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Factorization of graphs with common transversal sets

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Abstract. In this paper, S_k-factorization algorithm of an edge-disjoint sum of two K_k-factors of K_v is given.

1. Introduction

Let G and H be graphs. A spanning subgraph F of G is called an H-*factor* if and only if each component of F is isomorphic to H. If G is expressible as an edge-disjoint sum of H-factors, then this sum is called an H-*factorization* of G.
Let S_k be a *star* with k vertices. Let K_k and K_v be a *complete graph* with k vertices and v vertices, respectively.

2. Common transversal sets

We use common transversal sets on S_k-factorization of an edge-disjoint sum of two K_k-factors of K_v.
Consider a set N of v elements, where v=kt. Divide N into t subsets $A_1,A_2,\dots,A_t$ so that they are mutually disjoint subsets of same size k. And divide N into another t subsets $B_1,B_2,\dots,B_t$ so that they are mutually disjoint subsets of same size k. Let T be a t-element subsets of N. Then T ia called a *common transversal set* of $\{A_1,A_2,\dots,A_t\}$ and $\{B_1,B_2,\dots,B_t\}$ when it hold that $|T \cap A_j| = |T \cap B_j| = 1$, $1 \leqq j \leqq t$.

Lemma 1. *Let $\{A_1,A_2,\dots,A_t\}$ be a mutually disjoint partition of N and $\{B_1,B_2,\dots,B_t\}$ be another mutually disjoint partition of N. Then there exists a common transversal set T of $\{A_1,A_2,\dots,A_t\}$ and $\{B_1,B_2,\dots,B_t\}$.*

3. S_k-factorization algorithm of an edge-disjoint sum of two K_k-factors of K_v

For $k \geqq 3$, we have the following:

Lemma 2. *An edge-disjoint sum of two K_k-factors of K_v can be factorized into k S_k-factors.*

Proof. Let F_1 and F_2 be edge-disjoint K_k-factors of K_v. And let G be a sum of F_1 and F_2. Then $V(F_1)=V(F_2)=V(G)=V(K_v)$. Put $F_1=K_k^{(1)} \cup K_k^{(2)} \cup \ldots \cup K_k^{(t)}$ and $F_2=K_k^{(t+1)} \cup K_k^{(t+2)} \cup \ldots \cup K_k^{(2t)}$. And let $A_j=V(K_k^{(j)})$ and $B_j=V(K_k^{(t+j)})$, $1 \leqq j \leqq t$. Then $\{A_1,A_2,\ldots, A_t\}$ is a mutually disjoint partition of V(G) and $\{B_1,B_2,\ldots,B_t\}$ is another mutually disjoint partition of V(G). Let T_1 be a common transversal set of $\{A_1,A_2,\ldots,A_t\}$ and $\{B_1,B_2,\ldots,B_t\}$. Let T_j be a common transversal set of $\{A_1-T,A_2-T,\ldots,A_t-T\}$ and $\{B_1-T,B_2-T, \ldots,B_t-T\}$, where $T=T_1+T_2+\ldots+T_{j-1}$ $(2 \leqq j \leqq k)$. Consider 2k-2 subgraphs G_j of G such as $G_1=F_1$, $G_2=F_2$, $G_j= G_{j-2}-T_{j-2}$ $(3 \leqq j \leqq 2k-2)$, where $T_{k+i}=T_{k-i+1}$ $(1 \leqq i \leqq k-2)$. Consider 2k-2 subgraphs H_j of G such as $H_j=G_j-E(G_j-T_j)$ $(1 \leqq j \leqq 2k-4)$, $H_{2k-3}=G_{2k-3}$, $H_{2k-2}=G_{2k-2}$.
Note that every component of G_j is a complete graph with (2k-j+1)/2 vertices (j:odd) or (2k-j+2)/2 vertices (j:even) and that every component of H_j is a star with (2k-j+1)/2 vertices (j:odd) or (2k-j+2)/2 vertices (j:even).
Then we can construct k edge-disjoint S_k-factors $F_1',F_2',F_3, \ldots,F_k$ of G as follows:

$$F_1'=H_1,\ F_2'=H_2,\ F_j=H_j \cup H_{2k-j+1}\ (3 \leqq j \leqq k).$$

Therefore, it holds that $G=F_1'(+)F_2'(+)F_3(+)\ldots(+)F_k$, which is an S_k-factorization.
□

Note 1. The symbol (+) is used to denote the sum of factors.

As a resolvable BIBD(v,b,r,k, λ =1) is just a K_k-factorization of K_v, we have the following lemmas.

Lemma 3. *If there exists a resolvable BIBD(v,b,r,k, λ =1) (r:even), then K_v has an S_k-factorization.*

Lemma 4. *If there exists a resolvable BIBD(v,b,r,k, λ =1) (r:odd, m=v/k), then K(m;k) has an S_k-factorization.*

Note 2. K(m;k) is a complete multipartite graph with m partite sets of k vertices each.

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