<table>
<thead>
<tr>
<th>Title</th>
<th>Analysis of Variance of Partially Balanced Fractional $2^{m_1+m_2}$ Factorial Designs of Resolution IV (Combinatorial Aspects on the Analysis of Mathematical Models)</th>
</tr>
</thead>
<tbody>
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<td>Kuwada, Masahide</td>
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Kyoto University
Analysis of Variance of Partially Balanced Fractional
$2^{m_1+m_2}$ Factorial Designs of Resolution IV

広島大 総合科 柿田 正秀 (Masahide Kuwada)

Abstract

In this paper, attention is focused on the analysis of variance of partially balanced fractional $2^{m_1+m_2}$ factorial designs of resolution IV by using the algebraic structure. They can be obtained by partially balanced arrays with some conditions.

1. Introduction

A partially balanced array (PB-array), which is a special case of an asymmetrical balanced array of type 2 as introduced by Nishii [14], has been studied by several researchers (e.g., [4]). Necessary and sufficient conditions for the existence of a PB-array were obtained by Kuwada and Kuriki [10]. A PB-array yields a partially balanced fractional $2^{m_1+m_2}$ factorial ($2^{m_1+m_2}$-PBFF) design under some conditions (see [5,6]). However a $2^{m_1+m_2}$-PBFF design does not always mean a PB-array.

It is generally difficult to obtain the designs of resolution $2^l$ since there is a little information about the $l$-factor interactions. For earlier works on such designs, see for example, Kuwada and/or Matsuura [3,11], Margolin [12,13], Shirakura [17-20], Srivastava and/or Anderson [1,22], and Webb [23]. Especially, by using the triangular multidimensional partially bal-
anced (TMDPB) association scheme and its algebra, Shirakura [17] showed that a balanced array with index $\mu_3=0$ turns out to be a balanced fractional $2^n$ factorial design of resolution $2\ell$ under some conditions. Such a design permits to estimate all factorial effects up to the $(\ell-1)$-factor interactions and some linear combinations of the $\ell$-factor ones.

The analysis of variance (ANOVA) is a statistical technique for handling the data or observations derived from an experiment (cf. [9,15,16]). The ANOVA of $2^{m_1+m_2}$-PBFF designs of resolution V which are derived from PB-arrays has been studied by Kuwada [8]. In this paper, we present the ANOVA and the hypothesis testing of $2^{m_1+m_2}$-PBFF designs of resolution IV, which are PB-arrays. The designs considered here permit estimation of the general mean, all main effects and (A) all $\binom{m_1}{2}+\binom{m_2}{2}$ two-factor interactions and some linear combinations of the $m_1m_2$ ones, (B) all $\binom{m_1}{2}$ ones and some linear combinations of the $\binom{m_2}{2}$ ones and of the $m_1m_2$ ones, or (C) some linear combinations of the $\binom{m_k}{2}$ ones (k=1,2) and of the $m_1m_2$ ones (see [3,11]).

2. Preliminaries

Consider a factorial experiment with $m_1+m_2$ factors at two levels (0 and 1, say) of each, where $m_k>2$. Further consider the situation in which three-factor and higher order interactions are assumed to be negligible. The vector of unknown factorial effects to be estimated is then given by $(\theta'_0;\theta'_1;\theta'_2;\theta'_3;\theta'_4)$ ($=\theta'$, say), where $\theta'_0=\{\theta(0;0)\}$, $\theta'_1=\{\theta(u;0)\}$, $\theta'_2=\{\theta(0;u)\}$, $\theta'_3=\{\theta(u;v)\}$ and $\theta'_4=\{\theta(u_1,u_2;0)\}$, $\theta'_5=\{\theta(0;v_1,v_2)\}$ and $\theta'_6=\{\theta(u;v)\}$. Here $1\leq u \leq m_1$, $1\leq v \leq m_2$, $1\leq u_1<u_2 \leq m_1$ and $1\leq v_1<v_2 \leq m_2$, and $A'$ denotes the
transpose of a matrix A. Note that the total number of factorial
effects to be estimated is \(1+(m_1+m_2)\binom{m_1+m_2}{2} = \nu(m_1,m_2), \text{ say}\).
Let \([T^{(1)}; T^{(2)}] = T, \text{ say}\) be a fraction with N assemblies (or
treatment combinations), where \(T^{(k)}\)'s are \((0,1)\)-matrices of size
\(N \times m_k\). Then the ordinary linear model is given by
\[
y_T = E_T \theta + e_T, \tag{2.1}
\]
where \(y_T\) and \(E_T\) are the vector of N observations and the design
matrix of size \(N \times \nu(m_1,m_2)\), respectively, and \(e_T\) is an error vector
distributed as \(N(0_N, \sigma^2 I_N)\). Here \(0_p\) and \(I_p\) denote the \(p \times 1\) vector
with all zero and the identity matrix of order \(p\), respectively.
The normal equation for estimating \(\theta\) is given by \(M_T \hat{\theta} = E_T^T y_T\), where
\(M_T = E_T^T E_T\). If the information matrix \(M_T\) is nonsingular, the BLUE
of \(\theta\) and its variance-covariance matrix are given by \(\hat{\theta} = M_T^{-1} E_T^T y_T\)
and \(\text{Var}[\hat{\theta}] = \sigma^2 M_T^{-1}\), respectively.

