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Analysis of Variance of Partially Balanced Fractional $2^{m_1+m_2}$ Factorial Designs of Resolution IV

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Abstract

In this paper, attention is focused on the analysis of variance of partially balanced fractional $2^{m_1+m_2}$ factorial designs of resolution IV by using the algebraic structure. They can be obtained by partially balanced arrays with some conditions.

1. Introduction

A partially balanced array (PB-array), which is a special case of an asymmetrical balanced array of type 2 as introduced by Nishii [14], has been studied by several researchers (e.g., [4]). Necessary and sufficient conditions for the existence of a PB-array were obtained by Kuwada and Kuriki [10]. A PB-array yields a partially balanced fractional $2^{m_1+m_2}$ factorial ($2^{m_1+m_2}$-PBFF) design under some conditions (see [5,6]). However a $2^{m_1+m_2}$-PBFF design does not always mean a PB-array.

It is generally difficult to obtain the designs of resolution 2ℓ since there is a little information about the ℓ-factor interactions. For earlier works on such designs, see for example, Kuwada and/or Matsuura [3,11], Margolin [12,13], Shirakura [17-20], Srivastava and/or Anderson [1,22], and Webb [23]. Especially, by using the triangular multidimensional partially balanced (TMDPB) association scheme and its algebra, Shirakura [17] showed that a balanced array with index $\mu_\ell=0$ turns out to be a balanced fractional 2^m factorial design of resolution 2ℓ under some conditions. Such a design permits to estimate all factorial effects up to the (ℓ-1)-factor interactions and some linear combinations of the ℓ-factor ones.

The analysis of variance (ANOVA) is a statistical technique for handling the data or observations derived from an experiment (cf. [9,15,16]). The ANOVA of $2^{m_1+m_2}$-PBFF designs of resolution V which are derived from PB-arrays has been studied by Kuwada [8]. In this paper, we present the ANOVA and the hypothesis testing of $2^{m_1+m_2}$-PBFF designs of resolution IV, which are PB-arrays. The designs considered here permit estimation of the general mean, all main effects and (A) all $\binom{m_1}{2}+\binom{m_2}{2}$ two-factor interactions and some linear combinations of the m_1m_2 ones, (B) all $\binom{m_1}{2}$ ones and some linear combinations of the $\binom{m_2}{2}$ ones and of the m_1m_2 ones, or (C) some linear combinations of the $\binom{m_k}{2}$ ones (k=1,2) and of the m_1m_2 ones (see [3,11]).

2. Preliminaries

Consider a factorial experiment with m_1+m_2 factors at two levels (0 and 1, say) of each, where $m_k\geq2$. Further consider the situation in which three-factor and higher order interactions are assumed to be negligible. The vector of unknown factorial effects to be estimated is then given by $(\Theta'_{00};\Theta'_{10};\Theta'_{01};\Theta'_{20};\Theta'_{02};\Theta'_{11})$ ($=\Theta'$, say), where $\Theta'_{00}=(\{\theta(0;0)\})$, $\Theta'_{10}=(\{\theta(u;0)\})$, $\Theta'_{01}=(\{\theta(0;v)\})$, $\Theta'_{20}=(\{\theta(u_1u_2;0)\})$, $\Theta'_{02}=(\{\theta(0;v_1v_2)\})$ and $\Theta'_{11}=(\{\theta(u;v)\})$. Here $1\leq u\leq m_1$, $1\leq v\leq m_2$, $1\leq u_1<u_2\leq m_1$ and $1\leq v_1<v_2\leq m_2$, and A$'$ denotes the

transpose of a matrix A. Note that the total number of factorial effects to be estimated is $1+(m_1+m_2)+\binom{m_1+m_2}{2}$ $(=\nu(m_1m_2)$, say). Let $[T^{(1)};T^{(2)}](=T$, say) be a fraction with N assemblies (or treatment combinations), where $T^{(k)}$'s are (0,1)-matrices of size $N\times m_k$. Then the ordinary linear model is given by

$$\mathbf{y}_T = E_T\Theta + \mathbf{e}_T, \tag{2.1}$$

where $\mathbf{y}_T$ and E_T are the vector of N observations and the design matrix of size $N\times\nu(m_1m_2)$, respectively, and $\mathbf{e}_T$ is an error vector distributed as $\mathcal{N}(\mathbf{0}_N,\sigma^2 I_N)$. Here $\mathbf{0}_p$ and I_p denote the $p\times1$ vector with all zero and the identity matrix of order p, respectively. The normal equation for estimating Θ is given by $M_T\hat{\Theta}=E'_T\mathbf{y}_T$, where $M_T=E'_TE_T$. If the information matrix M_T is nonsingular, the BLUE of Θ and its variance-covariance matrix are given by $\hat{\Theta}=M_T^{-1}E'_T\mathbf{y}_T$ and $\mathrm{Var}[\hat{\Theta}]=\sigma^2M_T^{-1}$, respectively.

Suppose a relation of association is defined among the sets $\{(u_1\cdots u_{a_1};v_1\cdots v_{a_2})\}$, where $1\le u_1<\cdots<u_{a_1}\le m_1$ and $1\le v_1<\cdots<v_{a_2}\le m_2$, in such a way that $(u_1\cdots u_{a_1};v_1\cdots v_{a_2})$ and $(u'_1\cdots u'_{b_1};v'_1\cdots v'_{b_2})$ are the $(\alpha_1\alpha_2)th$ associates if

$$|\{u_1,\cdots,u_{a_1}\}\cap\{u'_1,\cdots,u'_{b_1}\}| = \min(a_1,b_1) - \alpha_1$$

and

$$|\{v_1,\cdots,v_{a_2}\}\cap\{v'_1,\cdots,v'_{b_2}\}| = \min(a_2,b_2) - \alpha_2,$$

where $|S|$ and $\min(a,b)$ denote the cardinality of a set S and the minimum value of integers a and b, respectively. The scheme thus defined is called the extended TMDPB (ETMDPB) association scheme (see [5]), which is regarded as a generalization of the TMDPB association scheme (e.g., [24,25]). Let $A_{\alpha_1\alpha_2}^{(a_1a_2,b_1b_2)}$ and $D_{\alpha_1\alpha_2}^{(a_1a_2,b_1b_2)}$ be the local association matrices of size $n(a_1a_2)\times$

$n(b_1b_2)$ and the ordered association matrices of order $\nu(m_1m_2)$ of the ETMDPB association scheme, respectively (see [5]), where $n(a_1a_2)=\binom{m_1}{a_1}\binom{m_2}{a_2}$. Further let $A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}=A^{\#(a_1,b_1)}_{\beta_1}\otimes A^{\#(a_2,b_2)}_{\beta_2}$, where $A^{\#(a,b)}_{\beta}$'s are the matrices which are linearly linked with the local association matrices $A^{(a,b)}_{\alpha}$ of the TMDPB association scheme (e.g., [25]), and $\otimes$ denotes the Kronecker product. A relationship between $A^{(a,b)}_{\alpha}$'s and $A^{\#(a,b)}_{\beta}$'s is given by

$$A^{(a,b)}_{\alpha} = \{A^{(b,a)}_{\alpha}\}' = \sum_{\beta} z^{(a,b)}_{\beta\alpha} A^{\#(a,b)}_{\beta} \quad \text{for } 0\le a\le b\le m$$

and

$$A^{\#(a,b)}_{\beta} = \{A^{\#(b,a)}_{\beta}\}' = \sum_{\alpha} z^{\beta\alpha}_{(a,b)} A^{(a,b)}_{\alpha} \quad \text{for } 0\le a\le b\le m,$$

where

$$z^{(a,b)}_{\beta\alpha} = \sum_{p=0}^{\alpha} (-1)^{p-\alpha}\binom{a-\beta}{p}\binom{a-p}{a-\alpha}\binom{m-a-\beta+p}{p}\left\{\binom{m-a-\beta}{b-a}\binom{b-\beta}{b-a}\right\}^{1/2}\Big/\binom{b-a+p}{p}$$
$$\text{for } a\le b,$$

