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<th>Title</th>
<th>Factorization of graphs with common transversal sets (Combinatorial Aspects on the Analysis of Mathematical Models)</th>
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<tbody>
<tr>
<td>Author(s)</td>
<td>Ushio, Kazuhiko</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録　1992年度　第802号　137-139</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1992-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/82869">http://hdl.handle.net/2433/82869</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Factorization of graphs with common transversal sets

近畿大学工　澳 和彦 (Kazuhiko Ushio)

Abstract. In this paper, \( S_k \)-factorization algorithm of an edge-disjoint sum of two \( K_k \)-factors of \( K_v \) is given.

1. Introduction

Let \( G \) and \( H \) be graphs. A spanning subgraph \( F \) of \( G \) is called an \( H \)-factor if and only if each component of \( F \) is isomorphic to \( H \). If \( G \) is expressible as an edge-disjoint sum of \( H \)-factors, then this sum is called an \( H \)-factorization of \( G \).

Let \( S_k \) be a star with \( k \) vertices. Let \( K_k \) and \( K_v \) be a complete graph with \( k \) vertices and \( v \) vertices, respectively.

2. Common transversal sets

We use common transversal sets on \( S_k \)-factorization of an edge-disjoint sum of two \( K_k \)-factors of \( K_v \).

Consider a set \( N \) of \( v \) elements, where \( v=kt \). Divide \( N \) into \( t \) subsets \( A_1,A_2,...,A_t \) so that they are mutually disjoint subsets of same size \( k \). And divide \( N \) into another \( t \) subsets \( B_1,B_2,...,B_t \) so that they are mutually disjoint subsets of same size \( k \). Let \( T \) be a \( t \)-element subsets of \( N \). Then \( T \) is called a common transversal set of \( \{A_1,A_2,...,A_t\} \) and \( \{B_1,B_2,...,B_t\} \) when it hold that \( |T \cap A_j| = |T \cap B_j| = 1, 1 \leq j \leq t \).

Lemma 1. Let \( \{A_1,A_2,...,A_t\} \) be a mutually disjoint partition of \( N \) and \( \{B_1,B_2,...,B_t\} \) be another mutually disjoint partition of \( N \). Then there exists a common transversal set \( T \) of \( \{A_1,A_2,...,A_t\} \) and \( \{B_1,B_2,...,B_t\} \).

3. \( S_k \)-factorization algorithm of an edge-disjoint sum of two \( K_k \)-factors of \( K_v \)

For \( k \geq 3 \), we have the following:
Lemma 2. An edge-disjoint sum of two $K_k$-factors of $K_v$ can be factorized into $k$ $S_k$-factors.

Proof. Let $F_1$ and $F_2$ be edge-disjoint $K_k$-factors of $K_v$. And let $G$ be a sum of $F_1$ and $F_2$. Then $V(F_1) = V(F_2) = V(G) = V(K_v)$. Put $F_1 = K_k^{(1)} \cup K_k^{(2)} \cup \ldots \cup K_k^{(t)}$ and $F_2 = K_k^{(t+1)} \cup K_k^{(t+2)} \cup \ldots \cup K_k^{(2t)}$. And let $A_j = V(K_k^{(j)})$ and $B_j = V(K_k^{(t+j)})$, $1 \leq j \leq t$. Then $\{A_1, A_2, \ldots, A_t\}$ is a mutually disjoint partition of $V(G)$ and $\{B_1, B_2, \ldots, B_t\}$ is another mutually disjoint partition of $V(G)$. Let $T_1$ be a common transversal set of $\{A_1, A_2, \ldots, A_t\}$ and $\{B_1, B_2, \ldots, B_t\}$. Let $T_j$ be a common transversal set of $\{A_1 - T_1, A_2 - T_2, \ldots, A_t - T_t\}$ and $\{B_1 - T_1, B_2 - T_2, \ldots, B_t - T_t\}$, where $T_1 = T_2 + \ldots + T_{t-1}$ $(2 \leq j \leq k)$. Consider $2k-2$ subgraphs $G_j$ of $G$ such as $G_1 = F_1$, $G_2 = F_2$, $G_3 = G_{j-2} - T_{j-2}$ $(3 \leq j \leq 2k-2)$, where $T_{k+i} = T_{k-i+1}$ $(1 \leq i \leq 2k-2)$. Consider $2k-2$ subgraphs $H_j$ of $G$ such as $H_1 = G_j - E(G_j - T_j)$ $(1 \leq j \leq 2k-4)$, $H_{2k-3} = G_{2k-3}$, $H_{2k-2} = G_{2k-2}$. Note that every component of $G_j$ is a complete graph with $(2k-j+1)/2$ vertices ($j$; odd) or $(2k-j+2)/2$ vertices ($j$; even) and that every component of $H_j$ is a star with $(2k-j+1)/2$ vertices ($j$; odd) or $(2k-j+2)/2$ vertices ($j$; even).

Then we can construct $k$ edge-disjoint $S_k$-factors $F_1', F_2', F_3', \ldots, F_k$ of $G$ as follows:

$F_1' = H_1$, $F_2' = H_2$, $F_j' = H_j \cup H_{2k-j+1}$ $(3 \leq j \leq k)$.

Therefore, it holds that $G = F_1' + F_2' + F_3' + \ldots + F_k$, which is an $S_k$-factorization. \qed

Note 1. The symbol (+) is used to denote the sum of factors.

As a resolvable BIBD($v, b, r, k, \lambda = 1$) is just a $K_k$-factorization of $K_v$, we have the following lemmas.

Lemma 3. If there exists a resolvable BIBD($v, b, r, k, \lambda = 1$) ($r$: even), then $K_v$ has an $S_k$-factorization.

Lemma 4. If there exists a resolvable BIBD($v, b, r, k, \lambda = 1$) ($r$: odd, $m = v/k$), then $K(m; k)$ has an $S_k$-factorization.

Note 2. $K(m; k)$ is a complete multipartite graph with $m$ partite sets of $k$ vertices each.
References