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<td>KAWATA, Shigeto</td>
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Kyoto University
1. Introduction

Let $G$ be a finite group and $k$ a field of characteristic $p > 0$. Let $\Gamma_s(kG)$ be the stable Auslander-Reiten quiver of the group algebra $kG$. By Webb's Theorem, the tree class of a connected component $\Delta$ of $\Gamma_s(kG)$ is restricted. We summarize results from [W, O1, Bt1, E-S] on the graph structure of connected components of $\Gamma_s(kG)$.

Theorem 1.1([W], [O1], [Bt1], [E-S]). Let $\Delta$ be a connected component of $\Gamma_s(kG)$. Then the tree class of $\Delta$ is $A_n$, $\tilde{A}_{1,2}$, $\tilde{B}_3$, $A_\infty$, $B_\infty$, $C_\infty$, $D_\infty$ or $A_\infty^\infty$. If $k$ is algebraically closed, then the tree class is not $B_\infty$ or $C_\infty$. Moreover if the tree class or the reduced graph of $\Delta$ is Euclidean, then the modules in $\Delta$ lie in a block whose defect group is a Klein four group $C_2 \times C_2$.

Moreover if $\Delta$ contains the trivial $kG$-module $k$, then the graph structure of $\Delta$ has been investigated [W, L, O1, E2].
Theorem 1.2([W], [L], [O1], [E2]). Let $\Delta_0$ be the connected component containing the trivial $kG$-module $k$ and $T$ the tree class of $\Delta_0$. Let $P$ be a Sylow $p$-subgroup of $G$. Then:

1. If $P$ is cyclic, then $T = A_n$ for some $n$.
2. If $P = C_2 \times C_2$ and $N_G(P) = C_G(P)$, then $T = \tilde{A}_{1,2}$.
3. If $P = C_2 \times C_2$ and $N_G(P) \neq C_G(P)$ but $k$ does not contain a primitive cube root of unity, then $T = \tilde{B}_3$.
4. If $P$ is a dihedral 2-group and neither (2) nor (3) holds, then $T = A_\infty$. Moreover if $P$ is dihedral of order at least 8, then $\Delta_0 \cong ZA_\infty$.
5. If $P$ is a semidihedral 2-group, then $T = D_\infty$ and $\Delta_0 \cong ZD_\infty$.
6. If $P$ is a generalized quaternion 2-group, then $T = A_\infty$ and $\Delta_0$ is a 2-tube.
7. $T = A_\infty$ and $\Delta_0 \cong ZA_\infty$ otherwise.

Here we study a connected component of $\Gamma_s(kG)$ containing an indecomposable $kG$-module whose $k$-dimension is not divided by $p$. Suppose that $M$ is an indecomposable $kG$-module and $p \nmid \dim_k M$. In Section 2, we will show that $M$ lies in a connected component isomorphic to $ZA_\infty$ if $k$ is an algebraically closed field of odd characteristic and a Sylow $p$-subgroup of $G$ is not cyclic. In Sections 3 and 4 we consider the situation where $p = 2$ and a Sylow 2-subgroup of $G$ is dihedral of order at least 8 or semidihedral. In Section 5 we make some remarks on tensoring the component containing the trivial $kG$-module $k$ with $M$.

The notation is almost standard. For an indecomposable non-projective $kG$-module $W$, we write $A(W)$ to denote the Auslander-Reiten sequence (AR-sequence) $0 \to \Omega^2W \to m(W) \to W \to 0$
terminating at $W$, where $\Omega$ is the Heller operator. The symbol $\otimes$ denotes tensor product over the coefficient field $k$. For an exact sequence of $kG$-modules $S : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and a $kG$-module $W$, we write $S \otimes W$ to denote the tensor sequence $0 \rightarrow A \otimes W \rightarrow B \otimes W \rightarrow C \otimes W \rightarrow 0$. For tensoring the AR-sequence with an indecomposable $kG$-module, see [A-C, B-C]. If an exact sequence of $kG$-modules $S$ is of the form $0 \rightarrow \Omega^2 W \oplus U' \rightarrow m(W) \oplus U \oplus U' \rightarrow W \oplus U \rightarrow 0$, where $W$ is an indecomposable non-projective $kG$-module, and $U$ and $U'$ are projective or 0, we say that $S$ is the AR-sequence $A(W)$ modulo projectives. Concerning some basic facts and terminologies used here, we refer to [Bn], [F] and [G].

2. $\mathbb{Z}_{\infty}$-Component

Throughout this section, we assume that

(#2) $k$ is algebraically closed and a Sylow $p$-subgroup $P$ of $G$ is not cyclic, dihedral, semidihedral or generalized quaternion.