$$z^{\beta\alpha}_{(a,b)} = \phi_{\beta} z^{(a,b)}_{\beta\alpha}\Big/\left\{\binom{m}{a}\binom{a}{\alpha}\binom{m-a}{b-a+\alpha}\right\} \quad \text{for } a\le b$$

and

$$\phi_{\beta} = \binom{m}{\beta} - \binom{m}{\beta-1}$$

(e.g., [7,21,25]). The matrices $A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}$ have the following properties:

$$A^{\#(a_1a_2,b_1b_2)}_{00} = [1/\{n(a_1a_2)\times n(b_1b_2)\}^{1/2}]G_{n(a_1a_2)\times n(b_1b_2)}, \tag{2.2}$$

$$\sum_{\beta_1\beta_2} A^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2} = I_{n(a_1a_2)}, \tag{2.3}$$

$$A^{\#(a_1a_2,c_1c_2)}_{\beta_1\beta_2} A^{\#(c_1c_2,b_1b_2)}_{\gamma_1\gamma_2} = \delta_{\beta_1\gamma_1}\delta_{\beta_2\gamma_2} A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2} \tag{2.4}$$

and

$$\operatorname{rank}(A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}) = \phi_{\beta_1}\times\phi_{\beta_2} \ (=\phi_{\beta_1\beta_2}, \text{ say}) \tag{2.5}$$

(see [5]), where $G_{p\times q}$ and δ_{pq} denote the $p\times q$ matrix with all unity and the Kronecker delta, respectively.

Let T be a PB-array of strength t_1+t_2 and size N having m_1+m_2 constraints, two levels, and index set $\{\mu(i_1i_2) \mid 0\le i_k\le t_k\le m_k\}$, where

$$t_k = \begin{cases} m_k & \text{if } m_k=2,3, \\ 4 & \text{if } m_k \geq 4. \end{cases} \tag{2.6}$$

This array is written as PBA$(N, m_1+m_2, 2, t_1+t_2; \{\mu(i_1 i_2)\})$ for brevity. The information matrix M_T associated with T, which is a PB-array with (2.6), can be expressed as

$$\begin{aligned} M_T &= \sum_{a_1 a_2} \sum_{b_1 b_2} \sum_{\alpha_1 \alpha_2} \gamma_{|a_1-b_1|+2\alpha_1, |a_2-b_2|+2\alpha_2} D^{(a_1 a_2, b_1 b_2)}_{\alpha_1 \alpha_2} \\ &= \sum_{a_1 a_2} \sum_{b_1 b_2} \sum_{\beta_1 \beta_2} \kappa^{a_1 a_2, b_1 b_2}_{\beta_1 \beta_2} D^{\#(a_1 a_2, b_1 b_2)}_{\beta_1 \beta_2}, \end{aligned} \tag{2.7}$$

where $D^{\#(a_1 a_2, b_1 b_2)}_{\beta_1 \beta_2}$'s are the matrices of order $\nu(m_1 m_2)$ which are given by some linear combinations of $D^{(a_1 a_2, b_1 b_2)}_{\alpha_1 \alpha_2}$'s,

$$\gamma_{j_1, j_2} = \sum_{i_1=0}^{t_1} \sum_{i_2=0}^{t_2} [\prod_{k=1}^{2} \{\sum_{p_k=0}^{j_k} (-1)^{p_k} \binom{j_k}{p_k} \binom{t_k-j_k}{i_k-j_k+p_k}\}] \mu(i_1 i_2)$$

and

$$\kappa^{a_1 a_2, b_1 b_2}_{\beta_1 \beta_2} = \sum_{\alpha_1 \alpha_2} [\prod_{k=1}^{2} \{z^{(a_k, b_k)}_{\beta_k \alpha_k}\}] \gamma_{|a_1-b_1|+2\alpha_1, |a_2-b_2|+2\alpha_2}$$

(see [5]).

3. $2^{m_1+m_2}$-PBFF designs of resolution IV

Throughout this paper, we consider a design, which is a PBA $(N, m_1+m_2, 2, t_1+t_2; \{\mu(i_1 i_2)\})$ with (2.6). Let $K_{\beta_1 \beta_2} = \| \kappa^{a_1 a_2, b_1 b_2}_{\beta_1 \beta_2} \|$ for $\beta_1 \beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11. Then a necessary and sufficient condition for the information matrix M_T to be nonsingular, i.e., T is of resolution V, is that every $K_{\beta_1 \beta_2}$ is positive definite (see [5]). Note that a PBA$(N, m_1+m_2, 2, t_1+t_2; \{\mu(i_1 i_2)\})$ yields a $2^{m_1+m_2}$-PBFF design of resolution V provided M_T is nonsingular. However the converse is not always true.

In this paper, we consider three cases as follows:

(A) $\det(K_{\beta_1 \beta_2}) \neq 0$ for $\beta_1 \beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), and $\det(K_{11})=0$,

(B) $\det(K_{\beta_1\beta_2})\neq 0$ for $\beta_1\beta_2=00,10,01,20$ (if $m_1\geq 4$), and $\det(K_{02})$ $=\det(K_{11})=0$ for $m_2\geq 4$

and

(C) $\det(K_{\beta_1\beta_2})\neq 0$ for $\beta_1\beta_2=00,10,01$, and $\det(K_{20})=\det(K_{02})$ $=\det(K_{11})=0$ for $m_k\geq 4$

(see [3,11]), where det(A) denotes the determinant of a matrix A. Let $\Psi'_A=(\Theta'_{00};\Theta'_{10};\Theta'_{01};\Theta'_{20};\Theta'_{02};(H^A_{11}\Theta_{11})')$, $\Psi'_B=(\Theta'_{00};\Theta'_{10};\Theta'_{01};\Theta'_{20};(H^B_{02}\times\Theta_{02})';(H^B_{11}\Theta_{11})')$ for $m_2\geq 4$ and $\Psi'_C=(\Theta'_{00};\Theta'_{10};\Theta'_{01};(H^C_{20}\Theta_{20})';(H^C_{02}\times\Theta_{02})';(H^C_{11}\Theta_{11})')$ for $m_k\geq 4$, where

$$H^A_{11} = h^A_{00}A^{\#(11,11)}_{00}+h^A_{10}A^{\#(11,11)}_{10}+h^A_{01}A^{\#(11,11)}_{01},$$

$$H^B_{11} = h^{B11}_{00}A^{\#(11,11)}_{00}+h^{B11}_{10}A^{\#(11,11)}_{10}+h^{B11}_{01}A^{\#(11,11)}_{01},$$

$$H^B_{02} = h^{B02}_{00}A^{\#(02,02)}_{00}+h^{B02}_{01}A^{\#(02,02)}_{01},$$

$$H^C_{11} = h^{C11}_{00}A^{\#(11,11)}_{00}+h^{C11}_{10}A^{\#(11,11)}_{10}+h^{C11}_{01}A^{\#(11,11)}_{01},$$

$$H^C_{02} = h^{C02}_{00}A^{\#(02,02)}_{00}+h^{C02}_{01}A^{\#(02,02)}_{01},$$

$$H^C_{20} = h^{C20}_{00}A^{\#(20,20)}_{00}+h^{C20}_{10}A^{\#(20,20)}_{10},$$

and $h^A_{\beta_1\beta_2}$'s, $h^{Ba_1a_2}_{\beta_1\beta_2}$'s and $h^{Ca_1a_2}_{\beta_1\beta_2}$'s are real constants. Then we have the following (see [3,11]):

Proposition 3.1. *Let* T *be a design which satisfies* Condition (A) ((B) *or* (C)). *Then* Ψ_A (Ψ_B *or* Ψ_C) *is an estimable function of* Θ, *and the* BLUE *of* Ψ_A (Ψ_B *or* Ψ_C) *is given by* $\hat{\Psi}_A=X_AE'_Ty_T$ ($\hat{\Psi}_B=X_BE'_Ty_T$ *or* $\hat{\Psi}_C=X_CE'_Ty_T$), *where* X_A (X_B *or* X_C) *is a matrix of order* $\nu(m_1m_2)$ *which satisfies* $X_AM_T=Z_A$ ($X_BM_T=Z_B$ *or* $X_CM_T=Z_C$), $Z_A=\mathrm{diag}(I_{\nu_A};H^A_{11})$ ($Z_B=\mathrm{diag}(I_{\nu_B};H^B_{02};H^B_{11})$ *or* $Z_C=\mathrm{diag}(I_{\nu_C};H^C_{20};H^C_{02};H^C_{11})$) *and* $\nu_A=1+m_1+m_2+\binom{m_1}{2}+\binom{m_2}{2}$ ($\nu_B=1+m_1+m_2+\binom{m_1}{2}$ *or* $\nu_C=1+m_1+m_2$).

