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Remarks on complete intersection

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Let \((R, \mathfrak{m})\) be a Noetherian local ring. The monomial conjecture asserts that for any integer \(n \geq 0\) and for any system of parameters \(a_1, \ldots, a_d\) of \(R\) we have

\[(a_1a_2\ldots a_d)^n \not\subseteq (a_1^n+1, \ldots, a_d^n+1).\]

The monomial conjecture holds if \(R\) contains a field or \(\dim R \leq 2\). The purpose of this note is to show that the monomial conjecture is equivalent to the following property (P) of Gorenstein local rings.

(P) An ideal of height 0 is \((0)\) if it is contained in a parameter ideal.

We begin with a reformulation of the monomial conjecture. Let \(a_1, \ldots, a_d\) be a system of parameters of a Noetherian local ring \((R, \mathfrak{m})\) and let \(a^n = (a_1^n, \ldots, a_d^n)\). We define an \(R\)-homomorphism

\[f_n : R/a_n \rightarrow R/a^{n+1}\]

by \(f_n(1) = a_1a_2\ldots a_d \mod a^{n+1}\). Then the direct limit of the direct system\

\[\{ R/a^n ; n = 1, 2, \ldots \} \]

is the local cohomology module \(H_m^d(R)\). Let

\[\Phi_n(a) : R/a_n \rightarrow H_m^d(R)\]

be the canonical homomorphism. Then \(\Phi_1(a) \neq 0\) if and only if

\[(a_1a_2\ldots a_d)^n \not\subseteq (a_1^{n+1}, \ldots, a_d^{n+1})\]
for all \( n \geq 0 \).

Let \( I \) be an ideal of \( R \) with \( \text{ht}(I) = 0 \). Since \((0 : I)\) is an \( R/I \)-module we have an isomorphism

\[
(0 : I) \cong (0 : I) \otimes_R R/I.
\]

Hence we get an \( R \)-homomorphism

\[
(0 : I) \otimes_R R/I \rightarrow R.
\]

Tensoring this with the direct system \( \{R/a^n : n = 1, 2, \ldots\} \), we get a commutative diagram

\[
\begin{array}{ccc}
(0 : I) \otimes_R R/(I, a^n) & \rightarrow & R/a^n \\
1 \otimes \Psi_n(a) & \downarrow & \downarrow \Phi_n(a) \\
(0 : I) \otimes_R H_m^q(R/I) & \rightarrow & H_m^q(R) \\
\end{array}
\]

where \( \Psi_n(a) : R/(I, a^n) \rightarrow H_m^q(R/I) \) is the canonical homomorphism. \( \Phi \) induces a homomorphism

\[
\partial^* : H_m^q(R/I) \rightarrow \text{Hom}_R((0 : I), H_m^q(R)).
\]

**Lemma.** If \( R \) is Gorenstein then \( \partial^* \) is an isomorphism.

**Proof.** Since \( R \) is Gorenstein, \( H_m^q(R) \) is isomorphic to the injective envelope of the residue field \( R/m \). The lemma follows from the local duality.

In order to characterize unmixed ideals in a Gorenstein local ring, we have:
Lemma 2. Let \((R,m)\) be a Gorenstein local ring and \(I\) an ideal of height 0. Then \(I\) is unmixed if and only if \((0 : (0 : I)) = I\).

Proof. If \((0 : (0 : I)) = I\) it is clear that \(I\) is unmixed. Conversely, suppose that \(I\) is unmixed. For any \(p \in \text{Ass}(R/I)\), \(R_p\) is a 0-dimensional Gorenstein local ring and we have \((0 : (0 : I))R_p = IR_p\). Since \(I\) is unmixed, we have \((0 : (0 : I)) = I\).

Now we are ready to prove:

Theorem 3. The following statements are equivalent:

1. The monomial conjecture holds.
2. Every Gorenstein local ring has the property \((P)\).

Proof. (1) \(\Rightarrow\) (2): Let \((R,m)\) be a Gorenstein local ring and \(I\) an unmixed ideal of height 0. Suppose that \(I \neq (0)\) and let \(J = (0 : I)\). Since \(I\) is unmixed we have \(I = (0 : J)\). By Lemma 2, let \(a = a_1, \ldots, a_d\) be a system of parameters of \(R\). We have a commutative diagram

\[
\begin{array}{ccc}
I \otimes_R R/I, a & \rightarrow & R/a \\
\downarrow & & \downarrow \\
I \otimes_R H^d_m(R/J) & \rightarrow & H^d_m(R).
\end{array}
\]

By Lemma 1, we have an isomorphism

\[H^d_m(R/J) \rightarrow \text{Hom}_R(I, H^d_m(R)).\]

The image \(\alpha\) of the identity of \(R/(I, a)\) in \(H^d_m(R/J)\) is not trivial by the monomial conjecture. \(\alpha\) induces a non-trivial homomorphism.
\[ \alpha^* : 1 \rightarrow H_\alpha^q(R) . \]

From the above commutative diagram, we see that \( 1 \) is not contained in \( a \).

(2) \( \Rightarrow \) (1): Suppose that the monomial conjecture is not true. Then, there is a Noetherian local ring \( A \) and a system of parameters of \( a_1, \ldots, a_d \) of \( A \) such that

\[(a_1a_2\ldots a_d)^n \subseteq (a_1^{n+1}, \ldots, a_d^{n+1})\]

for some \( n \geq 0 \). We may assume that \( A \) is a complete local domain. There is a Gorenstein local ring \( R \) and an ideal \( I \) of \( R \) such that \( A = R/I \). We can assume that \( \dim A = \dim R \). Since the monomial conjecture does not hold for \( A \), we have \( I \neq (0) \). Let \( J = (0 : I) \). If the monomial conjecture does not hold for \( R/I \), there is a system of parameters \( x_1, \ldots, x_d \) of \( R \) such that \( J \subset x \), by lemma 1. By assumption (2), we get \( J = (0) \), but this is impossible.

Corollary 4. Let \( R \) be a Gorenstein local ring containing a field and let \( I \) be an unmixed ideal of \( R \). Suppose that there is a system of parameters \( a_1, \ldots, a_d \) of \( R \) such that \( (a_1, \ldots, a_n) \subseteq I \subseteq (a_1, \ldots, a_d) \), with \( \text{ht}(I) = h \). Then \( I = (a_1, \ldots, a_n) \).

Example (J.R. Strooker and J. Stückrad). Let \( k \) be a field and \( R = k[[x,y,u,v]]/(xy - uv, y^2, yv) = k[[x,y,u,v]] \). Then \( R \) is a Cohen-Macaulay local ring but not Gorenstein. \( (y^2) \) is an unmixed ideal of \( R \) and contained in a parameter ideal \( (x + y, u) \).

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References
