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Non-Cohen-Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik's question

明大・理工 後藤四郎 (Shiro Goto)

Let $p$ be a prime ideal in a commutative Noetherian ring $A$ and put

$$R_s(p) := \sum_{n \geq 0} p^n t^n \subseteq A[t],$$

where $t$ denotes an indeterminate over $A$. Let me call it the symbolic Rees algebra of $p$. In my lecture, I'm interested in their ring-theoretic properties and especially, in the following two questions:

**Questions**
1. When is $R_s(p)$ a Noetherian ring, that is, when is $R_s(p)$ a finitely generated $A$-algebra?
2. When is $R_s(p)$ a Cohen-Macaulay or Gorenstein ring, provided that it is Noetherian?

Today I will answer these questions in the following special situation, that is,

Let $k$ be a field, and let $n_1$, $n_2$, and $n_3$ be positive integers with $\text{GCD}(n_1, n_2, n_3) = 1$. Let $A = A_k := k[[X, Y, Z]]$ be a formal power series ring over $k$ and let $\varphi : A \to k[[t]]$ be the $k$-algebra map defined by

$$\varphi(X) = t^{n_1}, \quad \varphi(Y) = t^{n_2}, \quad \text{and} \quad \varphi(Z) = t^{n_3}.$$
Let me denote by \( \mathbf{p} := \mathbf{p}_k(n_1, n_2, n_3) \) the kernel of \( \varphi \).

Then \( A \) is a regular local ring of dimension 3 and \( \mathbf{p} \) is a prime ideal in \( A \) of height 2. So in some sense, this is the simplest non-trivial case for the above questions. And my answer is

**Theorem 1** (with Nishida and Watanabe). Let \( m \) and \( n \) be positive integers such that \( n \geq 4 \) and \( 2m > n + 1 \). Let \( n_1 = 7m - 3 \), \( n_2 = 5mn - m - n \), and \( n_3 = 8n - 3 \). Assume that \( \text{GCD}(n_1, n_2, n_3) = 1 \) and let \( \mathbf{p} = \mathbf{p}_k(n_1, n_2, n_3) \). Then the symbolic Rees algebra \( R_S(\mathbf{p}) \) of \( \mathbf{p} \) is a Noetherian ring if and only if the characteristic of the ground field \( k \) is positive. When this is the case, \( R_S(\mathbf{p}) \) is not a Cohen-Macaulay ring.

The simplest example obtained by this theorem is the ideal

\[
\mathbf{p} = \mathbf{p}_k(18, 53, 29) \quad \text{(here } m = 3, \ n = 4 \text{)}
\]

\[
= I_2 \begin{bmatrix} X^4 & Y^2 & Z^5 \\ Y & Z^3 & X^7 \end{bmatrix}
\]

\[
= (Z^8 - X^7Y^2, X^{11} - YZ^5, Y^3 - X^4Z^3)_{A_k}.
\]

Therefore, if we consider the same prime ideal \( P = (Z^8 - X^7Y^2, X^{11} - YZ^5, Y^3 - X^4Z^3)_{B} \) inside of the polynomial ring \( B = k[X, Y, Z] \), the symbolic Rees algebra \( R_S(P) \) is a finitely generated \( k \)-algebra but not a Cohen-Macaulay ring if \( \text{ch } k \) is positive, and if \( \text{ch } k = 0 \), say \( k = \mathbb{Q} \), then it is not a finitely generated \( \mathbb{Q} \)-algebra.
Let me add one question:

**Question** What about the prime ideal

\[ \mathfrak{p} = \mathfrak{p}_k(11, 25, 21) \]

\[ = \mathfrak{I}_2 \left[ \begin{array}{ccc}
X^3 & Y^2 & Z^3 \\
Y & Z^2 & X^5 \\
\end{array} \right] \]

(that is, choose \( m = 2 \) and \( n = 3 \))? Of course, this ideal doesn't satisfy my condition. But by a theorem of Cutkosky you can easily check that \( R_s(\mathfrak{p}) \) is a Noetherian ring, if \( \text{ch } k > 0 \). However I couldn't know whether it is a Noetherian ring or not in the case where \( \text{ch } k = 0 \), though I believe that the answer is negative.