Note that a design satisfying Condition (A) ((B) or (C)) is of course of resolution IV.

4. Algebraic structure

It is empirically known that the main effects are more important than the two-factor interactions. Thus we are interested in testing the hypotheses such that there exist some linear combinations of the two-factor interactions or not. If they do not exist, we wish to test the hypotheses such that there exist another linear combinations of them (or some linear combinations of the main effects) or not, and so on.

Using the properties of $A^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2}$'s as in (2.3), the linear model (2.1) can be rewritten as

$$y_T = \sum_{\beta_1\beta_2} \sum_{a_1a_2} E_{a_1a_2} A^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2} \Theta_{a_1a_2} + e_T,$$

where $E_{a_1a_2}$'s are $N\times n(a_1a_2)$ submatrices of E_T corresponding to $\Theta_{a_1a_2}$, i.e., $E_T=[E_{00};E_{10};E_{01};E_{20};E_{02};E_{11}]$. By (2.2), (2.4) and (2.5), (i) every element of the vector $A^{\#(a_1a_2,a_1a_2)}_{00}\Theta_{a_1a_2}$ represents the average of $\Theta_{a_1a_2}$ for a_1a_2=00,10,01,20,02,11, (ii) the elements of $A^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2}\Theta_{a_1a_2}$ for $\beta_1\beta_2$=10,01,20 (if $m_1\geq 4$),02 (if $m_2\geq 4$),11 represent the contrasts between these effects and any two contrasts are orthogonal, and (iii) there exist $\phi_{\beta_1\beta_2}$ independent parametric functions of $\Theta_{a_1a_2}$ in $A^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2}\Theta_{a_1a_2}$, respectively (e.g., [17]).

Let $F^{a_1a_2,b_1b_2}_{\beta_1\beta_2}=E_{a_1a_2}A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}E'_{b_1b_2}$. Then by (2.4) and (2.7), we get the following (see [8]):

Lemma 4.1.

$$F^{a_1a_2,c_1c_2}_{\beta_1\beta_2}F^{d_1d_2,b_1b_2}_{\gamma_1\gamma_2} = \delta_{\beta_1\gamma_1}\delta_{\beta_2\gamma_2}\kappa^{c_1c_2,d_1d_2}_{\beta_1\beta_2}F^{a_1a_2,b_1b_2}_{\beta_1\beta_2}.$$

Let $K_{\beta_1\beta_2}(a_1a_2)$ be the matrices which are composed of the initial, $\cdots$, the a_1a_2th rows and the initial, $\cdots$, the a_1a_2th columns of $K_{\beta_1\beta_2}$. Further let $K_{\beta_1\beta_2}(a_1a_2)^{-1}=\|\eta_{\beta_1\beta_2}^{c_1c_2,d_1d_2}(a_1a_2)\|$, if $K_{\beta_1\beta_2}(a_1a_2)$ is nonsingular. In addition, let $K_{\beta_1\beta_2}(a_1a_2{}^*)^{-1}$ $=\|\eta_{\beta_1\beta_2}^{e_1e_2,f_1f_2}(a_1a_2{}^*)\|$, if $K_{\beta_1\beta_2}(a_1a_2{}^*)$ is nonsingular, where $K_{\beta_1\beta_2}(a_1a_2{}^*)$ is the matrix which is obtained by deleting the last row and the last column of $K_{\beta_1\beta_2}(a_1a_2)$, and $\eta_{\beta_1\beta_2}^{\beta_1\beta_2,\beta_1\beta_2}(\beta_1\beta_2{}^*)=0$ for $\beta_1\beta_2=00,10,01,20$ (if $m_1\geq4$), 02 (if $m_2\geq4$), 11.

Let

$$P_{\beta_1\beta_2}^{a_1a_2} = \sum_{c_1c_2}^{*\,a_1a_2}{}_{\beta_1\beta_2} \sum_{d_1d_2}^{*\,a_1a_2}{}_{\beta_1\beta_2} \eta_{\beta_1\beta_2}^{c_1c_2,d_1d_2}(a_1a_2)F_{\beta_1\beta_2}^{c_1c_2,d_1d_2}$$
$$- \sum_{e_1e_2}^{**\,a_1a_2}{}_{\beta_1\beta_2} \sum_{f_1f_2}^{**\,a_1a_2}{}_{\beta_1\beta_2} \eta_{\beta_1\beta_2}^{e_1e_2,f_1f_2}(a_1a_2{}^*)F_{\beta_1\beta_2}^{e_1e_2,f_1f_2},$$

where $\sum_{w_1w_2}^{*\,a_1a_2}{}_{\beta_1\beta_2}$ and $\sum_{s_1s_2}^{**\,a_1a_2}{}_{\beta_1\beta_2}$ are the summations over all the values of w_1w_2 and s_1s_2 such that (I) if $\beta_1\beta_2=00$ and (1) $a_1a_2=00$, then $w_1w_2=00$ and s_1s_2 vanishes, (2) $a_1a_2=10$, then $w_1w_2=00,10$ and $s_1s_2=00$, (3) $a_1a_2=01$, then $w_1w_2=00,10,01$ and $s_1s_2=00,10$, (4) $a_1a_2=20$, then $w_1w_2=00,10,01,20$ and $s_1s_2=00,10,01$, (5) $a_1a_2=02$, then $w_1w_2=00,10,01,20,02$ and $s_1s_2=00,10,01,20$, and (6) $a_1a_2=11$, then $w_1w_2=00,10,01,20,02,11$ and $s_1s_2=00,10,01,20,02$, (II) if $\beta_1\beta_2=10$ and (1) $a_1a_2=10$, then $w_1w_2=10$ and s_1s_2 vanishes, (2) $a_1a_2=20$ (if $m_1\geq3$), then $w_1w_2=10,20$ and $s_1s_2=10$, and (3) $a_1a_2=11$, then $w_1w_2=10,20$ (if $m_1\geq3$), 11 and $s_1s_2=10,20$ (if $m_1\geq3$), (III) if $\beta_1\beta_2=01$ and (1) $a_1a_2=01$, then $w_1w_2=01$ and s_1s_2 vanishes, (2) $a_1a_2=02$ (if $m_2\geq3$), then $w_1w_2=01,02$ and $s_1s_2=01$, and (3) $a_1a_2=11$, then $w_1w_2=01,02$ (if $m_2\geq3$), 11 and $s_1s_2=01,02$ (if $m_2\geq3$), (IV) if $\beta_1\beta_2=20$ ($m_1\geq4$) and $a_1a_2=20$, then $w_1w_2=20$ and s_1s_2 vanishes, (V) if $\beta_1\beta_2=02$ ($m_2\geq4$) and $a_1a_2=02$, then $w_1w_2=02$ and s_1s_2 vanishes, and (VII) if $\beta_1\beta_2=a_1a_2=11$, then $w_1w_2=11$ and s_1s_2 vanishes, respectively. Then the following can be proved easily (see [8]):

Lemma 4.2. (i) *The* $P^{a_1a_2}_{\beta_1\beta_2}$*'s are symmetric, mutually orthogonal and idempotent matrices.*

(ii) $\mathrm{rank}(P^{a_1a_2}_{\beta_1\beta_2}) = \phi_{\beta_1\beta_2}$.