Suppose a relation of association is defined among the sets
\((((u_1 \cdots u_{a_1} ; v_1 \cdots v_{a_2})\), where \(1 \leq u_1 < \cdots < u_{a_1} \leq m_1\) and \(1 \leq v_1 < \cdots < v_{a_2} \leq m_2\),
in such a way that \((u_1 \cdots u_{a_1} ; v_1 \cdots v_{a_2})\) and \((u'_1 \cdots u'_{b_1} ; v'_1 \cdots v'_{b_2})\)
are the \((a_1,a_2)\)th associates if
\[
|\{u_1, \cdots, u_{a_1}\} \cap \{u'_1, \cdots, u'_{b_1}\}| = \min(a_1, b_1) - \alpha_1
\]
and
\[
|\{v_1, \cdots, v_{a_2}\} \cap \{v'_1, \cdots, v'_{b_2}\}| = \min(a_2, b_2) - \alpha_2,
\]
where \(|S|\) and \(\min(a,b)\) denote the cardinality of a set \(S\) and the
minimum value of integers \(a\) and \(b\), respectively. The scheme thus
defined is called the extended TMDPB (ETMDPB) association scheme
(see [5]), which is regarded as a generalization of the TMDPB
association scheme (e.g., [24,25]). Let \(A_{a_1a_2}^{a_1a_2, b_1b_2}\) and
\(D_{a_1a_2}^{a_1a_2, b_1b_2}\) be the local association matrices of size \(n(a_1a_2) \times\)
n(b_1 b_2) and the ordered association matrices of order \( \nu(m_1 m_2) \) of the ETMDPB association scheme, respectively (see [5]), where \( n(a_1 a_2)=(m_1)(m_2) \). Further let \( A^{\#}(a_1 a_2, b_1 b_2) = A^{\#}(a_1, b_1) \otimes A^{\#}(a_2, b_2) \), where \( A^{\#}(a, b) \)'s are the matrices which are linearly linked with the local association matrices \( A^{a, b}_a \) of the TMDPB association scheme (e.g., [25]), and \( \otimes \) denotes the Kronecker product. A relationship between \( A^{a, b}_a \)'s and \( A^{\#}(a, b) \)'s is given by

\[
A^{a, b}_a = (A^{a, b}_a)^t = \sum_{c \in a} z^{a, b}_c A^{a, b}_c \quad \text{for } 0 \leq a \leq b \leq m,
\]

and

\[
A^{\#}(a, b) = (A^{a, b}_a)^t = \sum_{c \in a} z^{a, b}_c A^{a, b}_c \quad \text{for } 0 \leq a \leq b \leq m,
\]

where

\[
z^{a, b}_c = \sum_{p=0}^a (-1)^{p-a} (\begin{pmatrix} a-p \cr p \end{pmatrix}) (\begin{pmatrix} a-p-1 \cr p \end{pmatrix}) (\begin{pmatrix} b-a \cr p \end{pmatrix}) (\begin{pmatrix} b-a-1 \cr p \end{pmatrix}) \quad \text{for } a \leq b,
\]

\[
z^{c, a}_b = \phi_u z^{a, b}_c / (\begin{pmatrix} a \cr b \end{pmatrix}) (\begin{pmatrix} a-b \cr a-c \end{pmatrix}) \quad \text{for } a \leq b,
\]

and

\[
\phi_u = (-1)^{m-1}
\]

(e.g., [7, 21, 25]). The matrices \( A^{\#}(a_1 a_2, b_1 b_2) \) have the following properties:

\[
A^{\#}(a_1 a_2, b_1 b_2) = \frac{1}{n(a_1 a_2) \times n(b_1 b_2)} \text{G}_{n(a_1 a_2) \times n(b_1 b_2)}
\]

(2.2)

\[
\sum_{\beta_1 \beta_2} A^{\#}(a_1 a_2, a_1 a_2) = I_{n(a_1 a_2)}
\]

(2.3)

\[
A^{\#}(a_1 a_2, c_1 c_2) A^{\#}(c_1 c_2, b_1 b_2) = \delta_{\beta_1 \beta_1} \delta_{\beta_2 \beta_2} A^{\#}(a_1 a_2, b_1 b_2)
\]

(2.4)

and

\[
\text{rank}(A^{\#}(a_1 a_2, b_1 b_2)) = \phi_{\beta_1} \times \phi_{\beta_2} \quad (= \phi_{\beta_1 \beta_2}, \text{ say})
\]

(2.5)

(see [5]), where \( \text{G}_{p \times q} \) and \( \delta_{p \times q} \) denote the \( p \times q \) matrix with all unity and the Kronecker delta, respectively.

Let \( T \) be a PB-array of strength \( t_1 + t_2 \) and size \( N \) having \( m_1 + m_2 \) constraints, two levels, and index set \( \{ \mu(l_1 l_2) \mid 0 \leq i_1 \leq t_1, t_2 \leq m \} \).
where
\[ t_k = \begin{cases} m_k & \text{if } m_k = 2, 3, \\ 4 & \text{if } m_k \geq 4. \end{cases} \] (2.6)

This array is written as PBA\((N,m_1+m_2,2,t_1+t_2;\{\mu(1,1_2)\})\) for brevity. The information matrix \(M_T\) associated with \(T\), which is a PB-array with (2.6), can be expressed as
\[
M_T = \sum \alpha a_1^2 b_1 b_2 \sum \beta a_1^2 a_2 ^2 \sum \gamma a_1 b_1 2a_1 a_2 2a_1 a_2 \sum \delta a_1^2 b_1 b_2 D^\#(a_1 a_2 b_1 b_2) \beta \beta_2 \beta_2 \\
= \sum \alpha a_1^2 b_1 b_2 \sum \beta \beta_2 \beta_2 \sum \gamma \kappa a_1^2 a_2^2 b_1 b_2 D^\#(a_1 a_2 b_1 b_2) \beta \beta_2 \beta_2. \] (2.7)

where \(D^\#(a_1 a_2 b_1 b_2)\)'s are the matrices of order \(\nu(m_1 m_2)\) which are given by some linear combinations of \(D(\alpha a_1^2 b_1 b_2)\)'s,

\[ \gamma_{j1,j2} = \sum_{i_1=0}^{t_1} \sum_{k_2=0}^{t_2} \prod_{k_1=1}^{j_1} \prod_{k_2=0}^{j_2} (-1)^{p_k(j_k)}(1-k-J_k+p_k) \mu(1,1_2) \]

and

\[ \kappa_{\beta_1 \beta_2} = \sum_{k_1=1}^{2} \prod_{k_1=1}^{2} \left( z(a_k, b_k) \right) \gamma a_1 b_1 2a_1 a_2 2a_1 a_2 \]
(see [5]).