First of all, we show

Theorem 2.1. Suppose that $\Theta$ is a connected component of $\Gamma_s(kG)$ containing an indecomposable $kG$-module whose $k$-dimension is not divided by $p$. Then

(1) $\Theta$ is isomorphic to $\mathbb{Z}_\infty$ or $\mathbb{ZD}_\infty$.

(2) If $p$ is odd, then $\Theta$ is isomorphic to $\mathbb{Z}_\infty$. 

(3) All modules in $\Theta$ have the same vertex $P$.

Remark. The above (3) follows from [U, Theorem 4.3].

Let $M$ be an indecomposable $kG$-module with a Sylow $p$-subgroup $P$ of $G$ as vertex, and let $S$ be a $P$-source of $M$. Then $p \nmid \dim_k M$ if and only if $p \nmid \dim_k S$ from [B-C, Proposition 2.4].

**Proposition 2.2.** Let $M$ be an indecomposable $kG$-module such that $p \nmid \dim_k M$, and let $S$ be a $P$-source of $M$. Let $\Theta$ be the connected component of $\Gamma_S(kG)$ containing $M$, and let $\Xi$ be the connected component of $\Gamma_S(kP)$ containing $S$. Then

1. $\Theta$ is isomorphic to $ZA_{\infty}$ if and only if $\Xi$ is isomorphic to $ZA_{\infty}$.
2. $M$ lies at the end of $ZA_{\infty}$-component if and only if $S$ lies at the end of $ZA_{\infty}$-component.
3. Suppose that $\Theta$ is isomorphic to $ZA_{\infty}$ and $M$ lies at the end of $\Theta$. Let $M \rightarrow M_2 \rightarrow \cdots \rightarrow M_n \rightarrow \cdots$ is a maximal tree of $\Theta$ with an irreducible map $M_{n+1} \rightarrow M_n$ ($n \geq 1$). Then there is a $P$-source $S_n$ of $M_n$ ($n \geq 2$) such that $S \rightarrow S_2 \rightarrow \cdots \rightarrow S_n \rightarrow \cdots$ is a maximal tree of $\Xi$ with an irreducible map $S_{n+1} \rightarrow S_n$ ($n \geq 1$).

Now we give examples of indecomposable $kG$-modules lying at the ends of $ZA_{\infty}$-components.

**Proposition 2.3.** Let $M$ be an indecomposable $kG$-module whose $k$-dimension is not divided by $p$. Let $Q$ be a proper subgroup of $P$. Suppose that $M$ satisfies the following conditions (with respect to $Q$);
(1) The trivial $kQ$-module $k$ is a direct summand of $(M \otimes M) \downarrow_Q$ with multiplicity one;

(2) If $Q$ is generalized quaternion, then $\Omega^2 k \not| (M \otimes M) \downarrow_Q$.

Then $M$ lies at the end of $ZA_{\infty}$-component.

Remark. The above condition (1) is equivalent to the following condition: (1') We have an indecomposable direct sum decomposition $N \oplus (\oplus_i W_i)$ of $M \downarrow_Q$, where $p \nmid \dim_k N$ and $p \mid \dim_k W_i$ for all $i$.

From Proposition 2.3, we have following

Example 2.4. (1) Suppose that $p$ is odd. Let $M$ be an indecomposable $kG$-module with vertex $P$ and $S$ a $P$-source of $M$. Suppose that $\dim_k S = 2$. Then $M$ lies at the end of $ZA_{\infty}$-component.

(2) Suppose that $p \neq 3$. Let $M$ be an indecomposable $kG$-module with vertex $P$, and $S$ a $P$-source of $M$. Suppose that $\dim_k S = 3$. Then $M$ lies at the end of $ZA_{\infty}$-component.

Proof. There exists an element $x$ of $P$ such that $x$ does not act on $S$ trivially. Let $Q = \{x\}$. Then $S$ satisfies the conditions (with respect to $Q$) in Proposition 2.3.

Remark. In [E3], Erdmann proved that if $k$ is algebraically closed and a $p$-group $P$ is not cyclic, dihedral, semidihedral or generalized quaternion, then there are infinitely many $kP$-modules of dimension 2 or 3 lying at the ends of $ZA_{\infty}$-components ([E3], Propositions 4.2 and 4.4). Using this result, she consequently showed that for a block $B$ over an algebraically closed field, the stable Auslander-Reiten quiver $\Gamma_s(B)$ has infinitely many components of the
form $\mathbb{Z}A_{\infty}$ if a defect group of $B$ is not cyclic, dihedral, semidihedral or generalized quaternion.