First we consider a $2^{m_1+m_2}$-PBFF design which is a PB-array with Condition (A). If $N \geq \nu(m_1m_2)$, then there may exist a design of resolution V. However if $N=\nu(m_1m_2)$, there is no d.f. due to error. Thus we consider the case in which $\{3(m_1+m_2)+m_1^2+m_2^2\}/2$ $(=\nu^A(m_1m_2)$, say$)<N\leq\nu(m_1m_2)$. Let $P_e^A=I_N-{}_{\beta_1\beta_2}\Sigma^A{}_{a_1a_2}\Sigma P^{a_1a_2}_{\beta_1\beta_2}$, where the summation ${}_{\beta_1\beta_2}\Sigma^A$ is extended over all the values of $\beta_1\beta_2$ such that $\beta_1\beta_2=00,10,01,20$ (if $m_1\geq4$), 02 (if $m_2\geq4$). Then it follows from Lemma 4.2 that $(P_e^A)^2=P_e^A$, $P_e^AP^{a_1a_2}_{\beta_1\beta_2}=P^{a_1a_2}_{\beta_1\beta_2}P_e^A=0_{N\times N}$ and $\mathrm{rank}(P_e^A)=N-\nu^A(m_1m_2)$, where $0_{p\times q}$ denotes the $p\times q$ matrix with all zero. Let

$$\mathcal{R}^{a_1a_2}_{\beta_1\beta_2} = \mathcal{R}_{R(a_1a_2{}^*;\beta_1\beta_2)^{\perp}}(R(a_1a_2;\beta_1\beta_2)),$$

where

$$R(a_1a_2;\beta_1\beta_2) = [E_{\beta_1\beta_2}A^{\#(\beta_1\beta_2,\beta_1\beta_2)}_{\beta_1\beta_2};\cdots;E_{a_1a_2}A^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2}] \quad \text{for } \beta_1\beta_2=00,10,01,$$

$$R(20;20) = \begin{cases} \mathbf{0}_N & \text{if } m_1=2,3, \\ [E_{20}A^{\#(20,20)}_{20}] & \text{if } m_1\geq4, \end{cases}$$

$$R(02;02) = \begin{cases} \mathbf{0}_N & \text{if } m_2=2,3, \\ [E_{02}A^{\#(02,02)}_{02}] & \text{if } m_2\geq4 \end{cases}$$

and $R(a_1a_2{}^*;\beta_1\beta_2)$'s are the matrices which are obtained by deleting $E_{a_1a_2}A^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2}$ from $R(a_1a_2;\beta_1\beta_2)$, and $R(\beta_1\beta_2{}^*;\beta_1\beta_2)=\mathbf{0}_N$. Here $\mathcal{R}_{A^\perp}(B)$ is the orthocomplement subspace of $\mathcal{R}(A)$ relative to $\mathcal{R}(B)$ for the case $\mathcal{R}(A)\subset\mathcal{R}(B)$, where $\mathcal{R}(A)$ denotes the linear subspace spanned by the column vectors of a matrix A. Then Lemma

4.2 and the properties of P_e^A yield the following:

Theorem 4.1. *Let* T *be a* $2^{m_1+m_2}$-PBFF *design which is derived from a* PB-*array with* Condition (A). *Then we have*

$$\mathscr{R}^N = \begin{cases} \mathscr{R}_{00} \oplus \mathscr{R}_{10} \oplus \mathscr{R}_{01} \oplus \mathscr{R}_e^A & \textit{if } m_1, m_2 = 2, 3, \\ \mathscr{R}_{00} \oplus \mathscr{R}_{10} \oplus \mathscr{R}_{01} \oplus \mathscr{R}_{20} \oplus \mathscr{R}_e^A & \textit{if } m_1 \geq 4, m_2 = 2, 3, \\ \mathscr{R}_{00} \oplus \mathscr{R}_{10} \oplus \mathscr{R}_{01} \oplus \mathscr{R}_{02} \oplus \mathscr{R}_e^A & \textit{if } m_1 = 2, 3, m_2 \geq 4, \\ \mathscr{R}_{00} \oplus \mathscr{R}_{10} \oplus \mathscr{R}_{01} \oplus \mathscr{R}_{20} \oplus \mathscr{R}_{02} \oplus \mathscr{R}_e^A & \textit{if } m_1, m_2 \geq 4, \end{cases}$$

where $\mathscr{R}^N$ *is an* N-*dimensional vector space,* $\oplus$ *denotes the direct sum,* $\mathscr{R}_e^A = R_{E_T^\perp}$, *which is the orthocomplement subspace of* $\mathscr{R}(E_T)$ *relative to* $\mathscr{R}^N$, *and*

$$\mathscr{R}_{00} = \mathscr{R}_{00}^{00} \oplus \mathscr{R}_{00}^{10} \oplus \mathscr{R}_{00}^{01} \oplus \mathscr{R}_{00}^{20} \oplus \mathscr{R}_{00}^{02} \oplus \mathscr{R}_{00}^{11},$$

$$\mathscr{R}_{10} = \begin{cases} \mathscr{R}_{10}^{10} \oplus \mathscr{R}_{10}^{11} & \textit{if } m_1 = 2, \\ \mathscr{R}_{10}^{10} \oplus \mathscr{R}_{10}^{20} \oplus \mathscr{R}_{10}^{11} & \textit{if } m_1 \geq 3, \end{cases}$$

$$\mathscr{R}_{01} = \begin{cases} \mathscr{R}_{01}^{01} \oplus \mathscr{R}_{01}^{11} & \textit{if } m_2 = 2, \\ \mathscr{R}_{01}^{01} \oplus \mathscr{R}_{01}^{02} \oplus \mathscr{R}_{01}^{11} & \textit{if } m_2 \geq 3 \end{cases}$$

and

$$\mathscr{R}_{\beta_1\beta_2} = \mathscr{R}(R(\beta_1\beta_2;\beta_1\beta_2)) \textit{ for } \beta_1\beta_2 = 20 \ (\textit{if } m_1 \geq 4), 02 \ (\textit{if } m_2 \geq 4).$$

Next consider a $2^{m_1+m_2}$-PBFF design being a PB-array with Condition (B), and $\{3(m_1+2m_2)+m_1^2\}/2$ $(=\nu^B(m_1m_2)$, say$)<N\leq\nu^A(m_1m_2)$, where $m_2\geq 4$. Let $P_e^B = I_N - {}_{\beta}\Sigma^B_{\beta_1\beta_2}\Sigma_{a_1a_2} P^{a_1a_2}_{\beta_1\beta_2}$, where ${}_{\beta}\Sigma^B_{\beta_1\beta_2}$ is the summation over all the values of $\beta_1\beta_2$ such that $\beta_1\beta_2$=00,10,01,20 (if $m_1\geq 4$). Then $(P_e^B)^2 = P_e^B$, $P_e^B P^{a_1a_2}_{\beta_1\beta_2} = P^{a_1a_2}_{\beta_1\beta_2} P_e^B = 0_{N\times N}$ and $\mathrm{rank}(P_e^B) = N - \nu^B(m_1m_2)$.

