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Non-Cohen-Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik's question

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Let \( p \) be a prime ideal in a commutative Noetherian ring \( A \) and put

\[
R_S(p) := \sum_{n \geq 0} p^{(n)}t^n \subseteq A[t],
\]

where \( t \) denotes an indeterminate over \( A \). Let me call it the symbolic Rees algebra of \( p \). In my lecture, I'm interested in their ring-theoretic properties and especially, in the following two questions:

**Questions**

1. When is \( R_S(p) \) a Noetherian ring, that is, when is \( R_S(p) \) a finitely generated \( A \)-algebra?
2. When is \( R_S(p) \) a Cohen-Macaulay or Gorenstein ring, provided that it is Noetherian?

Today I will answer these questions in the following special situation, that is,

Let \( k \) be a field, and let \( n_1, n_2, \) and \( n_3 \) be positive integers with \( \text{GCD}(n_1, n_2, n_3) = 1 \). Let \( A = A_k := k[[X, Y, Z]] \) be a formal power series ring over \( k \) and let \( \varphi : A \to k[[t]] \) be the \( k \)-algebra map defined by

\[
\varphi(X) = t^{n_1}, \quad \varphi(Y) = t^{n_2}, \quad \text{and} \quad \varphi(Z) = t^{n_3}.
\]
Let me denote by \( p := p_k(n_1, n_2, n_3) \) the kernel of \( \varphi \).

Then \( A \) is a regular local ring of dimension 3 and \( p \) is a prime ideal in \( A \) of height 2. So in some sense, this is the simplest non-trivial case for the above questions. And my answer is

**Theorem 1** (with Nishida and Watanabe). Let \( m \) and \( n \) be positive integers such that \( n \geq 4 \) and \( 2m > n + 1 \). Let \( n_1 = 7m - 3 \), \( n_2 = 5mn - m - n \), and \( n_3 = 8n - 3 \). Assume that \( \gcd(n_1, n_2, n_3) = 1 \) and let \( p = p_k(n_1, n_2, n_3) \). Then the symbolic Rees algebra \( R_s(p) \) of \( p \) is a Noetherian ring if and only if the characteristic of the ground field \( k \) is positive. When this is the case, \( R_s(p) \) is not a Cohen-Macaulay ring.

The simplest example obtained by this theorem is the ideal

\[
p = p_k(18, 53, 29) \quad \text{(here } m = 3, n = 4 \text{)}
\]

\[
= I_2 \left[ \begin{array}{ccc}
X^4 & Y^2 & Z^5 \\
Y & Z^3 & X^7
\end{array} \right]
\]

\[
= (Z^8 - X^7Y^2, X^{11} - YZ^5, Y^3 - X^4Z^3)_{A_k}.
\]

Therefore, if we consider the same prime ideal \( P = (Z^8 - X^7Y^2, X^{11} - YZ^5, Y^3 - X^4Z^3)_{B} \) inside of the polynomial ring \( B = k[X, Y, Z] \), the symbolic Rees algebra \( R_s(P) \) is a finitely generated \( k \)-algebra but not a Cohen-Macaulay ring if \( \text{ch } k \) is positive, and if \( \text{ch } k = 0 \), say \( k = \mathbb{Q} \), then it is not a finitely generated \( \mathbb{Q} \)-algebra.
Let me add one question:

**Question** What about the prime ideal

\[ p = p_k(11, 25, 21) \]

\[ = I_2 \begin{bmatrix} x^3 & y^2 & z^3 \\ y & z^2 & x^5 \end{bmatrix} \]

(that is, choose \( m = 2 \) and \( n = 3 \))? Of course, this ideal doesn't satisfy my condition. But by a theorem of Cutkosky you can easily check that \( R_s(p) \) is a Noetherian ring, if \( \text{ch } k > 0 \).

However I couldn't know whether it is a Noetherian ring or not in the case where \( \text{ch } k = 0 \), though I believe that the answer is negative.

Now let me give a sketch of proof of the theorem. To do this I need a theorem due to Craig (Huneke). For a moment, let me assume that \((A, m)\) is a regular local ring of dimension 3 and \( p \) is a prime ideal in \( A \) of \( \dim A/p = 1 \).