3. \(2^{m_1+m_2}\)-PBFF designs of resolution IV

Throughout this paper, we consider a design, which is a PBA \((N,m_1+m_2,2,t_1+t_2;\{\mu(1,1_2)\})\) with (2.6). Let \(K_{\beta_1 \beta_2} = \|k_{\beta_1 \beta_2}\|\) for \(\beta_1 \beta_2 = 00, 10, 01, 20, 02, 12\) (if \(m_2 \geq 4\)). Then a necessary and sufficient condition for the information matrix \(M_T\) to be nonsingular, i.e., \(T\) is of resolution V, is that every \(K_{\beta_1 \beta_2}\) is positive definite (see [5]). Note that a PBA \((N,m_1+m_2,2,t_1+t_2;\{\mu(1,1_2)\})\) yields a \(2^{m_1+m_2}\)-PBFF design of resolution V provided \(M_T\) is nonsingular. However the converse is not always true.

In this paper, we consider three cases as follows:

(A) \(\det(K_{\beta_1 \beta_2}) \neq 0\) for \(\beta_1 \beta_2 = 00, 10, 01, 20, 02, 12\) (if \(m_2 \geq 4\)).
and \( \det(K_{11}) = 0 \).

(B) \( \det(K_{\beta_1 \beta_2}) = 0 \) for \( \beta_1 \beta_2 = 00, 10, 01, 20 \) (if \( m_2 \geq 4 \)), and \( \det(K_{02}) = \det(K_{11}) = 0 \) for \( m_2 \geq 4 \)

and

(C) \( \det(K_{\beta_1 \beta_2}) = 0 \) for \( \beta_1 \beta_2 = 00, 10, 01 \), and \( \det(K_{20}) = \det(K_{02}) \)

(see [3,11]), where \( \det(A) \) denotes the determinant of a matrix \( A \).

Let \( \Psi_A = (\Theta_0; \Theta_1; \Theta_2; \Theta_3; (H_1^{\alpha}; \Theta_1')) \), \( \Psi_B = (\Theta_0; \Theta_1; \Theta_2; \Theta_3; (H_2^{\alpha}; \Theta_2')) \), \( \Psi_C = (\Theta_0; \Theta_1; \Theta_2; (H_0^{\alpha}; \Theta_0'); (H_1^{\alpha}; \Theta_1')) \) for \( m_2 \geq 4 \) and \( \Psi_C = (\Theta_0; \Theta_1; \Theta_2; (H_0^{\alpha}; \Theta_0'); (H_1^{\alpha}; \Theta_1')) \) for \( m_4 \geq 4 \), where

\[
\begin{align*}
H_A^{\alpha} &= h_0^{\alpha}A^{\alpha}_0 + h_1^{\alpha}A^{\alpha}_1 + h_2^{\alpha}A^{\alpha}_2 + h_3^{\alpha}A^{\alpha}_3, \\
H_B^{\alpha} &= h_0^{\beta}A^{\beta}_0 + h_1^{\beta}A^{\beta}_1 + h_2^{\beta}A^{\beta}_2 + h_3^{\beta}A^{\beta}_3, \\
H_C^{\alpha} &= h_0^{\alpha}A^{\alpha}_0 + h_1^{\alpha}A^{\alpha}_1 + h_2^{\alpha}A^{\alpha}_2 + h_3^{\alpha}A^{\alpha}_3,
\end{align*}
\]

and \( h_0^{\beta}, h_1^{\beta}, h_2^{\beta}, h_0^{\alpha}, h_1^{\alpha}, h_2^{\alpha} \)'s are real constants. Then we have the following (see [3,11]):

**Proposition 3.1.** Let \( T \) be a design which satisfies Condition (A) ((B) or (C)). Then \( \Psi_A \) (\( \Psi_B \) or \( \Psi_C \)) is an estimable function of \( \Theta \), and the BLUE of \( \Psi_A \) (\( \Psi_B \) or \( \Psi_C \)) is given by \( \hat{\Theta}_A = X_AE_TY_T \) (\( \hat{\Theta}_B = X_BE_TY_T \) or \( \hat{\Theta}_C = X_CE_TY_T \)), where \( X_A \) (\( X_B \) or \( X_C \)) is a matrix of order \( \nu(m_1 m_2) \) which satisfies \( X_AM_T = Z_A \) (\( X_BM_T = Z_B \) or \( X_CM_T = Z_C \)), \( Z_A = \text{diag}(I_{\nu_A}; H_A^{\alpha}) \)

\( Z_B = \text{diag}(I_{\nu_B}; H_B^{\alpha}; H_B^{\beta}) \) or \( Z_C = \text{diag}(I_{\nu_C}; H_C^{\alpha}; H_C^{\beta}; H_C^{\gamma}) \) and \( \nu_A = m_1 + m_2 \)

\((m_1) + (m_2) \) or \( \nu_B = m_1 + m_2 \) or \( \nu_C = m_1 + m_2 \).

Note that a design satisfying Condition (A) ((B) or (C)) is of course of resolution IV.
4. Algebraic structure

It is empirically known that the main effects are more important than the two-factor interactions. Thus we are interested in testing the hypotheses such that there exist some linear combinations of the two-factor interactions or not. If they do not exist, we wish to test the hypotheses such that there exist another linear combinations of them (or some linear combinations of the main effects) or not, and so on.