3. Dihedral 2-group

In this section we consider the following situation:

(#3) $k$ is an algebraically closed field of characteristic 2 and a Sylow 2-subgroup $P$ of $G$ is dihedral of order at least 8.

Let $\Delta_0$ be the connected component containing the trivial $kG$-module $k$. Then $\Delta_0$ is isomorphic to $\mathbb{Z}A_{\infty}$ by Theorem 1.2. It is known that all modules in $\Delta_0$ are endotrivial $kG$-modules (see, e.g., [Bt2]). Hence the following holds.

Proposition 3.1. Assume (#3). Let $M$ be an odd dimensional indecomposable $kG$-module. Let $\Theta$ be the connected component of $\Gamma_s(kG)$ containing $M$ and $\Delta_0$ the connected component containing $k$. Then $\Theta$ is isomorphic to $\mathbb{Z}A_{\infty}$ and tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$. Moreover all modules in $\Theta$ have the same vertex $P$.

4. Semidihedral 2-group

Throughout this section, we assume that
(#4) $k$ is an algebraically closed field of characteristic 2 and a Sylow 2-subgroup $P$ of $G$ is semidihedral.

Let $A_0$ be the connected component of $\Gamma_s(kP)$ containing the trivial $kP$-module $k$. Then $A_0$ is isomorphic to $ZD_{\infty}$ (see [E2, p.76, II. 10.7 Remark]). Thus a part of $A_0$ is as follows for some indecomposable $kG$-modules $H_2, H_3$ and $I$.

\[
\begin{array}{cccc}
\Omega^2k & \xrightarrow{\cdot} & \Omega^2H_2 & \xrightarrow{\cdot} \Omega^2I & \xrightarrow{\cdot} H_2 & \xrightarrow{\cdot} I & \xrightarrow{\cdot} \Omega^2H_3
\end{array}
\]

Let $P = \langle x, y \mid x^2 = y^{2^n - 1} = 1, \ y^x = y^{-1+2^{n-2}} \rangle$ and $\mathcal{X} = \{ x \}$. Then an $\mathcal{X}$-projective cover resolution of $k$ is $0 \to \Omega_\mathcal{X} k \to (k \downarrow_{\mathcal{X}'} \uparrow P) \to k \to 0$, where $(k \downarrow_{\mathcal{X}'} \uparrow P) \to k$ is a canonical epimorphism and $\Omega_\mathcal{X} k$ is its kernel. Concerning some basic facts on relative projective cover, we refer to [Kn, T, O2].

In [O2], Okuyama showed the following

Theorem 4.1[O2]. With the same assumption and notations as above,
(1) $I \cong \Omega(\Omega_{\mathcal{X}} k)$ and $I$ is an endotrivial $kP$-module.
(2) $I$ is self-dual and odd dimensional.
(3) If $I'$ is self-dual, odd dimensional and indecomposable, then
I' \equiv k \text{ or } I.

Applying Theorem 4.1, we have

Lemma 4.2. Let S be an odd dimensional indecomposable kP-module. Then \( S \not\equiv S \otimes 1 \).

If S is an odd dimensional indecomposable kP-module, then the projective-free part \( S' \) of \( S \otimes 1 \) is odd dimensional indecomposable and \( S \neq S' \) by Theorem 4.1 and Lemma 4.2. Moreover it follows that the projective-free part of \( S \otimes H_2 \) is indecomposable. Therefore the following holds.

Proposition 4.3. Let S be an odd dimensional indecomposable kP-module and \( \Xi \) the connected component of \( \Gamma_s(kP) \) containing S. Then

1. \( \Xi \) is isomorphic to \( ZD_{\infty} \).
2. All indecomposable kP-modules in \( \Xi \) have the same vertex P.

Remark. The above (2) follows from [E1, Theorem A].

Let \( k \rightarrow H_2 \rightarrow H_3 \rightarrow \cdots \rightarrow H_n \rightarrow \cdots \) be a maximal tree of \( \Lambda_0 \).

If S is an odd dimensional indecomposable kG-module, then the projective-free part \( S_n \) of \( H_n \otimes S \) is indecomposable and the tensor sequence \( A(H_n) \otimes S \) is the AR-sequence \( A(S_n) \) modulo projectives. Hence the following holds.
Lemma 4.4. Let $S$ be an odd dimensional indecomposable $kP$-module and $\Xi$ the connected component of $\Gamma_S(kP)$ containing $S$. Then tensoring with $S$ induces a graph isomorphism from $\Delta_0$ onto $\Xi$.