Now let me give a sketch of proof of the theorem. To do this I need a theorem due to Craig (Huneke). For a moment, let me assume that \((A, m)\) is a regular local ring of dimension \( 3 \) and \( \mathfrak{p} \) is a prime ideal in \( A \) of \( \text{dim } A/\mathfrak{p} = 1 \).

**Theorem 2** (C. Huneke). If there exist two elements \( f \in \mathfrak{p}^{(k)} \) and \( g \in \mathfrak{p}^{(l)} \) with positive integers \( k, l \) such that the equality

\[ \ell_A(A/(x, f, g)) = kl \cdot \ell_A(A/\mathfrak{p} + xA) \]

holds for some (and hence for any) element \( x \in m \setminus \mathfrak{p} \), then the symbolic Rees algebra \( R_s(\mathfrak{p}) \) is a Noetherian ring. If the field \( A/m \) is infinite, the converse is also true.

By this theorem, Huneke showed that \( R_s(\mathfrak{p}) \) is a Noetherian ring for \( \mathfrak{p} = \mathfrak{p}_k(n_1, n_2, n_3) \), if \( \min \{n_1, n_2, n_3\} \leq 4 \).
If \( R = R_s(p) \) is a Noetherian ring, then you can easily get an isomorphism \( K_R \cong R(-1) \). Therefore \( R \) is a Gorenstein ring, once it is Cohen-Macaulay. To check the Cohen-Macaulay property of \( R \), you have the following

**Theorem 3** (__, Nishida and Shimoda). Let \( f \) and \( g \) be the elements in the above theorem. Then the following two conditions are equivalent.

1. The symbolic Rees algebra \( R_s(p) \) is a Gorenstein ring.
2. For any integer \( 1 \leq n \leq k + l - 2 \), the ring \( A/(f, g) + p^{(n)} \) is a Cohen-Macaulay ring.

When this is the case, the rings \( A/(f) + p^{(n)} \), \( A/(g) + p^{(n)} \), and \( A/(f, g) + p^{(n)} \) are Cohen-Macaulay for all \( n \geq 1 \), and we have the equality

\[
R_s(p) = A[(p^{(n)}t^n)_{1 \leq n \leq k + l - 2}, ft^k, gt^l].
\]

Using this criterion, you can show that \( R_s(p) \) is a Gorenstein ring for \( p = p_k(n_1, n_2, n_3) \), if \( \min\{n_1, n_2, n_3\} \leq 4 \). But in general, the Cohen-Macaulay property of \( R_s(p) \) depends on the characteristic of the ground field. Let me give one example:

**Example** Let \( p = p_k(7, 11, 13) \). Then \( R_s(p) \) is always a Noetherian ring, but it is a Gorenstein ring if and only if \( \text{ch } k \neq 2, 3 \).

Now let's start the proof of the theorem. In what follows, let \( m \) and \( n \) be positive integers such that \( n \geq 4 \) and \( 2m > n \).
+ 1. Let $n_1 = 7m - 3$, $n_2 = 5mn - m - n$, and $n_3 = 8n - 3$. We assume that $\gcd(n_1, n_2, n_3) = 1$. Then

$$p = p_k(n_1, n_2, n_3)$$

$$= I_2 \left[ \begin{array}{ccc} x^n & y^2 & z^{2m-1} \\ y & z^m & x^{2n-1} \end{array} \right].$$

Let $a = z^{3m-1} - x^{2n-1}y^2$, $b = x^{3n-1} - yz^{2m-1}$, and $c = y^3 - x^n z^m$. Then $p = (a, b, c)$ and we have two equations

$$x^{n_a} + y^2 b + z^{2m-1} c = 0,$$

$$y a + z^m b + x^{2n-1} c = 0.$$

I claim that

**Lemma** There exist elements $d_2 \in p^{(2)}$, and $d_3, d_3' \in p^{(3)}$ such that $d_2 = z^5 m^{-2}$, $d_3 = z^7 m^{-2}$, $d_3' = y^8 z^{2m-2} \pmod{X}$, and

$$X d_3 + Y b c^2 + Z d_3' = 0.$$

**Proof.** First of all, consider two expressions of $-Y^2 ab$:

$$-Y^2 ab = Y b (-Y a) = Y b (z^m b + x^{2n-1} c)$$

$$= a (-Y^2 b) = a (x^n a + z^{2m-1} c).$$

And you get

$$x^n (a^2 - x^{n-1} Y b c) = z^m (Y b^2 - z^{m-1} ac);$$

hence there exists an element $d_2$ of $A$ such that

$$x^n d_2 = Y b^2 - z^{m-1} ac,$$ and
\[ Z^{m_2} d_2 = a^2 - x^{n-1} Y bc. \]