Using the properties of $A_{\beta_1\beta_2}^{\#(a_1a_2,a_1a_2)}$, as in (2.3), the linear model (2.1) can be rewritten as

$$y_\tau = \sum_{\beta_1\beta_2} \sum_{a_1a_2} \sum_{E_{a_1a_2}} A_{\beta_1\beta_2}^{\#(a_1a_2,a_1a_2)} \Theta_{a_1a_2} + e_\tau,$$

where $E_{a_1a_2}$'s are $N \times n(a_1a_2)$ submatrices of $E_\tau$ corresponding to $\Theta_{a_1a_2}$, i.e., $E_\tau = [E_{00}; E_{01}; E_{02}; E_{10}; E_{11}].$ By (2.2), (2.4) and (2.5), (i) every element of the vector $A_{\beta_1\beta_2}^{\#(a_1a_2,a_1a_2)} \Theta_{a_1a_2}$ represents the average of $\Theta_{a_1a_2}$ for $a_1a_2=00, 01, 02, 10, 11, 12,$ (ii) the elements of $A_{\beta_1\beta_2}^{\#(a_1a_2,a_1a_2)} \Theta_{a_1a_2}$ for $\beta_1\beta_2=10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11 represent the contrasts between these effects and any two contrasts are orthogonal, and (iii) there exist $\phi_{\beta_1\beta_2}$ independent parametric functions of $\Theta_{a_1a_2}$ in $A_{\beta_1\beta_2}^{\#(a_1a_2,a_1a_2)} \Theta_{a_1a_2},$ respectively (e.g., [17]).

Let $P_{\beta_1\beta_2}^{a_1a_2,b_1b_2} = E_{a_1a_2}^{\#(a_1a_2,b_1b_2)} E_{b_1b_2}^{\#(a_1a_2,b_1b_2)}$. Then by (2.4) and (2.7), we get the following (see [8]):

**Lemma 4.1.**

$$E_{a_1a_2,c_1c_2,d_1d_2,b_1b_2}^{\beta_1\beta_2} = \delta_{\beta_1\beta_2}^{c_1c_2,d_1d_2} E_{a_1a_2,b_1b_2}^{\beta_1\beta_2}. $$

Let $K_{\beta_1\beta_2}^{a_1a_2}$ be the matrices which are composed of the
initial, \ldots, the $a_1a_2$th rows and the initial, \ldots, the $a_1a_2$th columns of $K_{\beta_1\beta_2}$. Further let $K_{\beta_1\beta_2}(a_1a_2)^{-1} = \|\eta_{\beta_1\beta_2}\|$, if $K_{\beta_1\beta_2}(a_1a_2)$ is nonsingular. In addition, let $K_{\beta_1\beta_2}(a_1a_2^x)^{-1} = \|\eta_{\beta_1\beta_2}\|$, if $K_{\beta_1\beta_2}(a_1a_2^x)$ is nonsingular, where $K_{\beta_1\beta_2}(a_1a_2^x)$ is the matrix which is obtained by deleting the last row and the last column of $K_{\beta_1\beta_2}(a_1a_2)$, and $\eta_{\beta_1\beta_2}$, $\beta_1\beta_2(x\beta_2^x)=0$ for $\beta_1\beta_2=00,10,01,20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11.

Let
\[
p_{1a_2} = \frac{a_1a_2}{C_{1}\beta_1\beta_2} \text{ and } \frac{a_1a_2}{S_{1}\beta_1\beta_2}
\]
where $C_{1}\beta_1\beta_2$ and $S_{1}\beta_1\beta_2$ are the summations over all the values of $w_1w_2$ and $s_1s_2$ such that (I) if $\beta_1\beta_2=00$ and (1) $a_1a_2=00$, then $w_1w_2=00$ and $s_1s_2$ vanishes, (2) $a_1a_2=10$, then $w_1w_2=00,10$ and $s_1s_2$ =00, (3) $a_1a_2=01$, then $w_1w_2=00,10,01$ and $s_1s_2=00,10$, (4) $a_1a_2=20$, then $w_1w_2=00,10,01,20$ and $s_1s_2=00,10,01$, (5) $a_1a_2=02$, then $w_1w_2=00,10,01,20,02$ and $s_1s_2=00,10,01,20$, and (6) $a_1a_2=11$, then $w_1w_2=00,10,01,20,02,11$ and $s_1s_2=00,10,01,20,02$, (II) if $\beta_1\beta_2=10$ and (1) $a_1a_2=10$, then $w_1w_2=10$ and $s_1s_2$ vanishes, (2) $a_1a_2=20$ (if $m_1 \geq 3$), then $w_1w_2=10,20$ and $s_1s_2=10$, and (3) $a_1a_2=11$, then $w_1w_2=10,20$ (if $m_1 \geq 3$), 11 and $s_1s_2=10,20$ (if $m_1 \geq 3$), (III) if $\beta_1\beta_2=01$ and (1) $a_1a_2=01$, then $w_1w_2=01$ and $s_1s_2$ vanishes, (2) $a_1a_2=02$ (if $m_2 \geq 3$), then $w_1w_2=01,02$ and $s_1s_2=01$, and (3) $a_1a_2=11$, then $w_1w_2=01,02$ (if $m_2 \geq 3$), 11 and $s_1s_2=01,02$ (if $m_2 \geq 3$), (IV) if $\beta_1\beta_2=20$ (if $m_1 \geq 4$) and $a_1a_2=20$, then $w_1w_2=20$ and $s_1s_2$ vanishes, (V) if $\beta_1\beta_2=02$ (if $m_2 \geq 4$) and $a_1a_2=02$, then $w_1w_2=02$ and $s_1s_2$ vanishes, and (VII) if $\beta_1\beta_2= a_1a_2=11$, then $w_1w_2=11$ and $s_1s_2$ vanishes, respectively. Then the
following can be proved easily (see [8]):

Lemma 4.2. (i) The $P^{a_1a_2}_{\beta_1\beta_2}$'s are symmetric, mutually orthogonal and idempotent matrices.

(ii) $\text{rank}(P^{a_1a_2}_{\beta_1\beta_2}) = \phi_{\beta_1\beta_2}$.