Using [Ka1, Theorem and Ka2, Theorem], we obtain

Proposition 4.5. Let $M$ be an odd dimensional indecomposable $kG$-module and $\Theta$ the connected component containing $M$. Let $\Delta_0$ be the connected component containing the trivial $kG$-module $k$. Then

1. $\Theta$ is isomorphic to $\mathbb{Z}D_{\infty}$ and tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$.

2. All indecomposable $kG$-modules in $\Theta$ have the same vertex $P$.

5. Remarks on tensoring with a certain module

Suppose that $M$ is an indecomposable $kG$-module and $p \nmid \dim_k M$. Let $\Theta$ be the connected component of $\Gamma_S(kG)$ containing $M$ and $\Delta_0$ the connected component containing the trivial $kG$-module $k$. If a Sylow $p$-subgroup $P$ of $G$ is dihedral of order at least 8 or semidihedral, then tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$ as we have seen in Propositions 3.1 and 4.5.

In this section we consider on tensoring modules in $\Delta_0$ with $M$ under the same hypothesis as in Section 2. Throughout this section, we assume that
(#2) $k$ is algebraically closed and a Sylow $p$-subgroup $P$ of $G$ is not cyclic, dihedral, semidihedral or generalized quaternion.

Hence the connected component $\Delta_0$ of $\Gamma_s(kG)$ containing the trivial $kG$-module $k$ is of the form $ZA_\infty$ by Theorem 1.2.

Proposition 5.1. Suppose that $M$ is indecomposable $kG$-module and $p \nmid \dim_k M$. Let $\Theta$ be the connected component of $\Gamma_s(kG)$ containing $M$. Let $S$ be a $P$-source of $M$ and $\Xi$ the connected component of $\Gamma_s(kP)$ containing $S$. Suppose that $\Theta$ is isomorphic to $ZA_\infty$ and $M$ lies at the end of $\Theta$. Then the following are equivalent.

1. Tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$.
2. Tensoring with $S$ induces a graph isomorphism from the connected component of $\Gamma_s(kP)$ containing the trivial $kP$-module $k$ onto $\Xi$.

Note that the hypothesis of Proposition 5.1 implies that $\Xi \cong ZA_\infty$ and $S$ lies at the end of $\Xi$ by Proposition 2.2.

Example 5.2. Let $M$ be a trivial source module with vertex $P$. Let $\Theta$ be the connected component of $\Gamma_s(kG)$ containing $M$. Then $\Theta$ is isomorphic to $ZA_\infty$ and $M$ lies at the end of $\Theta$. Moreover tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$.

We consider an indecomposable $kG$-module $M$ lying at the end of its connected component $\Theta$ isomorphic to $ZA_\infty$. In the following,
we give conditions which imply that tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$.

Proposition 5.3. Let $M$ be an indecomposable $kG$-module with $p \nmid \dim_k M$, and let $\Theta$ be the connected component of $\Gamma_s(kG)$ containing $M$. Suppose that $M$ lies at the end of $\Theta$ and $M \otimes M^* \cong k \Theta (\otimes_i W_i)$, where each $W_i$ is indecomposable and $p \mid \dim_k W_i$. Then tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$.

Example 5.4. Suppose that $M$ is an endotrivial $kG$-module. Let $\Theta$ be the connected component containing $M$. Then $M$ satisfies the condition in Proposition 5.5. Hence tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$.

Remark. Without the assumption (2), if $M$ is an endotrivial $kG$-module, then tensoring with $M$ induces a graph isomorphism from the connected component containing the trivial $kG$-module onto the connected component containing $M$ (Bt2, Theorem 2.3]). For related results on endotrivial modules, see also [Bt2].

Proposition 5.5. Let $M$ be an indecomposable $kG$-module with $p \nmid \dim_k M$, and let $\Theta$ be the connected component of $\Gamma_s(kG)$ containing $M$. Let $Q$ be a proper subgroup of $P$. Suppose that $M$ satisfies the conditions (with respect to $Q$) in Proposition 2.3. Then tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto $\Theta$. 
Example 5.6. (1) Suppose that $p$ is odd. Let $M$ be an indecomposable $kG$-module with vertex $P$ and $S$ a $P$-source of $M$. Suppose that $\dim_k S = 2$. Then tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto the connected component containing $M$.

(2) Suppose that $p = 2$. Let $M$ be an indecomposable $kG$-module with vertex $P$ and $S$ a $P$-source of $M$. Suppose that $\dim_k S = 3$. Then tensoring with $M$ induces a graph isomorphism from $\Delta_0$ onto the connected component containing $M$.

References