Of course, \( d_2 \) is in \( p^{(2)} \). To get \( Yd_2 \), consider

\[
-Yd_2 = d_2(-Ya) \\
= d_2(Z^{m_2} b + x^{2n-1} c) \\
= b \cdot Z^{m_2} d_2 + x^{n-1} c \cdot Xn_2 \\
= b(a^2 - x^{n-1} Y bc) + x^{n-1} c(Yb^2 - Z^{m-1} ac) \\
= -a(-ab + x^{n-1} Z^{m-1} c^2). 
\]

Thus \( Yd_2 = -ab + x^{n-1} Z^{m-1} c \) and we have two equations:

\[ Yd_2 = -ab + x^{n-1} Z^{m-1} c^2, \]

\[ Z^{m_2} d_2 = a^2 - x^{n-1} Y bc. \]

We compare two expressions of \( a^2 b \):

\[
a^2 b = b(Z^{m_2} d_2 + x^{n-1} Y bc) \\
= a(-Yd_2 + x^{n-1} Z^{m-1} c^2). 
\]

Then we have

\[ Z^{m-1}(-Zbd_2 + x^{n-1} ac^2) = Y(a d_2 + x^{n-1} bc^2). \]

and so we get an element \( d_3 \in p^{(3)} \) such that

\[ Yd_3 = -Zbd_2 + x^{n-1} ac^2. \]

As \( Yd_2 = -ab \mod (X) \), we know

\[ Yd_2 = -Z^{3m-1}(-Yz^{2m-1}); \]
hence \( d_2 = Z^{5m-2} \mod (X) \). As \( Yd_3 = -Zb_2 \mod (X) \), we get
\[
Yd_3 = -Z(-YZ^{2m-1})Z^{5m-2} \mod (X);
\]
hence \( d_3 = Z^{7m-2} \mod (X) \). Notice that 
\[
Yd_3 = x^{n-1}ac^2 = x^{n-1}(-x^{2n-1}y^2)(y^3)^2 \mod (Z)
\]
and we have
\[
d_3 = -x^{3n-2}y^7 \mod (Z),
\]
so that
\[
Xd_3 + Ybc^2 = X(-x^{3n-2}y^7) + Yx^{3n-1}(y^3)^2 = 0 \mod (Z).
\]
Thus there is an element \( d_3' \) of \( p^{(3)} \) such that
\[
Xd_3 + Ybc^2 + Zd_3' = 0.
\]
Clearly \( d_3' = Y^8Z^{2m-2} \mod (X) \). This proves the lemma.

**Proposition** \( p^{(2)} = p^2 + (d_2), \quad p^{(3)} = pp^{(2)} + (d_3, d_3') \), and \( p^{(4)} = pp^{(3)} + (p^{(2)})^2 \).

For example, let \( I = p^2 + (d_2) \). Then as \( (X) + I = (X) + (Z^{3m-1}, YZ^{2m-1}, Y^3)^2 + (Z^{5m-2}), \) you have
\[
l_A(A/(X) + I) = 3 \cdot (7m - 3)
\]
\[
= 3 \cdot l_A(A/(X) + p).
\]
On the other hand, because \( l_A(A/(X) + p^{(2)}) = e_{X_A}(A/p^{(2)}) = l_A(A/(X) + p) \cdot l_{A_p}(A/p^2A) = 3 \cdot l_A(A/(X) + p) \), you get that
\[ l_A(A/(X) + I) = l_A(A/(X) + p^{(2)}) \] hence \((X) + I = (X) + p^{(2)}\), because \(I \subseteq p^{(2)}\). Consequently \(p^{(2)} = I + (X) \cap p^{(2)} = I + X p^{(2)}\). Thus we have \(p^{(2)} = I\) by Nakayama’s lemma. Similarly you can show that \(p^{(3)} = pp^{(2)} + (d_3, d_3')\). As

\[
l_A(A/(X) + p^{(4)}) = e_{XA}(A/p^{(4)}) = l_A(A/(X) + p) \cdot l_A(A/p^4A_p) = 10 \cdot l_A(A/(X) + p) < l_A(A/(X) - pp^{(3)} + (p^{(2)})^2),
\]

we have

\(p^{(4)} = pp^{(3)} + (p^{(2)})^2\).

