<table>
<thead>
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<th>Title</th>
<th>Factorization of graphs with common transversal sets (Combinatorial Aspects on the Analysis of Mathematical Models)</th>
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</thead>
<tbody>
<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1992), 802: 137-139</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1992-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/82869">http://hdl.handle.net/2433/82869</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Factorization of graphs with common transversal sets

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Abstract. In this paper, $S_k$-factorization algorithm of an edge-disjoint sum of two $K_k$-factors of $K_v$ is given.

1. Introduction

Let $G$ and $H$ be graphs. A spanning subgraph $F$ of $G$ is called an $H$-factor if and only if each component of $F$ is isomorphic to $H$. If $G$ is expressible as an edge-disjoint sum of $H$-factors, then this sum is called an $H$-factorization of $G$. Let $S_k$ be a star with $k$ vertices. Let $K_k$ and $K_v$ be a complete graph with $k$ vertices and $v$ vertices, respectively.

2. Common transversal sets

We use common transversal sets on $S_k$-factorization of an edge-disjoint sum of two $K_k$-factors of $K_v$.

Consider a set $N$ of $v$ elements, where $v=kt$. Divide $N$ into $t$ subsets $A_1, A_2, \ldots, A_t$ so that they are mutually disjoint subsets of same size $k$. And divide $N$ into another $t$ subsets $B_1, B_2, \ldots, B_t$ so that they are mutually disjoint subsets of same size $k$. Let $T$ be a $t$-element subsets of $N$. Then $T$ is called a common transversal set of $\{A_1, A_2, \ldots, A_t\}$ and $\{B_1, B_2, \ldots, B_t\}$ when it hold that $|T \cap A_j| = |T \cap B_j| = 1$, $1 \leq j \leq t$.

Lemma 1. Let $(A_1, A_2, \ldots, A_t)$ be a mutually disjoint partition of $N$ and $(B_1, B_2, \ldots, B_t)$ be another mutually disjoint partition of $N$. Then there exists a common transversal set $T$ of $(A_1, A_2, \ldots, A_t)$ and $(B_1, B_2, \ldots, B_t)$.

3. $S_k$-factorization algorithm of an edge-disjoint sum of two $K_k$-factors of $K_v$

For $k \geq 3$, we have the following:
Lemma 2. An edge-disjoint sum of two $K_k$-factors of $K_v$ can be factorized into $k$ $S_k$-factors.

Proof. Let $F_1$ and $F_2$ be edge-disjoint $K_k$-factors of $K_v$. And let $G$ be a sum of $F_1$ and $F_2$. Then $V(F_1)=V(F_2)=V(G)=V(K_v)$. Put $F_1=K_k^{(1)} \cup K_k^{(2)} \cup \ldots \cup K_k^{(t)}$ and $F_2=K_k^{(t+1)} \cup K_k^{(t+2)} \cup \ldots \cup K_k^{(2t)}$. And let $A_j=V(K_k^{(j)})$ and $B_j=V(K_k^{(t+j)})$, $1 \leq j \leq t$. Then $\{A_1,A_2,\ldots,A_t\}$ is a mutually disjoint partition of $V(G)$ and $\{B_1,B_2,\ldots,B_t\}$ is another mutually disjoint partition of $V(G)$. Let $T_1$ be a common transversal set of $\{A_1,A_2,\ldots,A_t\}$ and $\{B_1,B_2,\ldots,B_t\}$. Let $T_j$ be a common transversal set of $\{A_1-T,A_2-T,\ldots,A_t-T\}$ and $\{B_1-T,B_2-T,\ldots,B_t-T\}$, where $T=T_1+T_2+\ldots+T_{t-1}$ ($2 \leq j \leq k$). Consider $2k-2$ subgraphs $G_j$ of $G$ such as $G_1=F_1$, $G_2=F_2$, $G_3=G_{j-2}-T_j$, $j \leq 2k-2$, $2 \leq j \leq 2k-2$, $2 \leq j \leq 2k-2$. Consider $2k-2$ subgraphs $H_j$ of $G$ such as $H_j=G_j-\{G_j-T_j\}$ ($1 \leq j \leq 2k-4$), $H_{2k-3}=G_{2k-3}$, $H_{2k-3}=G_{2k-2}$.

Note that every component of $G_j$ is a complete graph with $(2k-j+1)/2$ vertices ($j$ odd) or $(2k-j+2)/2$ vertices ($j$ even) and that every component of $H_j$ is a star with $(2k-j+1)/2$ vertices ($j$ odd) or $(2k-j+2)/2$ vertices ($j$ even).

Then we can construct $k$ edge-disjoint $S_k$-factors $F_1,F_2,F_3,\ldots,F_k$ of $G$ as follows:

$F_1=H_1$, $F_2,H_2$, $F_j=H_j \cup H_{2k-j+1}$ \mbox{ (3 $\leq j \leq k$)}.

Therefore, it holds that $G=F_1(+)F_2(+)F_3(+)\ldots(+)F_k$, which is an $S_k$-factorization.

Note 1. The symbol (+) is used to denote the sum of factors.

As a resolvable BIBD$(v,b,r,k,\lambda=1)$ is just a $K_k$-factorization of $K_v$, we have the following lemmas.

Lemma 3. If there exists a resolvable BIBD$(v,b,r,k,\lambda=1)$ ($r$: even), then $K_v$ has an $S_k$-factorization.

Lemma 4. If there exists a resolvable BIBD$(v,b,r,k,\lambda=1)$ ($r$: odd, $m=v/k$), then $K(m;k)$ has an $S_k$-factorization.

Note 2. $K(m;k)$ is a complete multipartite graph with $m$ partite sets of $k$ vertices each.
References


