** Let \( M \) be an indecomposable \( kG \)-module with \( p \nmid \dim_k M \), and let \( \Theta \) be the connected component of \( \Gamma_s(kG) \) containing \( M \). Let \( Q \) be a proper subgroup of \( P \). Suppose that \( M \) satisfies the conditions (with respect to \( Q \)) in Proposition 2.3. Then tensoring with \( M \) induces a graph isomorphism from \( \Delta_0 \) onto \( \Theta \).
Example 5.6. (1) Suppose that $p$ is odd. Let $M$ be an indecomposable $kG$-module with vertex $P$ and $S$ a $P$-source of $M$. Suppose that $\dim_k S = 2$. Then tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto the connected component containing $M$.

(2) Suppose that $p = 2$. Let $M$ be an indecomposable $kG$-module with vertex $P$ and $S$ a $P$-source of $M$. Suppose that $\dim_k S = 3$. Then tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto the connected component containing $M$.

References


