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On Eigenvalues of Cartan Matrices for Finite Groups

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1. Introduction

This note aims to report our joint work [K-W] on eigenvalues of the Cartan matrix. There are few known results about it (e.g. [M]), but the concept of eigenvalues of the Cartan matrix was implicit in Wallace's problem [W]. In order to state our main results we will fix some notation. Let G be a finite group and F an algebraically closed field of characteristic $p>0$. Let B be a block of the group algebra FG with defect group D. Let $F_1,\cdots,F_l$ be the simple FG-modules in B and $U_1,\cdots,U_l$ the corresponding principal indecomposable FG-modules, where $l=l(B)$ is the number of non-isomorphic simple FG-modules in B. We represent f_i , u_i the dimension of F_i , U_i , respectively. The Cartan matrix C_B of B is a nonnegative indecomposable $l\times l$ matrix (e.g. [F, Theorem I.16.7]), and so C_B has the Frobenius root $\rho(C_B)$. Since $C_B={}^tD_BD_B$ where D_B is the decomposition matrix of B, any eigenvalue of C_B is a positive real number, and hence $\rho(C_B)$ is the largest eigenvalue of C_B . For an estimation of $\rho(C_B)$, we have the following

Theorem 1. Let u be the dimension of the principal indecomposable FG-module corresponding to the trivial FG-module and $O_p(G)$ the maximal normal p-subgroup of G. Then we have

$$| O_p(G) | \leqq \rho(C_B) \leqq u .$$

Furthermore, $| O_p(G) | = \rho(C_B)$ if and only if D is normal in G, and $\rho(C_B) = u$ if and only if $u_i = uf_i$ for $i = 1, \cdots, l(B)$.

Theorem 2. If G is p-solvable, then we have

$$\rho(C_B) \leqq | D | .$$

Furthermore, equality holds in the above inequality if and only if every irreducible Brauer character in B has height 0.

2. The proofs of Theorems

We will use the same notation as in the previous section. The following lemma directly follows from the theory of Frobenius roots but it provides an essential tool for an estimation of $\rho(C_B)$.

Lemma 3. $\min\{u_i/f_i\} \leqq \rho(C_B) \leqq \max\{u_i/f_i\}$.

Moreover, if one of the equalities holds, then both hold.

Proof. Since $C_B\,{}^t(f_1, \cdots, f_l) = {}^t(u_1, \cdots, u_l)$, the result follows from the theory of Frobenius roots.

Proof of Theorem 1. By a theorem of Brauer-Nesbitt [F, Lemma IV.4.5], we have $u_i \leqq uf_i$ for all i. So the second inequality follows from Lemma 3.

On the first inequality, set $\overline{G} = G/O_p(G)$ and let $\overline{U}_i$ be a projective cover of a simple B-module F_i as an $F\overline{G}$-module and denote by $\overline{u}_i$ its dimension. Then we have $u_i = | O_p(G) | \overline{u}_i$ by a result of Fong [F, Lemma X.3.1]. Therefore $u_i \geqq | O_p(G) | f_i$. By using Lemma 3, this implies $| O_p(G) | \leqq \rho(C_B)$. Since $O_p(G) \subseteq D$, it follows from the following Lemmas 4 and 5 that the equality holds if and only if D is normal in G. For the proofs of Lemmas 4 and 5, we refer to [K-W].

Lemma 4. If there exists a constant c such that $u_i = cf_i$ for all i, then $c = |D|$.

Lemma 5. If D is normal in G, then $\rho(C_B) = |D|$.

Proof of Theorem 2. Since G is p-solvable, it follows from Fong's theorem [F, Lemma X.3.2] that $u_i = f_i |D| / p^{e_i}$, where e_i is the height of the Brauer character afforded by the simple FG-module F_i. Since e_i is a nonnegative integer, we have $u_i / f_i \leqq |D|$ and equality holds if and only if $e_i = 0$. Using Lemma 3, we get the desired result.