First we consider a $2^{m_1+m_2}$-PBFF design which is a PB-array with Condition (A). If $N \geq \nu(m_1m_2)$, then there may exist a design of resolution V. However if $N = \nu(m_1m_2)$, there is no d.f. due to error. Thus we consider the case in which $\{3(m_1+m_2)+m_1^2+m_2^2\}/2
(=\nu(m_1m_2), \text{say}) < \nu(m_1m_2)$. Let $P_0^A = I_N - \sum_{\beta_1\beta_2} P^{a_1a_2}_{\beta_1\beta_2}$, where the summation $\sum_{\beta_1\beta_2}$ is extended over all the values of $\beta_1\beta_2$ such that $\beta_1\beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4, 02$ (if $m_2 \geq 4$). Then it follows from Lemma 4.2 that $(P_0^A)^2 = P_0^A$, $P_0^A P^{a_1a_2}_{\beta_1\beta_2} P_0^A = 0_{N \times N}$ and $\text{rank}(P_0^A) = \nu(m_1m_2)$, where $0_{p \times q}$ denotes the $p \times q$ matrix with all zero. Let

$$R^{a_1a_2}_{\beta_1\beta_2} = R(a_1a_2^x; \beta_1\beta_2) \setminus (R(a_1a_2; \beta_1\beta_2)),$$

where

$$R(a_1a_2; \beta_1\beta_2) = \left[ \begin{array}{c} E_{\beta_1\beta_2} A^{\#}(\beta_1\beta_2, \beta_1\beta_2); \cdots; E_{a_1a_2} A^{\#}(a_1a_2, a_1a_2) \end{array} \right]$$

for $\beta_1\beta_2 = 00, 10, 01, 20$.

$$R(20; 20) = \left\{ \begin{array}{ll} 0_N & \text{if } m_1 = 2, 3, \\ [E_{20} A^{\#}(20, 20)] & \text{if } m_1 \geq 4. \end{array} \right.$$ 

$$R(02; 02) = \left\{ \begin{array}{ll} 0_N & \text{if } m_2 = 2, 3, \\ [E_{02} A^{\#}(02, 02)] & \text{if } m_2 \geq 4. \end{array} \right.$$ 

and $R(a_1a_2^x; \beta_1\beta_2)$'s are the matrices which are obtained by deleting $E_{a_1a_2} A^{\#}(a_1a_2, a_1a_2)$ from $R(a_1a_2; \beta_1\beta_2)$, and $R(\beta_1\beta_2^x; \beta_1\beta_2) = 0_N$.

Here $\mathcal{R}_A(B)$ is the orthocomplement subspace of $\mathcal{R}(A)$ relative to $\mathcal{R}(B)$ for the case $\mathcal{R}(A) \subset \mathcal{R}(B)$, where $\mathcal{R}(A)$ denotes the linear subspace spanned by the column vectors of a matrix $A$. Then Lemma
4.2 and the properties of $P_\alpha$ yield the following:

**Theorem 4.1.** Let $T$ be a $2^{m_1+m_2}$-PBFF design which is derived from a PB-array with Condition (A). Then we have

$$\mathcal{R}^N = \begin{cases} \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_0 \oplus \mathcal{R}_0^\alpha & \text{if } m_1, m_2 = 2, 3, \\ \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_0 \oplus \mathcal{R}_2 \oplus \mathcal{R}_0^\alpha & \text{if } m_1 \geq 4, m_2 = 2, 3, \\ \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_0 \oplus \mathcal{R}_0 \oplus \mathcal{R}_2 \oplus \mathcal{R}_0^\alpha & \text{if } m_1 = 2, 3, m_2 \geq 4, \\ \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_0 \oplus \mathcal{R}_2 \oplus \mathcal{R}_2 \oplus \mathcal{R}_0^\alpha & \text{if } m_1, m_2 \geq 4, \end{cases}$$

where $\mathcal{R}^N$ is an $N$-dimensional vector space, $\oplus$ denotes the direct sum, $\mathcal{R}_0^\alpha = \mathcal{R}_{E_1}$, which is the orthocomplement subspace of $\mathcal{R}(E_1)$ relative to $\mathcal{R}^N$, and

$$\mathcal{R}_0 = \{ \mathcal{R}_0 \cap \mathcal{R}_0 \} \quad \text{if } m_1 = 2, \quad \mathcal{R}_0 = \{ \mathcal{R}_0 \cap \mathcal{R}_0 \} \quad \text{if } m_1 \geq 3, \quad \mathcal{R}_0 = \{ \mathcal{R}_0 \cap \mathcal{R}_0 \} \quad \text{if } m_2 = 2, \quad \mathcal{R}_0 = \{ \mathcal{R}_0 \cap \mathcal{R}_0 \} \quad \text{if } m_2 \geq 3$$

and

$$\mathcal{R}_{\beta_1 \beta_2} = \mathcal{R}(R(\beta_1 \beta_2; \beta_1 \beta_2)) \text{ for } \beta_1 \beta_2 = 20 \text{ (if } m_1 \geq 4), 02 \text{ (if } m_2 \geq 4).$$

Next consider a $2^{m_1+m_2}$-PBFF design being a PB-array with Condition (B), and $(3(m_1+2m_2)+m_1^3)/2 = \nu_B(m_1m_2)$, say $N = \nu_A(m_1m_2)$, where $m_2 \geq 4$. Let $P_0^B = I_N - \sum_{\beta_1 \beta_2 a_1 a_2 \beta_1 \beta_2} p_{a_1 a_2}^{\beta_1 \beta_2}$, where $\sum_{\beta_1 \beta_2}^{B}$ is the summation over all the values of $\beta_1 \beta_2$ such that $\beta_1 \beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$). Then $(P_0^B)^2 = P_0^B$, $P_0^B p_{a_1 a_2}^{\beta_1 \beta_2} p_{a_1 a_2}^{\beta_1 \beta_2} = 0_{N \times N}$ and rank$(P_0^B) = N - \nu_B(m_1m_2)$.