**Corollary** The ring \(A/(c) + p^{(3)}\) is not Cohen-Macaulay.

In fact, notice that

\[
l_A(A/(X, c) + p^{(3)}) = 3 \cdot (7m - 3) + 1 > e_{XA}(A/(c) + p^{(3)}) = 3 \cdot (7m - 3);
\]

hence \(A/(c) + p^{(3)}\) cannot be a Cohen-Macaulay ring.

Now let me assume that \(ch k = p > 0\). First of all, assume that \(p \geq 3\) and write \(p = 2q + 1\) (hence \(q \geq 1\)). Then by the equations

\[ Xd_3 + Yb^2 + Zd_3' = 0, \]

we get

\[ 0 = Xp d_3 + Yb b p c^2 p \mod (Zp) = Xp d_3 + (Y^2 b) Yb q^{q+1} c^2 p. \]
As \( X^n a + Y^2 b + Z^{2m-1} c = 0 \), we furthermore have

\[
0 = X^p d_3^p + (-1)^q \sum_{i=0}^{q} \binom{q}{i} X^{n(q-i)} Y^Z(2m-1)i_a q^{-i_b q+1} c^2 p^i
\]

\[
= X^p d_3^p + \sum_{(2m-1)i < p} \binom{q}{i} X^{n(q-i)} Y^Z(2m-1)i_a q^{-i_b q+1} c^2 p^i
\]

mod \( (Z^p) \).

Now recall that \( 2m > n + 1 \) and \( n \geq 4 \). Then we have \( n(q - i) \geq p \) or \( (2m - 1)i \geq p \) for each \( 0 \leq i \leq q \).
(In fact, if \( n(q - i) < p \) and \( (2m - 1)i < p \), then we get \( n(q - i) \leq 2q \) and \( (2m - 1)i \leq 2q \) so that \( nq + (2m - n - 1)i \leq 4q \). Hence we must have \( n = 4 \) and \( i = 0 \) and so \( n(q - i) = 4q \leq 2q \), which is impossible.) Thus

\[
0 = X^p \left\{ d_3^p + \sum_{(2m-1)i < p} \binom{q}{i} X^{n(q-i)} Y^Z(2m-1)i_a q^{-i_b q+1} c^2 p^i \right\}
\]

mod \( (Z^p) \) and thus we have an element \( h \in d_{(3p)} \) such that

\[
Z^p h = d_3^p + \sum_{(2m-1)i < p} \binom{q}{i} X^{n(q-i)} Y^Z(2m-1)i_a q^{-i_b q+1} c^2 p^i
\]

As \( Z^p h = d_3^p = Z^{(7m-2)p} \) mod \( (X, c) \), we get \( h = Z^{(7m-3)p} \)
mod \( (X, c) \). Thus we have the following
**Lemma** There exists an element \( h \in p^{(3p)} \) such that \( h = z^{(7m-3)p} \mod (X, c). \)

(You can prove this lemma also in the case \( p = 2 \).)

Now recall Huneke's theorem. First we take \( f = c \) and \( g = h \). Then

\[
\ell_A(A/(X, c, h)) = \ell_A(A/(X, c, Z^{(7m-3)p}))
= \ell_A(A/(X, Y^3, Z^{(7m-3)p}))
= 3p \cdot (7m-3)
= 1 \cdot 3p \cdot \ell_A(A/(X) + p).
\]

Hence \( R_s(p) \) is a Noetherian ring by Theorem 2. Because \( A/(c) + p^{(3)} \) is not a Cohen-Macaulay ring, \( R_s(p) \) cannot be Cohen-Macaulay by Theorem 3.

To study the case of \( ch k = 0 \), we need further information in the case where \( ch k = p > 0 \). Let \( F = \{ 0 < \ell \in Z \mid \exists g \in p^{(l)} \} \) such that \( \ell_A(A/(X, c, g)) = \ell \cdot \ell_A(A/(X) + p) \). Then \( 3p \in F \).