Theorem 4.2. *For a* $2^{m_1+m_2}$-PBFF *design* T, *which is a* PB-*array with* Condition (B), *we have*

$$\mathscr{R}^N = \begin{cases} \mathscr{R}_{00} \oplus \mathscr{R}_{10} \oplus \mathscr{R}_{01} \oplus \mathscr{R}_e^B & \textit{if } m_1 = 2, 3, \\ \mathscr{R}_{00} \oplus \mathscr{R}_{10} \oplus \mathscr{R}_{01} \oplus \mathscr{R}_{20} \oplus \mathscr{R}_e^B & \textit{if } m_1 \geq 4, \end{cases}$$

where $\mathscr{R}_{\beta_1\beta_2}$*'s for* $\beta_1\beta_2$=00,10,01,20 (*if* $m_1\geq 4$) *are given in* Theorem

4.1, *and* $\mathscr{R}_e^B = \mathscr{R}_{E_T^\perp}$.

Finally consider a $2^{m_1+m_2}$-PBFF design which is derived from a PB-array with Condition (C), where $3(m_1+m_2)$ $(=\nu^C(m_1m_2)$, say$)<N\leq \nu^B(m_1m_2)$ and $m_k\geq 4$. Let $P_e^C = I_N - {}_{\beta_1\beta_2}\Sigma^C \sum_{a_1a_2} P_{\beta_1\beta_2}^{a_1a_2}$, where ${}_{\beta_1\beta_2}\Sigma^C$ is the summation over all the values of $\beta_1\beta_2$ such that $\beta_1\beta_2=00,10,01$. Then $(P_e^C)^2=P_e^C$, $P_e^C P_{\beta_1\beta_2}^{a_1a_2} = P_{\beta_1\beta_2}^{a_1a_2} P_e^C = 0_{N\times N}$ and $\mathrm{rank}(P_e^C) = N-\nu^C(m_1m_2)$.

Theorem 4.3. *Let* T *be a* $2^{m_1+m_2}$-PBFF *design which is a* PB-*array with* Condition (C). *Then*

$$\mathscr{R}^N = \mathscr{R}_{00} \oplus \mathscr{R}_{10} \oplus \mathscr{R}_{01} \oplus \mathscr{R}_e^C,$$

where $\mathscr{R}_{\beta_1\beta_2}$*'s* $(\beta_1\beta_2=00,10,01)$ *are the same as in* Theorem 4.1, *and* $\mathscr{R}_e^C = \mathscr{R}_{E_T^\perp}$.

5. ANOVA and hypothesis testing

We first consider the ANOVA and the hypothesis testing of $2^{m_1+m_2}$-PBFF designs of resolution IV satisfying Condition (A). Let $S_{\beta_1\beta_2}^{a_1a_2} = y_T' P_{\beta_1\beta_2}^{a_1a_2} y_T$ and $S_e^A = y_T' P_e^A y_T$. Then by Theorem 4.1, we have the following:

Theorem 5.1. *Let* T *be a* $2^{m_1+m_2}$-PBFF *design which is a* PB-*array with* Condition (A) *and* $\nu^A(m_1m_2)<N\leq\nu(m_1m_2)$. *Then we have*

$$y_T' y_T = {}_{\beta_1\beta_2}\Sigma^A \sum_{a_1a_2} S_{\beta_1\beta_2}^{a_1a_2} + S_e^A.$$

Theorem 5.2. *For a design* T *of* Theorem 5.1, *an unbiased estimator of* σ^2 *is given by*

$$\hat{\sigma}^2 = S_e^A/\{N-\nu^A(m_1m_2)\}.$$

The noncentrality parameters, say, $\lambda_{\beta_1\beta_2}^{a_1a_2}/\sigma^2$, of the quadratic forms $y_T' P_{\beta_1\beta_2}^{a_1a_2} y_T/\sigma^2$ are defined by $\mathscr{E}[y_T'] P_{\beta_1\beta_2}^{a_1a_2} \mathscr{E}[y_T]/\sigma^2$, where

$\mathscr{E}[y]$ denotes the expected value of a random vector $\mathbf{y}$. Let

$$c_{00}(p_1p_2,q_1q_2;a_1a_2)$$

$$= \begin{cases} \sum\limits_{c_1c_2}^{a_1a_2}{}^{(*)00} \sum\limits_{d_1d_2}^{a_1a_2}{}^{(*)00} \eta_{00}^{c_1c_2,d_1d_2}(a_1a_2)\kappa_{00}^{p_1p_2,c_1c_2}\kappa_{00}^{d_1d_2,q_1q_2} \\ \quad - \sum\limits_{e_1e_2}^{a_1a_2}{}^{(**)00} \sum\limits_{f_1f_2}^{a_1a_2}{}^{(**)00} \eta_{00}^{e_1e_2,f_1f_2}(a_1a_2{}^*)\kappa_{00}^{p_1p_2,e_1e_2}\kappa_{00}^{f_1f_2,q_1q_2} \\ \quad \text{if } (1)\ p_1p_2=q_1q_2=11 \quad \text{for } a_1a_2=11, \\ \quad\quad (2)\ p_1p_2,q_1q_2=02,11 \quad \text{for } a_1a_2=02, \\ \quad\quad (3)\ p_1p_2,q_1q_2=20,02,11 \quad \text{for } a_1a_2=20, \\ \quad\quad (4)\ p_1p_2,q_1q_2=01,20,02,11 \quad \text{for } a_1a_2=01, \\ \quad\quad (5)\ p_1p_2,q_1q_2=10,01,20,02,11 \quad \text{for } a_1a_2=10, \\ \quad\quad (6)\ p_1p_2,q_1q_2=00,10,01,20,02,11 \quad \text{for } a_1a_2=00, \\ 0 \quad \text{otherwise,} \end{cases}$$

where $\sum\limits_{w_1w_2}^{a_1a_2}{}^{(*)00}$ and $\sum\limits_{s_1s_2}^{a_1a_2}{}^{(**)00}$ are extended over all the values of w_1w_2 and s_1s_2 such that if $a_1a_2=11$, then $w_1w_2=11$ and s_1s_2 vanishes, if $a_1a_2=02$, then $w_1w_2=02,11$ and $s_1s_2=11$, if $a_1a_2=20$, then $w_1w_2=20,02,11$ and $s_1s_2=02,11$, if $a_1a_2=01$, then $w_1w_2=01,20,02,11$ and $s_1s_2=20,02,11$, if $a_1a_2=10$, then $w_1w_2=10,01,20,02,11$ and $s_1s_2=01,20,02,11$, and if $a_1a_2=00$, then $w_1w_2=00,10,01,20,02,11$ and $s_1s_2=10,01,20,02,11$, respectively. Let

$$c_{10}(p_1p_2,q_1q_2;a_1a_2)$$

$$= \begin{cases} \sum\limits_{c_1c_2}^{a_1a_2}{}^{(*)10} \sum\limits_{d_1d_2}^{a_1a_2}{}^{(*)10} \eta_{10}^{c_1c_2,d_1d_2}(a_1a_2)\kappa_{10}^{p_1p_2,c_1c_2}\kappa_{10}^{d_1d_2,q_1q_2} \\ \quad - \sum\limits_{e_1e_2}^{a_1a_2}{}^{(**)10} \sum\limits_{f_1f_2}^{a_1a_2}{}^{(**)10} \eta_{10}^{e_1e_2,f_1f_2}(a_1a_2{}^*)\kappa_{10}^{p_1p_2,e_1e_2}\kappa_{10}^{f_1f_2,q_1q_2} \\ \quad \text{if } (1)\ p_1p_2=q_1q_2=11 \quad \text{for } a_1a_2=11, \\ \quad\quad (2)\ p_1p_2,q_1q_2=20\ (m_1\geq 3),11 \quad \text{for } a_1a_2=20, \\ \quad\quad (3)\ p_1p_2,q_1q_2=10,20\ (m_1\geq 3),11 \quad \text{for } a_1a_2=10, \\ 0 \quad \text{otherwise,} \end{cases}$$

where $\sum\limits_{w_1w_2}^{a_1a_2}{}^{(*)10}$ and $\sum\limits_{s_1s_2}^{a_1a_2}{}^{(**)10}$ are the summations over all the values of w_1w_2 and s_1s_2 such that $w_1w_2=11$ and s_1s_2 vanishes if $a_1a_2=11$,