**Theorem 4.2.** For a $2^{m_1+m_2}$-PBFF design $T$, which is a PB-array with Condition (B), we have

$$\mathcal{R}^N = \begin{cases} \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_0 \oplus \mathcal{R}_0^\alpha & \text{if } m_1 = 2, 3, \\ \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_0 \oplus \mathcal{R}_2 \oplus \mathcal{R}_0^\alpha & \text{if } m_1 \geq 4, \end{cases}$$

where $\mathcal{R}_{\beta_1 \beta_2}$'s for $\beta_1 \beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$) are given in Theorem
Finally consider a $2^{m_1+m_2}$-PBFF design which is derived from a PB-array with Condition (C), where $3(m_1+m_2)$ (= $v^C(m_1,m_2)$, say) $< N \leq v^B(m_1,m_2)$ and $m_k \geq 4$. Let $P^C \equiv I_N-\sum_\beta_1 \beta_2 a_1 a_2 P^A a_2$, where $\sum_\beta_1 \beta_2$ is the summation over all the values of $\beta_1 \beta_2$ such that $\beta_1 \beta_2 = 00, 10, 01$. Then $(P^C)^2 = P^C$, $P^C a_1 a_2 = P^A a_2 P^C = 0_{N \times N}$ and rank$(P^C) = N - v^C(m_1,m_2)$.

**Theorem 4.3.** Let $T$ be a $2^{m_1+m_2}$-PBFF design which is a PB-array with Condition (C). Then

$$R^N = R_{00} \oplus R_{10} \oplus R_{01} \oplus R^C,$$

where $R_{\beta_1 \beta_2}$'s ($\beta_1 \beta_2 = 00, 10, 01$) are the same as in Theorem 4.1, and $R^C = R_{E_t}^A$.

### 5. ANOVA and hypothesis testing

We first consider the ANOVA and the hypothesis testing of $2^{m_1+m_2}$-PBFF designs of resolution IV satisfying Condition (A). Let $S_{\beta_1 \beta_2} = y_1 P^A a_2 y_7$ and $S^A = y_1 P^A y_7$. Then by Theorem 4.1, we have the following:

**Theorem 5.1.** Let $T$ be a $2^{m_1+m_2}$-PBFF design which is a PB-array with Condition (A) and $v^A(m_1,m_2) < N \leq v(m_1,m_2)$. Then we have

$$y_1 y_7 = \sum_\beta \sum a_1 a_2 S_{\beta_1 \beta_2} + S^A.$$

**Theorem 5.2.** For a design $T$ of Theorem 5.1, an unbiased estimator of $\sigma^2$ is given by

$$\hat{\sigma}^2 = S^A / \{N - v^A(m_1,m_2)\}.$$  

The noncentrality parameters, say, $\lambda_{\beta_1 \beta_2} / \sigma^2$, of the quadratic forms $y_1 P^A a_2 y_7 / \sigma^2$ are defined by $\gamma[y_1] P^A a_2 \gamma[y_7] / \sigma^2$, where
\( \mathcal{E}[y] \) denotes the expected value of a random vector \( y \). Let

\[
\begin{align*}
&c_{00}(p_1 p_2, q_1 q_2; a_1 a_2) \\
&= \begin{cases} \\
&\frac{a_1 a_2}{c_1 c_2} \frac{a_1 a_2}{d_1 d_2} (a_1 a_2) \kappa_{00} \eta_{00} p_2, c_1 c_2, d_1 d_2, q_1 q_2 \\
&- \frac{a_1 a_2}{e_1 e_2} \frac{a_1 a_2}{f_1 f_2} (a_1 a_2) \kappa_{00} \eta_{00} p_2, e_1 e_2, f_1 f_2, q_1 q_2 \\
&\frac{a_1 a_2}{g_1 g_2} \frac{a_1 a_2}{h_1 h_2} (a_1 a_2) \kappa_{00} \eta_{00} p_2, g_1 g_2, h_1 h_2, q_1 q_2 \\
&\text{if (1) } p_1 p_2 = q_1 q_2 = 11 \text{ for } a_1 a_2 = 11, \\
&\text{ (2) } p_1 p_2, q_1 q_2 = 02, 11 \text{ for } a_1 a_2 = 02, \\
&\text{ (3) } p_1 p_2, q_1 q_2 = 20, 02, 11 \text{ for } a_1 a_2 = 20, \\
&\text{ (4) } p_1 p_2, q_1 q_2 = 01, 20, 02, 11 \text{ for } a_1 a_2 = 01, \\
&\text{ (5) } p_1 p_2, q_1 q_2 = 10, 01, 20, 02, 11 \text{ for } a_1 a_2 = 10, \\
&\text{ (6) } p_1 p_2, q_1 q_2 = 00, 10, 01, 20, 02, 11 \text{ for } a_1 a_2 = 00, \\
&\text{ otherwise,}
\end{cases}
\]

where \( \sum_{w_1 w_2} \) and \( \sum_{s_1 s_2} \) are extended over all the values of \( w_1 w_2 \) and \( s_1 s_2 \) such that if \( a_1 a_2 = 11 \), then \( w_1 w_2 = 11 \) and \( s_1 s_2 \) vanishes, if \( a_1 a_2 = 02 \), then \( w_1 w_2 = 02, 11 \) and \( s_1 s_2 = 11 \), if \( a_1 a_2 = 20 \), then \( w_1 w_2 = 20, 02, 11 \) and \( s_1 s_2 = 02, 11 \), if \( a_1 a_2 = 01 \), then \( w_1 w_2 = 01, 20, 02, 11 \) and \( s_1 s_2 = 20, 02, 11 \), if \( a_1 a_2 = 10 \), then \( w_1 w_2 = 10, 01, 20, 02, 11 \) and \( s_1 s_2 = 01, 20, 02, 11 \), and if \( a_1 a_2 = 00 \), then \( w_1 w_2 = 00, 10, 01, 20, 02, 11 \) and \( s_1 s_2 = 10, 01, 20, 02, 11 \), respectively. Let