**Theorem 2** (C. Huneke). If there exist two elements \( f \in p^{(k)} \) and \( g \in p^{(l)} \) with positive integers \( k, l \) such that the equality

\[ l_A(A/(x, f, g)) = kl \cdot l_A(A/p + xA) \]

holds for some (and hence for any) element \( x \in m \setminus p \), then the symbolic Rees algebra \( R_s(p) \) is a Noetherian ring. If the field \( A/m \) is infinite, the converse is also true.

By this theorem, Huneke showed that \( R_s(p) \) is a Noetherian ring for \( p = p_k(n_1, n_2, n_3) \), if \( \min(n_1, n_2, n_3) \leq 4 \).
If $R = R_s(p)$ is a Noetherian ring, then you can easily get an isomorphism $K_R = R(-1)$. Therefore $R$ is a Gorenstein ring, once it is Cohen-Macaulay. To check the Cohen-Macaulay property of $R$, you have the following

**Theorem 3** (, Nishida and Shimoda). Let $f$ and $g$ be the elements in the above theorem. Then the following two conditions are equivalent.

1. The symbolic Rees algebra $R_s(p)$ is a Gorenstein ring.
2. For any integer $1 \leq n \leq k + l - 2$, the ring $A/(f, g) + p^{(n)}$ is a Cohen-Macaulay ring.

When this is the case, the rings $A/(f) + p^{(n)}$, $A/(g) + p^{(n)}$, and $A/(f, g) + p^{(n)}$ are Cohen-Macaulay for all $n \geq 1$, and we have the equality

$$R_s(p) = A[(p^{(n)}t^n)_{1 \leq n \leq k + l - 2}, ft^k, gt^l].$$

Using this criterion, you can show that $R_s(p)$ is a Gorenstein ring for $p = p_k(n_1, n_2, n_3)$, if $\min \{n_1, n_2, n_3\} \leq 4$. But in general, the Cohen-Macaulay property of $R_s(p)$ depends on the characteristic of the ground field. Let me give one example:

**Example** Let $p = p_k(7, 11, 13)$. Then $R_s(p)$ is always a Noetherian ring, but it is a Gorenstein ring if and only if $\text{ch } k \neq 2, 3$.

Now let's start the proof of the theorem. In what follows, let $m$ and $n$ be positive integers such that $n \geq 4$ and $2m > n$
+ 1. Let $n_1 = 7m - 3$, $n_2 = 5mn - m - n$, and $n_3 = 8n - 3$. We assume that $\text{GCD}(n_1, n_2, n_3) = 1$. Then

$$
p = p_k(n_1, n_2, n_3)
$$

$$
= I_2 \begin{bmatrix} x^n & y^2 & z^{2m-1} \\
Y & Z^m & x^{2n-1} \end{bmatrix}.
$$

Let $a = z^{3m-1} - x^{2n-1}y^2$, $b = x^{3n-1} - yz^{2m-1}$, and $c = y^3 - x^n z^m$. Then $p = (a, b, c)$ and we have two equations

$$
x^n a + y^2 b + z^{2m-1} c = 0,
$$

$$
Y a + Z^m b + x^{2n-1} c = 0.
$$

I claim that

**Lemma** There exist elements $d_2 \in p^{(2)}$, and $d_3, d_3' \in p^{(3)}$ such that $d_2 = z^{5m-2}$, $d_3 = z^{7m-2}$, $d_3' = y^8 z^{2m-2} \mod (X)$, and

$$
X d_3 + Y b c^2 + Z d_3' = 0.
$$

**Proof.** First of all, consider two expressions of $-y^2 ab$:

$$-y^2 ab = Y b (-Y a) = Y b (Z^m b + x^{2n-1} c)
$$

$$= a (-y^2 b) = a (x^n a + z^{2m-1} c).
$$

And you get

$$x^n (a^2 - x^{n-1} Y b c) = z^m (Y b^2 - Z^{m-1} ac),
$$

hence there exists an element $d_2$ of $A$ such that

$$x^n d_2 = Y b^2 - Z^{m-1} ac,$$
\[ Z^{m_d_2} = a^2 - X^{n-1}Ybc. \]