3. A conjecture

Let $t(B)$ be the nilpotency index of the radical $J(B)$ of B. The following propositions suggest the close relation between $t(B)$ and $\rho(C_B)$.

Proposition 6. If D is cyclic, then $t(B) \leqq \rho(C_B) \leqq |D|$.

Furthermore, one of the equalities holds if and only if the Brauer tree of B is a star and the exceptional vertex, if it exists, is at the center.

Proposition 7. If D is normal in G, then $t(B) \leqq \rho(C_B)$ and equality holds if and only if D is cyclic.

From many examples the following seems to be true.

Conjecture 8. For any finite group G and for any block B of FG, $t(B) \leqq \rho(C_B)$.

Remark 9. By Erdmann's works [E], we can verify Conjecture 8 for blocks with dihedral, quaternion or semidihedral defect groups. Let B be the principal 2-block of a finite group with an abelian Sylow 2-subgroup. Then a result of Landrock and Michler [L-M] implies that Conjecture 8 is true for B. For the principal

2-block of the alternating groups A_n ($6 \leqq n \leqq 9$) we can also verify Conjecture 8 by Benson's [B1], [B2].

4. The dimension of the radical $J(B)$

The next result shows that there are some relations between $\dim J(B)$ and the eigenvalues of C_B.

Theorem 10. Let $\rho(C_B)$, $\mu(C_B)$ be the largest and smallest eigenvalue of C_B, respectively. Then the following hold.

(1) $\dim B\,(1-1/\mu(C_B)) \leqq \dim J(B) \leqq \dim B\,(1-1/\rho(C_B))$.

(2) $\dim B\,(1-1/\mu(C_B)) = \dim J(B)$ if and only if $l(B)=1$.

(3) $\dim J(B) = \dim B\,(1-1/\rho(C_B))$ if and only if $u_i = cf_i$ ($i=1,\cdots,l$) for some constant c.

The above result is a slight generalization of the well-known inequality of Brauer and Nesbitt ([F, Theorem VI.4.10]).

Remark 11. By Theorem 10 and the results of Sections 1 and 2, if one of the following holds, then $\dim J(B) \leqq \dim B\,(1-1/|D|)$.

(a) G is p-solvable, (b) D is normal in G, (c) D is cyclic.

5. Problems

It seems very likely that $\rho(C_B)$ strongly restricts the structure of the block B (in particular the defect group D). For this direction, we will mention some problems.

Problem A. Does it hold that $|D| = \rho(C_B)$ if $\rho(C_B)$ is an integer?

Problem B. Does $|D|$ divide the norm of $\rho(C_B)$?

Problem C. Does $\rho(C_B)$ determine $|D|$ uniquely?

Since $|D| / \rho(C_B)$ is always an algebraic integer, Problem A is a special case of Problem B. If D is cyclic, Problems A and C are resolved in the affirmative.

References

[B1] D.J. Benson, The Loewy structure of the projective indecomposable modules for A_8 in characteristic 2. Comm. in Algebra, 11(1983), 1359-1432.

[B2] D.J. Benson, The Loewy structure of the projective indecomposable modules for A_9 in characteristic 2. Comm. in Algebra, 11(1983), 1433-1453.

[E] K. Erdmann, "Blocks of Tame Representation Type and Related Algebras". Lecture Notes Math., 1428, Springer, Berlin, 1990.

[F] W. Feit, "The Representation Theory of Finite Groups". North-Holland, Amsterdam, 1982.

[K-W] M. Kiyota and T. Wada, Some remarks on eigenvalues of Cartan matrix in finite groups. submitted.

[L-M] P. Landrock and G.O. Michler, Principal 2-blocks of the simple groups of Ree type. Trans. Amer. Math. Soc., 260(1980), 83-111.

[M] K. Motose, On radicals of principal blocks. Hokkaido Math. J., 6(1977), 255-259.

[W] D.A.R. Wallace, On the radical of a group algebra. Proc. Amer. Math. Soc., 12(1961), 133-137.