$w_1w_2=20$ (if $m_1\geq 3$),11 and $s_1s_2=11$ if $a_1a_2=20$, and $w_1w_2=10,20$ (if $m_1\geq 3$),11 and $s_1s_2=20$ (if $m_1\geq 3$),11 if $a_1a_2=10$, respectively. Let

$$c_{01}(p_1p_2,q_1q_2;a_1a_2) = \begin{cases} \sum^{a_1a_2}_{c_1c_2}{}^{(*)01} \sum^{a_1a_2}_{d_1d_2}{}^{(*)01} \eta_{01}^{c_1c_2,d_1d_2}(a_1a_2)\kappa_{01}^{p_1p_2,c_1c_2}\kappa_{01}^{d_1d_2,q_1q_2} \\ \quad - \sum^{a_1a_2}_{e_1e_2}{}^{(**)01} \sum^{a_1a_2}_{f_1f_2}{}^{(**)01} \eta_{01}^{e_1e_2,f_1f_2}(a_1a_2{}^*)\kappa_{01}^{p_1p_2,e_1e_2}\kappa_{01}^{f_1f_2,q_1q_2} \\ \quad \text{if (1) } p_1p_2=q_1q_2=11 \quad \text{for } a_1a_2=11, \\ \quad\quad \text{(2) } p_1p_2,q_1q_2=02\ (m_2\geq 3),11 \quad \text{for } a_1a_2=02, \\ \quad\quad \text{(3) } p_1p_2,q_1q_2=01,02\ (m_2\geq 3),11 \quad \text{for } a_1a_2=01, \\ 0 \quad \text{otherwise,} \end{cases}$$

where $\sum^{a_1a_2}_{w_1w_2}{}^{(*)01}$ and $\sum^{a_1a_2}_{s_1s_2}{}^{(**)01}$ are extended over all the values of w_1w_2 and s_1s_2 such that $w_1w_2=11$ and s_1s_2 vanishes if $a_1a_2=11$, $w_1w_2=02$ (if $m_2\geq 3$),11 and $s_1s_2=11$ if $a_1a_2=02$, and $w_1w_2=01,02$ (if $m_2\geq 3$),11 and $s_1s_2=02$ (if $m_2\geq 3$),11 if $a_1a_2=01$, respectively. Further let

$$c_{20}(p_1p_2,q_1q_2;a_1a_2) = \begin{cases} \kappa_{20}^{20,20} & \text{if } p_1p_2=q_1q_2=a_1a_2=20, \\ 0 & \text{otherwise,} \end{cases}$$

if $\det(K_{20})\neq 0$ and $m_1\geq 4$, and

$$c_{02}(p_1p_2,q_1q_2;a_1a_2) = \begin{cases} \kappa_{02}^{02,02} & \text{if } p_1p_2=q_1q_2=a_1a_2=02, \\ 0 & \text{otherwise,} \end{cases}$$

if $\det(K_{02})\neq 0$ and $m_2\geq 4$. Then the following yields:

Theorem 5.3. *Let* T *be a design of* Theorem 5.1, *then the non-centrality parameters of the quadratic forms* $y_T'P^{a_1a_2}_{\beta_1\beta_2}y_T/\sigma^2$ *for* $\beta_1\beta_2=00,10,01,20$ *(if* $m_1\geq 4$*)*,02 *(if* $m_2\geq 4$*) are*

$$\lambda^{a_1a_2}_{\beta_1\beta_2}/\sigma^2 = \sum_{p_1p_2}\sum_{q_1q_2} \{c_{\beta_1\beta_2}(p_1p_2,q_1q_2;a_1a_2)/\sigma^2\} \times \Theta'_{p_1p_2}A^{\#(p_1p_2,q_1q_2)}_{\beta_1\beta_2}\Theta_{q_1q_2}.$$

Let $H^{a_1a_2}_{\beta_1\beta_2}$ be the hypotheses such that $A^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2}\Theta_{a_1a_2}=\mathbf{0}_{n(a_1a_2)}$ (if they exist). We are first interested in testing the hypotheses $H^{11}_{\beta_1\beta_2}$ against $K^{11}_{\beta_1\beta_2}$ ($\beta_1\beta_2$=00,10,01), H^{20}_{20} against K^{20}_{20} (if $m_1 \geq 4$), and H^{02}_{02} against K^{02}_{02} (if $m_2 \geq 4$), where $K^{a_1a_2}_{\beta_1\beta_2}$'s are the hypotheses that $A^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2}\Theta_{a_1a_2} \neq \mathbf{0}_{n(a_1a_2)}$. Next, if H^{11}_{10} (or H^{11}_{01}) is accepted, we then consider the testing hypothesis H^{20}_{10} (if $m_1 \geq 3$) or H^{10}_{10} (if $m_1=2$) against H^{11}_{10} (or H^{02}_{01} (if $m_2 \geq 3$) or H^{01}_{01} (if $m_2=2$) against H^{11}_{01}). If H^{11}_{00} is accepted, then we consider H^{02}_{00} against H^{11}_{00}. Third, if H^{20}_{10} (if $m_1 \geq 3$) (or H^{02}_{01} (if $m_2 \geq 3$)) is accepted, then consider H^{10}_{10} against H^{20}_{10} (or H^{01}_{01} against H^{02}_{01}), and if H^{02}_{00} is accepted, consider H^{20}_{00} against H^{02}_{00}. If H^{20}_{00} is accepted, consider H^{01}_{00} against H^{20}_{00}, and lastly if H^{01}_{00} is accepted, then consider H^{10}_{00} against H^{01}_{00}. This method is the so-called nested test procedure (e.g., [2]). Notice that Theorem 5.3 implies that $\bigcap^{a_1a_2}_{b_1b_2} H^{b_1b_2}_{\beta_1\beta_2}$ is accepted if and only if $\lambda^{a_1a_2}_{\beta_1\beta_2}=0$, where $\bigcap^{a_1a_2}_{b_1b_2} H^{b_1b_2}_{\beta_1\beta_2}$ denotes the intersection of $H^{b_1b_2}_{\beta_1\beta_2}$'s such that the running indices b_1b_2 have the same values as w_1w_2 of $\sum^{a_1a_2\,(*)}_{w_1w_2}\beta_1\beta_2$ for $\beta_1\beta_2$=00,10,01, and as $\beta_1\beta_2$ for $\beta_1\beta_2$=20 (if $m_1 \geq 4$),02 (if $m_2 \geq 4$). The test statistics for the nested method are given by

(i) for $\beta_1\beta_2$=00,

$$\frac{S^{11}_{00}/\phi_{00}}{S^A_e/\{N-\nu^A(m_1m_2)\}} \;(=F^{A\,11}_{\;\;00},\ \text{say}), \tag{5.1}$$

$$\frac{S^{02}_{00}/\phi_{00}}{\{S^A_e+S^{11}_{00}\}/\{N-\nu^A(m_1m_2)+\phi_{00}\}} \;(=F^{A\,02}_{\;\;00},\ \text{say}), \tag{5.2}$$

$$\frac{S^{20}_{00}/\phi_{00}}{\{S^A_e+S^{11}_{00}+S^{02}_{00}\}/\{N-\nu^A(m_1m_2)+2\phi_{00}\}} \;(=F^{A\,20}_{\;\;00},\ \text{say}), \tag{5.3}$$

$$\frac{S^{01}_{00}/\phi_{00}}{\{S^A_e+S^{11}_{00}+S^{02}_{00}+S^{20}_{00}\}/\{N-\nu^A(m_1m_2)+3\phi_{00}\}} \;(=F^{A\,01}_{\;\;00},\ \text{say}) \tag{5.4}$$

and

$$\frac{S^{10}_{00}/\phi_{00}}{\{S^{A}_{e}+S^{11}_{00}+S^{02}_{00}+S^{20}_{00}+S^{01}_{00}\}/\{N-\nu^{A}(m_1m_2)+4\phi_{00}\}} \quad (=F^{A\,10}_{\ 00},\ \text{say}), \quad (5.5)$$