Of course, \( d_2 \) is in \( \mathcal{P}^{(2)} \). To get \( Yd_2 \), consider

\[ -Yd_2 = d_2(- Ya) \]
\[ = d_2(Z^{m_b} + X^{2n-1}c) \]
\[ = b \cdot Z^{m_d_2} + X^{n-1}c \cdot X^{n_d_2} \]
\[ = b(a^2 - X^{n-1}Ybc) + X^{n-1}c(Yb^2 - Z^{m-1}ac) \]
\[ = - a(- ab + X^{n-1}Z^{m-1}c^2). \]

Thus \( Yd_2 = - ab + X^{n-1}Z^{m-1}c \) and we have two equations:

\[ Yd_2 = - ab + X^{n-1}Z^{m-1}c^2, \]
\[ Z^{m_d_2} = a^2 - X^{n-1}Ybc. \]

We compare two expressions of \( a^2b \):

\[ a^2b = b(Z^{m_d_2} + X^{n-1}Ybc) \]
\[ = a(- Yd_2 + X^{n-1}Z^{m-1}c^2). \]

Then we have

\[ Z^{m-1}(- Zbd_2 + X^{n-1}ac^2) = Y( a d_2 + X^{n-1}bc^2). \]

and so we get an element \( d_3 \in \mathcal{P}^{(3)} \) such that

\[ Yd_3 = - Zbd_2 + X^{n-1}ac^2. \]

As \( Yd_2 = - ab \mod (X) \), we know

\[ Yd_2 = - Z^{3m-1}(- YZ^{2m-1}); \]
hence $d_2 = Z^{5m-2} \mod (X)$. As $Yd_3 = -Zbd_2 \mod (X)$, we get

$$Yd_3 = -Z(-YZ^{2m-1})Z^{5m-2} \mod (X);$$

hence $d_3 = Z^{7m-2} \mod (X)$. Notice that $Yd_3 = X^{n-1}ac^2 = X^{n-1}(X^{2n-1}Y^2)(Y^3)^2 \mod (Z)$ and we have

$$d_3 = -X^{3n-2}Y^7 \mod (Z),$$

so that

$$Xd_3 + Ybc^2 = X(-X^{3n-2}Y^7) + Y \cdot X^{3n-1} \cdot (Y^3)^2 \equiv 0 \mod (Z).$$

Thus there is an element $d_3'$ of $p(3)$ such that

$$Xd_3 + Ybc^2 + Zd_3' = 0.$$ 

Clearly $d_3' = Y^8Z^{2m-2} \mod (X)$. This proves the lemma.

**Proposition** $p^{(2)} = p^2 + (d_2)$, $p^{(3)} = pp^{(2)} + (d_3, d_3')$, and $p^{(4)} = pp^{(3)} + (p^{(2)})^2$.

For example, let $I = p^2 + (d_2)$. Then as $(X) + I = (X) + (Z^{3m-1}, YZ^{2m-1}, Y^3)^2 + (Z^{5m-2})$, you have

$$l_A(A/(X) + I) = 3 \cdot (7m - 3)$$

$$= 3 \cdot l_A(A/(X) + p).$$

On the other hand, because $l_A(A/(X) + p^{(2)}) = e_{X_A}(A/p^{(2)}) = l_A(A/(X) + p) \cdot l_A(A/p^2A) = 3 \cdot l_A(A/(X) + p)$, you get that
\[ \ell_A(A/(X) + I) = \ell_A(A/(X) + p^{(2)}) ; \text{ hence } (X) + I = (X) + p^{(2)}, \]
because \( I \subseteq p^{(2)} \). Consequently \( p^{(2)} = I + (X) \cap p^{(2)} = I + X \cdot p^{(2)} \). Thus we have \( p^{(2)} = I \) by Nakayama's lemma. Similarly you can show that \( p^{(3)} = pp^{(2)} + (d_3, d_3') \). As
\[
\ell_A(A/(X) + p^{(4)}) = e_{\Delta}(A/p^{(4)}) \\
= \ell_A(A/(X) + p) \cdot \ell_A(A/p^4A_p) \\
= 10 \cdot \ell_A(A/(X) + p) \\
< \ell_A(A/(X) + pp^{(3)} + (p^{(2)})^2),
\]
we have
\[ p^{(4)} = pp^{(3)} + (p^{(2)})^2. \]

**Corollary** The ring \( A/(c) + p^{(3)} \) is not Cohen-Macaulay.

In fact, notice that
\[
\ell_A(A/(X, c) + p^{(3)}) = 3 \cdot (7m - 3) + 1 \\
> e_{\Delta}(A/(c) + p^{(3)}) \\
= 3 \cdot (7m - 3);
\]
hence \( A/(c) + p^{(3)} \) cannot be a Cohen-Macaulay ring.