\[
\begin{align*}
&c_{10}(p_1 p_2, q_1 q_2; a_1 a_2) \\
&= \begin{cases} \\
&\frac{a_1 a_2}{c_1 c_2} \frac{a_1 a_2}{d_1 d_2} (a_1 a_2) \kappa_{10} \eta_{10} p_2, c_1 c_2, d_1 d_2, q_1 q_2 \\
&- \frac{a_1 a_2}{e_1 e_2} \frac{a_1 a_2}{f_1 f_2} (a_1 a_2) \kappa_{10} \eta_{10} p_2, e_1 e_2, f_1 f_2, q_1 q_2 \\
&\frac{a_1 a_2}{g_1 g_2} \frac{a_1 a_2}{h_1 h_2} (a_1 a_2) \kappa_{10} \eta_{10} p_2, g_1 g_2, h_1 h_2, q_1 q_2 \\
&\text{if (1) } p_1 p_2 = q_1 q_2 = 11 \text{ for } a_1 a_2 = 11, \\
&\text{ (2) } p_1 p_2, q_1 q_2 = 20 (m_1 \geq 3), 11 \text{ for } a_1 a_2 = 20, \\
&\text{ (3) } p_1 p_2, q_1 q_2 = 10, 20 (m_1 \geq 3), 11 \text{ for } a_1 a_2 = 10, \\
&\text{ otherwise,}
\end{cases}
\]

where \( \sum_{w_1 w_2} \) and \( \sum_{s_1 s_2} \) are the summations over all the values of \( w_1 w_2 \) and \( s_1 s_2 \) such that \( w_1 w_2 = 11 \) and \( s_1 s_2 \) vanishes if \( a_1 a_2 = 11 \),
\( w_1w_2 = 20 \) (if \( m_1 \geq 3 \)), \( 11 \) and \( s_1s_2 = 11 \) if \( a_1a_2 = 20 \), and \( w_1w_2 = 10, 20 \) (if \( m_1 \geq 3 \)), \( 11 \) and \( s_1s_2 = 20 \) (if \( m_1 \geq 3 \)), \( 11 \) if \( a_1a_2 = 10 \), respectively. Let

\[
\begin{align*}
&c_0(p_1p_2, q_1q_2; a_1a_2)
\end{align*}
\]

where \( \gamma(x)^{0,1}_{k_00} \) and \( \gamma(x)^{1,0}_{k_00} \) are extended over all the values of \( w_1w_2 \) and \( s_1s_2 \) such that \( w_1w_2 = 11 \) and \( s_1s_2 \) vanishes if \( a_1a_2 = 11 \), \( w_1w_2 = 02 \) (if \( m_2 \geq 3 \)), \( 11 \) and \( s_1s_2 = 11 \) if \( a_1a_2 = 02 \), and \( w_1w_2 = 01, 02 \) (if \( m_2 \geq 3 \)), \( 11 \) if \( a_1a_2 = 01 \), respectively. Further let

\[
\begin{align*}
&c_20(p_1p_2, q_1q_2; a_1a_2) = \begin{cases} 
\kappa_{0, 2}^{0, 20} & \text{if } p_1p_2 = q_1q_2 = a_1a_2 = 20, \\
0 & \text{otherwise},
\end{cases}
\end{align*}
\]

if \( \det(K_{20}) \neq 0 \) and \( m_1 \geq 4, \) and

\[
\begin{align*}
&c_02(p_1p_2, q_1q_2; a_1a_2) = \begin{cases} 
\kappa_{0, 2}^{0, 02} & \text{if } p_1p_2 = q_1q_2 = a_1a_2 = 02, \\
0 & \text{otherwise},
\end{cases}
\end{align*}
\]

if \( \det(K_{02}) \neq 0 \) and \( m_2 \geq 4 \). Then the following yields:

**Theorem 5.3.** Let \( T \) be a design of Theorem 5.1, then the non-centrality parameters of the quadratic forms \( y^T \beta_1 \beta_2 y / \sigma^2 \) for \( \beta_1 \beta_2 = 00, 10, 01, 20 \) (if \( m_1 \geq 4 \)), \( 02 \) (if \( m_2 \geq 4 \)) are

\[
\begin{align*}
\lambda_{\beta_1 \beta_2}^{a_1a_2} / \sigma^2 = \sum_{p_1p_2} \sum_{q_1q_2} \{ c \beta_1 \beta_2 (p_1p_2, q_1q_2; a_1a_2) / \sigma^2 \}
\end{align*}
\]

\[
\times \Theta^T \mathbb{A}^#(p_1p_2, q_1q_2) \Theta q_1q_2.
\]