(ii) for $\beta_1\beta_2$=10,

$$\frac{S^{11}_{10}/\phi_{10}}{S^{A}_{e}/\{N-\nu^{A}(m_1m_2)\}} \quad (=F^{A\,11}_{\ 10},\ \text{say}), \quad (5.6)$$

$$\frac{S^{20}_{10}/\phi_{10}}{\{S^{A}_{e}+S^{11}_{10}\}/\{N-\nu^{A}(m_1m_2)+\phi_{10}\}} \quad (=F^{A\,20}_{\ 10},\ \text{say})\ (\text{if } m_1\geq 3) \quad (5.7a)$$

$$(\text{or} \quad \frac{S^{10}_{10}/\phi_{10}}{\{S^{A}_{e}+S^{11}_{10}\}/\{N-\nu^{A}(m_1m_2)+\phi_{10}\}} \quad (=F^{A\,10}_{\ 10},\ \text{say})\ (\text{if } m_1=2)) \quad (5.7b)$$

and

$$\frac{S^{10}_{10}/\phi_{10}}{\{S^{A}_{e}+S^{11}_{10}+S^{20}_{10}\}/\{N-\nu^{A}(m_1m_2)+2\phi_{10}\}} \quad (=F^{A\,10}_{\ 10},\ \text{say})\ (\text{if } m_1\geq 3) \quad (5.8)$$

(iii) for $\beta_1\beta_2$=01,

$$\frac{S^{11}_{01}/\phi_{01}}{S^{A}_{e}/\{N-\nu^{A}(m_1m_2)\}} \quad (=F^{A\,11}_{\ 01},\ \text{say}), \quad (5.9)$$

$$\frac{S^{02}_{01}/\phi_{01}}{\{S^{A}_{e}+S^{11}_{01}\}/\{N-\nu^{A}(m_1m_2)+\phi_{01}\}} \quad (=F^{A\,02}_{\ 01},\ \text{say})\ (\text{if } m_2\geq 3) \quad (5.10)$$

$$(\text{or} \quad \frac{S^{01}_{01}/\phi_{01}}{\{S^{A}_{e}+S^{11}_{01}\}/\{N-\nu^{A}(m_1m_2)+\phi_{01}\}} \quad (=F^{A\,01}_{\ 01},\ \text{say})\ (\text{if } m_2=2))$$

and

$$\frac{S^{01}_{01}/\phi_{01}}{\{S^{A}_{e}+S^{11}_{01}+S^{02}_{01}\}/\{N-\nu^{A}(m_1m_2)+2\phi_{01}\}} \quad (=F^{A\,01}_{\ 01},\ \text{say})\ (\text{if } m_2\geq 3) \quad (5.11)$$

(iv) for $\beta_1\beta_2$=20 and $m_1\geq 4$,

$$\frac{S^{20}_{20}/\phi_{20}}{S^{A}_{e}/\{N-\nu^{A}(m_1m_2)\}} \quad (=F^{A\,20}_{\ 20},\ \text{say}) \quad (5.12)$$

and (v) for $\beta_1\beta_2$=02 and $m_2\geq 4$,

$$\frac{S^{02}_{02}/\phi_{02}}{S^{A}_{e}/\{N-\nu^{A}(m_1m_2)\}} \quad (=F^{A\,02}_{\ 02},\ \text{say}).$$

All of them have *F* distributions, and the nesting procedure is continued until a significant test is obtained for each $\beta_1\beta_2$. Note that $F^{A\,\alpha_1\alpha_2}_{\ \beta_1\beta_2}$'s are central or noncentral *F* distributions with

$\phi_{\beta_1\beta_2}$ and $\{N-\nu^A(m_1m_2)\}+\tau^A(a_1a_2;\beta_1\beta_2)\phi_{\beta_1\beta_2}$ d.f., and noncentrality parameters $\lambda^{a_1a_2}_{\beta_1\beta_2}/\sigma^2$ depending on which $\bigcap_{b_1b_2}^{a_1a_2} H^{b_1b_2}_{\beta_1\beta_2}$ are true, where $\tau^A(a_1a_2;\beta_1\beta_2)$'s are some integers as above.

Next consider the ANOVA and the hypothesis testing of $2^{m_1+m_2}$-PBFF designs of resolution IV which satisfy Condition (B).

Theorem 5.4. *Let* T *be a* $2^{m_1+m_2}$-PBFF *design which is a* PB-*array with* Condition (B) *and* $\nu^B(m_1m_2)<N\leq\nu^A(m_1m_2)$. *Then*

$$y'_T y_T = \sum_{\beta_1\beta_2}^{B} \sum_{a_1a_2} S^{a_1a_2}_{\beta_1\beta_2} + S^B_e,$$

where $S^B_e=y'_T P^B_e y_T$.

Theorem 5.5. *For a design* T *of* Theorem 5.4, *an unbiased estimator of* σ^2 *is*

$$\hat{\sigma}^2 = S^B_e/\{N-\nu^B(m_1m_2)\}.$$

Theorem 5.6. *Let* T *be a design of* Theorem 5.4. *Then the noncentrality parameters of the quadratic forms* $y'_T P^{a_1a_2}_{\beta_1\beta_2} y_T/\sigma^2$ *for* $\beta_1\beta_2$=00,10,01,20 (*if* $m_1\geq4$) *are given by*

$$\lambda^{a_1a_2}_{\beta_1\beta_2}/\sigma^2 = \sum_{p_1p_2}\sum_{q_1q_2} \{c_{\beta_1\beta_2}(p_1p_2,q_1q_2;a_1a_2)/\sigma^2\} \times\Theta'_{p_1p_2} A^{\#(p_1p_2,q_1q_2)}_{\beta_1\beta_2} \Theta_{q_1q_2}.$$

We now consider the hypotheses $H^{11}_{\beta_1\beta_2}$ against $K^{11}_{\beta_1\beta_2}$ for $\beta_1\beta_2$ =00,10,01, H^{20}_{20} against K^{20}_{20} (if $m_1\geq4$). Next if H^{11}_{10} (or H^{11}_{01}) is accepted, consider the testing hypothesis H^{20}_{10} (if $m_1\geq3$) or H^{10}_{10} (if $m_1=2$) against H^{11}_{10} (or H^{02}_{01} against H^{11}_{01}). If H^{11}_{00} is accepted, then consider H^{02}_{00} against H^{11}_{00}. Third, if H^{20}_{10} (if $m_1\geq3$) (or H^{02}_{01}) is accepted, then consider H^{10}_{10} against H^{20}_{10} (or H^{01}_{01} against H^{02}_{01}), and if H^{02}_{00} is accepted, consider H^{20}_{00} against H^{02}_{00}. If H^{20}_{00} is accepted, consider H^{01}_{00} against H^{20}_{00}, and lastly if H^{01}_{00} is accepted, then consider H^{10}_{00} against H^{01}_{00}. Note that Theorem 5.6 means that

${}_{b_1^{\cap}b_2}^{a_1a_2}H_{\beta_1\beta_2}^{b_1b_2}$ is accepted if and only if $\lambda_{\beta_1\beta_2}^{a_1a_2}=0$. The test statistics, say $F^{B}{}_{\beta_1\beta_2}^{a_1a_2}$, for the nested method are given by replacing S_e^A and $\nu^A(m_1m_2)$ of (5.1) through (5.12) with S_e^B and $\nu^B(m_1m_2)$, respectively. The $F^{B}{}_{\beta_1\beta_2}^{a_1a_2}$'s have *F* distributions similar to $F^{A}{}_{\beta_1\beta_2}^{a_1a_2}$'s.

We finally consider the ANOVA and the hypothesis testing of $2^{m_1+m_2}$-PBFF designs satisfying Condition (C).