Now let me assume that \( \text{ch } k = p > 0 \). First of all, assume that \( p \geq 3 \) and write \( p = 2q + 1 \) (hence \( q \geq 1 \)). Then by the equations
\[
X d_3 + Ybc^2 + Zd_3' = 0,
\]
we get
\[
0 = xp d_3^p + yp b pc^{2p} \mod (2p) \\
= xp d_3^p + (y^2 b)q yb^{q+1} c^{2p}.
\]
As \( X^n a + Y^{2b} + Z^{2m-1} c = 0 \), we furthermore have

\[
0 = X^p d_3^p + (-1)^q \sum_{i=0}^{q} \binom{q}{i} X^{n(q-i)} Y Z^{(2m-1)i} a^{q-i} b^{q+1} c^{2p+i}
\]

\[
= X^p d_3^p + (-1)^q \sum_{(2m-1)i < p} \binom{q}{i} X^{n(q-i)} Y Z^{(2m-1)i} a^{q-i} b^{q+1} c^{2p+i}
\]

\( \text{mod } (Z^p) \).

Now recall that \( 2m > n + 1 \) and \( n \geq 4 \). Then we have \( n(q-i) \geq p \) or \( (2m-1)i \geq p \) for each \( 0 \leq i \leq q \).

(In fact, if \( n(q-i) < p \) and \( (2m-1)i < p \), then we get \( n(q-i) \leq 2q \) and \( (2m-1)i \leq 2q \) so that \( nq + (2m-n-1)i \leq 4q \). Hence we must have \( n = 4 \) and \( i = 0 \) and so \( n(q-i) = 4q \leq 2q \), which is impossible.) Thus

\[
0 = X^p \left\{ d_3^p + (-1)^q \sum_{(2m-1)i < p} \binom{q}{i} X^{n(q-i)-p} Y Z^{(2m-1)i} a^{q-i} b^{q+1} c^{2p+i} \right\}
\]

\( \text{mod } (Z^p) \) and thus we have an element \( h \in \mathbb{F}^{(3p)} \) such that

\[
Z^p h = d_3^p + (-1)^q \sum_{(2m-1)i < p} \binom{q}{i} X^{n(q-i)-p} Y Z^{(2m-1)i} a^{q-i} b^{q+1} c^{2p+i}
\]

As \( Z^p h = d_3^p = Z^{(7m-2)p} \mod (X, c) \), we get \( h = Z^{(7m-3)p} \mod (X, c) \). Thus we have the following
Lemma There exists an element \( h \in \mathfrak{p}^{(3p)} \) such that \( h = z^{(7m-3)p} \mod (X, c) \).

(You can prove this lemma also in the case \( p = 2 \).)

Now recall Huneke's theorem. First we take \( f = c \) and \( g = h \). Then

\[
\lambda_A(A/(X, c, h)) = \lambda_A(A/(X, c, Z^{(7m-3)p}))
\]
\[
= \lambda_A(A/(X, Y^3, Z^{(7m-3)p}))
\]
\[
= 3p \cdot (7m-3)
\]
\[
= 1 \cdot 3p \cdot \lambda_A(A/(X) + \mathfrak{p}) .
\]

Hence \( R_s(\mathfrak{p}) \) is a Noetherian ring by Theorem 2. Because \( A/(c) + \mathfrak{p}^{(3)} \) is not a Cohen-Macaulay ring, \( R_s(\mathfrak{p}) \) cannot be Cohen-Macaulay by Theorem 3.

To study the case of \( \operatorname{ch} k = 0 \), we need further information in the case where \( \operatorname{ch} k = p > 0 \). Let \( F = \{ 0 < \ell \in \mathbb{Z} \mid \exists g \in \mathfrak{p}^{(\ell)} \) such that \( \lambda_A(A/(X, c, g)) = \ell \cdot \lambda_A(A/(X) + \mathfrak{p}) \} \). Then \( 3p \in F \).