Let \( \ell_0 = \min F \) and choose \( g_0 \in p^{(\ell_0)} \) so that \( \ell_A(A/(X, c, g_0)) = \ell_0 \cdot \ell_A(A/(X) + p) \). Then we have

**Lemma**

1. \( \ell_0 | \ell \) for all \( \ell \in F \).
2. \( R_s(p) = A[(p^{(n)}t^n)_{1 \leq n \leq \ell_0 - 1}, \text{ct}, g_0^{\ell_0}] \).
3. \( g_0^{\ell_0} \) is not contained in \( A[(p^{(n)}t^n)_{1 \leq n \leq \ell_0 - 1}] \).

Let me use this lemma without proof. First, we have by (1) that \( \ell_0 | 3p \); hence \( \ell_0 = 1, 3, p, \) or \( 3p \). But if \( \ell_0 = p, \) then
we have by (2) that \( R_s(p) = A[p, p^{(2)}t^2, p^{(3)}t^3] \), which is impossible because \( p^{(4)} \neq pp^{(3)} + (p^{(2)})^2 \). Thus \( t_0 \geq p \) and by (3) we get \( g_0 t_0 \) is not contained in \( A[p^{(n)}t^n]_{1 \leq n \leq t_0 - 1} \).

This means, to generate the A-algebra \( R_s(p) \), you need at least one new element of degree \( \geq p \), depending on the characteristic \( p = \text{ch} \ k \). On the other hand, if \( R_s(p) \) were a Noetherian ring in the case where \( \text{ch} \ k = 0 \), say \( k = \mathbb{Q} \), then because everything is defined over \( \mathbb{Z} \), you can find a system of generators for the algebra \( R_s(p_Q) \) so that passing to the field \( k = \mathbb{Z}/p\mathbb{Z} \) for \( p >> 0 \), the system still generates the algebra \( R_s(p_k) \) (see the theorem below). This is impossible, because you need at least one new element of degree \( \geq p \). Thus \( R_s(p_Q) \) cannot be a Noetherian ring for our example \( p \).

Let me state the required theorem more explicitly.

**Theorem.** Let \( C = \mathbb{Z}[X, Y, Z] \) and let \( I = \text{Ker} \ (\varphi : C \to \mathbb{Z}[t]) \) where \( \varphi(X) = t^{n_1} \), \( \varphi(Y) = t^{n_2} \), and \( \varphi(Z) = t^{n_3} \). Then if \( R_s(p) \) is a Noetherian ring for the prime ideal \( p = p_Q^{(n_1, n_2, n_3)} \) in \( \mathbb{Q}[X, Y, Z] \), there exist positive integers \( l \) and \( N \) and elements \( f \) and \( g \) of \( I^{(l)} \) such that for all prime numbers \( p \geq N \), we have

1. \( I^{(l)} A_k = p_k^{(l)} \) and
2. \( l_{A_k} (A_k/(X, f, g)A_k) = l^2 \cdot l_{A_k} (A_k/(X) + p_k) \),

where \( k = \mathbb{Z}/p\mathbb{Z} \).

Here \( A_k = k[[X, Y, Z]] \) and \( p_k = p_k^{(n_1, n_2, n_3)} \).

Before closing my talk, let me give a few open problems.
Problems Let $p = p_k(n_1, n_2, n_3)$ and $n = \min \{n_1, n_2, n_3\}$.
(1) $\text{ch} \ k = p > 0 \Rightarrow R_{s}(p)$ is a Noetherian ring?
(2) $\text{ch} \ k = 0$ and $R_{s}(p)$ is Noetherian $\Rightarrow R_{s}(p)$ is a Gorenstein ring?
(3) $n \leq 8, n \neq 7 \Rightarrow R_{s}(p)$ is a Gorenstein ring? (For $p = p_k(9, 10, 13)$ you can show that $R_{s}(p)$ is a Noetherian ring but not Cohen-Macaulay, if $\text{ch} \ k = 2, 3, 7$.)
(4) $n = 5 \Rightarrow R_{s}(p)$ is a Noetherian ring?
(5) $n = 6 \Rightarrow R_{s}(p)$ is a Gorenstein ring? (The Noetherian property of this case was guaranteed by Cutkosky.)
(6) $p = p_k(11, 16, 13)$ $\Rightarrow$ ?????

References

25.  