Theorem 5.7. *Let* T *be a* $2^{m_1+m_2}$-PBFF *design which is a* PB-*array with* Condition (C) *and* $\nu^C(m_1m_2)<N\leq\nu^B(m_1m_2)$. *Then we have*

$$y_T'y_T = \sum_{\beta_1\beta_2}^{C}\sum_{a_1a_2} S_{\beta_1\beta_2}^{a_1a_2} + S_e^C,$$

where $S_e^C=y_T'P_e^Cy_T$.

Theorem 5.8. *Let* T *be a design of* Theorem 5.7, *then an unbiased estimator of* σ^2 *is given by*

$$\hat{\sigma}^2 = S_e^C/\{N-\nu^C(m_1m_2)\}.$$

Theorem 5.9. *For a design* T *of* Theorem 5.7, *the noncentrality parameters of the quadratic forms* $y_T'P_{\beta_1\beta_2}^{a_1a_2}y_T/\sigma^2$ $(\beta_1\beta_2=00,10,01)$ *are*

$$\lambda_{\beta_1\beta_2}^{a_1a_2}/\sigma^2 = \sum_{p_1p_2}\sum_{q_1q_2}\{c_{\beta_1\beta_2}(p_1p_2,q_1q_2;a_1a_2)/\sigma^2\}$$
$$\times\Theta_{p_1p_2}'A_{\beta_1\beta_2}^{\#(p_1p_2,q_1q_2)}\Theta_{q_1q_2}.$$

Consider the testing hypotheses $H_{\beta_1\beta_2}^{11}$ against $K_{\beta_1\beta_2}^{11}$ for $\beta_1\beta_2$ =00,10,01. Next if H_{10}^{11} (or H_{01}^{11}) is accepted, then consider the testing hypothesis H_{10}^{20} against H_{10}^{11} (or H_{01}^{02} against H_{01}^{11}). If H_{00}^{11} is accepted, then consider H_{00}^{02} against H_{00}^{11}. Third if H_{10}^{20} (or H_{01}^{02}) is accepted, consider H_{10}^{10} against H_{10}^{20} (or H_{01}^{01} against H_{01}^{02}), and if H_{00}^{02} is accepted, consider H_{00}^{20} against H_{00}^{02}. If H_{00}^{20} is accepted, consider H^{01}_{00} against H^{20}_{00}, and lastly if H^{01}_{00} is accepted, consider H^{10}_{00} against H^{01}_{00}. Note that Theorem 5.9 implies that $\bigcap_{b_1 b_2}^{a_1 a_2} H^{b_1 b_2}_{\beta_1 \beta_2}$ is accepted if and only if $\lambda^{a_1 a_2}_{\beta_1 \beta_2}=0$. The test statistics, say $F^{Ca_1 a_2}_{\beta_1 \beta_2}$, for the nested method are given by replacing S^A_e and $\nu^A(m_1 m_2)$ of (5.1) through (5.11) with S^C_e and $\nu^C(m_1 m_2)$, respectively. The $F^{Ca_1 a_2}_{\beta_1 \beta_2}$'s have F distributions similar to $F^{Aa_1 a_2}_{\beta_1 \beta_2}$'s.

References

[1] D.A. Anderson and J.N. Srivastava (1972), "Resolution IV designs of the $2^m \times 3$ series", *J. Roy. Statist. Soc.*, **B34**, 377-384.

[2] T.W. Anderson (1962), "The choice of the degree of a polynomial regression as a multiple decision problem", *Ann. Math. Statist.*, **33**, 255-265.

[3] M. Kuwada (1986), "Optimal partially balanced fractional $2^{m_1+m_2}$ factorial designs of resolution IV", *Ann. Inst. Statist. Math.*, **A38**, 343-351.

[4] M. Kuwada (1986), "Note on some upper bounds on the constraints for some partially balanced arrays with two symbols", *J. Combin. Inform. System Sci.*, **11**(2-4), 138-146.

[5] M. Kuwada (1988), "A-optimal partially balanced fractional $2^{m_1+m_2}$ factorial designs of resolution V, with $4 \le m_1+m_2 \le 6$", *J. Statist. Plann. Inference*, **18**, 177-193.

[6] M. Kuwada (1988), "On the characteristic polynomials of the information matrices of partially balanced fractional $2^{m_1+m_2}$ factorial designs of resolution VII", *J. Japan Statist.*

Soc., **18**, 77-86.

[7] M. Kuwada (1990), "On triangular multidimensional partially balanced association schemes and their algebras, with applications to balanced fractional 2^m factorial designs", TR#**267**, *Statistical Research Group*, Hiroshima University.

[8] M. Kuwada (1991), "An analysis of variance of partially balanced fractional $2^{m_1+m_2}$ factorial designs of resolution V", *J. Japan Statist. Soc.*, **21**, 43-59.

[9] M. Kuwada (1991), "Analysis of variance of balanced fractional factorial designs", (to appear in) *Discrete Math.*

[10] M. Kuwada and S. Kuriki (1986), "Some existence conditions for partially balanced arrays with 2 symbols", *Discrete Math.*, **61**, 221-233.

[11] M. Kuwada and M. Matsuura (1984), "Further results on partially balanced fractional $2^{m_1+m_2}$ factorial designs of resolution IV", *J. Japan Statist. Soc.*, **14**, 69-83.

[12] B.H. Margolin (1969), "Resolution IV fractional factorial designs", *J. Roy. Statist. Soc.*, **B31**, 514-523.

[13] B.H. Margolin (1969), "Results on factorial designs of resolution IV for the 2^n and 2^n3^m series", *Technometrics* **11**, 431-444.

[14] R. Nishii (1981), "Balanced fractional $r^m \times s^n$ factorial designs and their analysis", *Hiroshima Math. J.*, **11**, 379-413.

[15] H. Scheffé (1959), *The Analysis of Variance*, Wiley.

[16] G.A.F. Seber (1980), *The Linear Hypothsesis; A General Theory* (Second Edition) Charles Griffin & Company Ltd.

[17] T. Shirakura (1976), "Balanced fractional 2^m factorial designs of even resolution obtained from balanced arrays of strength 2ℓ with index $\mu_\ell=0$", *Ann. Statist.*, **4**, 723-735.

[18] T. Shirakura (1977), "Contributions to balanced fractional 2^m factorial designs derived from balanced arrays of strength 2ℓ", *Hiroshima Math. J.*, **7**, 217-285.

[19] T. Shirakura (1979), "Optimal balanced fractional 2^m factorial designs of resolution IV derived from balanced arrays of strength four", *J. Japan Statist. Soc.*, **9**, 19-27.

[20] T. Shirakura (1980), "Necessary and sufficient condition for a balanced array of strength 2ℓ to be a balanced fractional 2^m factorial design of resolution 2ℓ", *Austral. J. Statist.*, **22**, 69-74.

[21] T. Shirakura and M. Kuwada (1976), "Covariance matrices of the estimates for balanced fractional 2^m factorial designs of resolution $2\ell+1$", *J. Japan Statist. Soc.*, **6**, 27-31.

[22] J.N. Srivastava and D.A. Anderson (1970), "Optimal fractional factorial plans for main effects orthogonal to two-factor interactions: 2^m series", *J. Amer. Statist. Assoc.*, **65**, 828-843.

[23] S. Webb (1968), "Non-orthogonal designs of even resolution", *Technometrics* **10**, 291-299.

[24] S. Yamamoto, T. Shirakura and M. Kuwada (1975), "Balanced arrays of strength 2ℓ and balanced fractional 2^m factorial designs", *Ann. Inst. Statist. Math.*, **27**, 143-157.

[25] S. Yamamoto, T. Shirakura and M. Kuwada (1976), "Characteristic polynomials of the information matrices of balanced fractional 2^m factorial designs of higher $(2\ell+1)$ resolution", *Essays in Probability and Statistics* (Eds., S. Ikeda et al.), Shinko Tsusho, Tokyo, 73-94.