Let \( \ell_0 = \min F \) and choose \( g_0 \in \mathfrak{p}^{(\ell_0)} \) so that \( \lambda_A(A/(X, c, g_0)) = \ell_0 \cdot \lambda_A(A/(X) + \mathfrak{p}) \). Then we have

Lemma

1. \( \ell_0 | \ell \) for all \( \ell \in F \).
2. \( R_s(\mathfrak{p}) = A[\{ \mathfrak{p}^{(n)}t^n \} \}_{1 \leq n \leq \ell_0 \cdot 1}, \text{ct, } g_0 t^{\ell_0}] \).
3. \( g_0 t^{\ell_0} \) is not contained in \( A[\{ \mathfrak{p}^{(n)}t^n \} \}_{1 \leq n \leq \ell_0 \cdot 1}] \).

Let me use this lemma without proof. First, we have by (1) that \( \ell_0 | 3p \); hence \( \ell_0 = 1, 3, p, \) or \( 3p \). But if \( \ell_0 = p, 3p \), then
we have by (2) that
\[ R_S(p) = A[pt, p(2)t^2, p(3)t^3], \]
which is impossible because \( p(4) \neq pp(3) + (p(2))^2 \). Thus \( l_0 \geq p \) and by (3) we get \( g_0 t^l_0 \) is not contained in \( A[(p(n)t^n)_{1 \leq n \leq l_0 - 1}] \).

This means, to generate the \( A \)-algebra \( R_S(p) \), you need at least one new element of degree \( \geq p \), depending on the characteristic \( p = \text{ch} \ k \). On the other hand, if \( R_S(p) \) were a Noetherian ring in the case where \( \text{ch} \ k = 0 \), say \( k = \mathbb{Q} \), then because everything is defined over \( \mathbb{Z} \), you can find a system of generators for the algebra \( R_S(pQ) \) so that passing to the field \( k = \mathbb{Z}/p\mathbb{Z} \) for \( p \gg 0 \), the system still generates the algebra \( R_S(p_k) \) (see the theorem below). This is impossible, because you need at least one new element of degree \( \geq p \). Thus \( R_S(pQ) \) cannot be a Noetherian ring for our example \( p \).

Let me state the required theorem more explicitly.

**Theorem.** Let \( C = \mathbb{Z}[X, Y, Z] \) and let \( I = \text{Ker} \ (\varphi : C \to \mathbb{Z}[t]) \)
where \( \varphi(X) = t^{n_1}, \ \varphi(Y) = t^{n_2}, \) and \( \varphi(Z) = t^{n_3} \). Then if \( R_S(p) \)
is a Noetherian ring for the prime ideal \( p = pQ(n_1, n_2, n_3) \) in \( \mathbb{Q}[[X, Y, Z]] \), there exist positive integers \( l \) and \( N \) and elements \( f \) and \( g \) of \( I(l) \) such that for all prime numbers \( p \geq N \), we have

1. \( I(l) A_k = p_k(l) \) and
2. \( l A_k(A_k/(X, f, g)A_k) = l^2 \cdot l A_k(A_k/(X + p_k)) \),

where \( k = \mathbb{Z}/p\mathbb{Z} \).

Here \( A_k = k[[X, Y, Z]] \) and \( p_k = p_k(n_1, n_2, n_3) \).

Before closing my talk, let me give a few open problems.
**Problems** Let $\mathbf{p} = \mathbf{p}_k(n_1, n_2, n_3)$ and $n = \min\{n_1, n_2, n_3\}$.

1. $\text{ch } k = p > 0 \implies R_s(\mathbf{p})$ is a Noetherian ring?

2. $\text{ch } k = 0$ and $R_s(\mathbf{p})$ is Noetherian $\implies R_s(\mathbf{p})$ is a Gorenstein ring?

3. $n \leq 8$, $n \neq 7 \implies R_s(\mathbf{p})$ is a Gorenstein ring? (For $\mathbf{p} = \mathbf{p}_k(9, 10, 13)$ you can show that $R_s(\mathbf{p})$ is a Noetherian ring but not Cohen-Macaulay, if $\text{ch } k = 2, 3, 7$.)

4. $n = 5 \implies R_s(\mathbf{p})$ is a Noetherian ring?

5. $n = 6 \implies R_s(\mathbf{p})$ is a Gorenstein ring? (The Noetherian property of this case was guaranteed by Cutkosky.)

6. $\mathbf{p} = \mathbf{p}_k(11, 16, 13) \implies \ ????$

**References**


25.