Let \( n^{a_1a_2} \beta_1 \beta_2 \) be the hypotheses such that \( \mathbb{A}^#(a_1a_2, a_1a_2) \Theta a_1a_2 = \Theta 0_n(a_1a_2) \) (if they exist). We are first interested in testing the
hypotheses $H_{\beta_1 \beta_2}^{a_1 a_2}$ against $K_{\beta_1 \beta_2}^{a_1 a_2} (\beta_1 \beta_2=00,10,01)$, $H_{\beta_1 \beta_2}^{a_1 a_2}$ against $K_{\beta_1 \beta_2}^{a_1 a_2}$ (if $m_1 \geq 4$), and $H_{\beta_1 \beta_2}^{a_1 a_2}$ against $K_{\beta_1 \beta_2}^{a_1 a_2}$ (if $m_2 \geq 4$), where $K_{\beta_1 \beta_2}^{a_1 a_2}$'s are the hypotheses that $A_{\beta_1 \beta_2}^{(a_1 a_2,a_1 a_2)} \theta_{a_1 a_2} \neq 0(n(a_1 a_2))$. Next, if $H_{\beta_1 \beta_2}^{a_1 a_2}$ (or $H_{\beta_1 \beta_2}^{a_1 a_2}$) is accepted, we then consider the testing hypothesis $H_{\beta_1 \beta_2}^{a_1 a_2}$ (if $m_1 \geq 3$) or $H_{\beta_1 \beta_2}^{a_1 a_2}$ (if $m_1 = 2$) against $H_{\beta_1 \beta_2}^{a_1 a_2}$ (or $H_{\beta_1 \beta_2}^{a_1 a_2}$ (if $m_2 \geq 3$) or $H_{\beta_1 \beta_2}^{a_1 a_2}$ (if $m_2 = 2$) against $H_{\beta_1 \beta_2}^{a_1 a_2}$). If $H_{\beta_1 \beta_2}^{a_1 a_2}$ is accepted, then we consider $H_{\beta_1 \beta_2}^{a_1 a_2}$ against $H_{\beta_1 \beta_2}^{a_1 a_2}$. Third, if $H_{\beta_1 \beta_2}^{a_1 a_2}$ (if $m_1 \geq 3$) (or $H_{\beta_1 \beta_2}^{a_1 a_2}$ (if $m_2 \geq 3$)) is accepted, then consider $H_{\beta_1 \beta_2}^{a_1 a_2}$ against $H_{\beta_1 \beta_2}^{a_1 a_2}$ (or $H_{\beta_1 \beta_2}^{a_1 a_2}$ against $H_{\beta_1 \beta_2}^{a_1 a_2}$), and if $H_{\beta_1 \beta_2}^{a_1 a_2}$ is accepted, consider $H_{\beta_1 \beta_2}^{a_1 a_2}$ against $H_{\beta_1 \beta_2}^{a_1 a_2}$, and lastly if $H_{\beta_1 \beta_2}^{a_1 a_2}$ is accepted, then consider $H_{\beta_1 \beta_2}^{a_1 a_2}$ against $H_{\beta_1 \beta_2}^{a_1 a_2}$. This method is the so-called nested test procedure (e.g., [2]). Notice that Theorem 5.3 implies that $b_{\beta_1 \beta_2}^{a_1 a_2}$ is accepted if and only if $a_{\beta_1 \beta_2}^{a_1 a_2} = 0$, where $b_{\beta_1 \beta_2}^{a_1 a_2}$ denotes the intersection of $b_{\beta_1 \beta_2}^{a_1 a_2}$'s such that the running indices $b_{\beta_1 \beta_2}$ have the same values as $w_1 w_2$ of $\sum_{j=1}^{\infty} \beta_1 \beta_2$ for $\beta_1 \beta_2=00,10,01$, and as $\beta_1 \beta_2$ for $\beta_1 \beta_2=20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$).

The test statistics for the nested method are given by

\begin{align}
S_{\beta_1 \beta_2}^{a_1 a_2} / \phi_{\beta_1 \beta_2}^{a_1 a_2} &= \left(=F_{\beta_1 \beta_2}^{a_1 a_2}, \text{say}\right), \\
S_{\beta_1 \beta_2}^{a_1 a_2} / \phi_{\beta_1 \beta_2}^{a_1 a_2} &= \left(=F_{\beta_1 \beta_2}^{a_2}, \text{say}\right), \\
S_{\beta_1 \beta_2}^{a_1 a_2} / \phi_{\beta_1 \beta_2}^{a_2} &= \left(=F_{\beta_1 \beta_2}^{a_2}, \text{say}\right), \\
S_{\beta_1 \beta_2}^{a_1 a_2} / \phi_{\beta_1 \beta_2}^{a_2} &= \left(=F_{\beta_1 \beta_2}^{a_2}, \text{say}\right),
\end{align}

and
(ii) for $\beta_1 \beta_2 = 10$,

$$\frac{S_2^0 / \phi_0}{S_A / (N - \nu^A(m_1m_2))} = (F^A_{1,0}, \text{ say}), \quad (5.6)$$

$$\frac{S_0^0 / \phi_1}{S_A / (N - \nu^A(m_1m_2) + \phi_{10})} = (F^A_{1,0}, \text{ say}) \quad (\text{if } m_1 \geq 3) \quad (5.7a)$$

(or)

$$\frac{S_0^0 / \phi_1}{S_A / (N - \nu^A(m_1m_2) + \phi_{10})} = (F^A_{1,0}, \text{ say}) \quad (\text{if } m_1 = 2) \quad (5.7b)$$

and

$$\frac{S_0^0 / \phi_1}{S_A / (N - \nu^A(m_1m_2) + 2\phi_{10})} = (F^A_{1,0}, \text{ say}) \quad (\text{if } m_1 \geq 3) \quad (5.8)$$

(iii) for $\beta_1 \beta_2 = 01$,

$$\frac{S_0^1 / \phi_0}{S_A / (N - \nu^A(m_1m_2))} = (F^A_{0,1}, \text{ say}), \quad (5.9)$$

$$\frac{S_0^1 / \phi_0}{S_A / (N - \nu^A(m_1m_2) + \phi_{01})} = (F^A_{0,1}, \text{ say}) \quad (\text{if } m_2 \geq 3) \quad (5.10)$$

(or)

$$\frac{S_0^1 / \phi_0}{S_A / (N - \nu^A(m_1m_2) + \phi_{01})} = (F^A_{0,1}, \text{ say}) \quad (\text{if } m_2 = 2)$$

and

$$\frac{S_0^1 / \phi_0}{S_A / (N - \nu^A(m_1m_2) + 2\phi_{01})} = (F^A_{0,1}, \text{ say}) \quad (\text{if } m_2 \geq 3) \quad (5.11)$$

(iv) for $\beta_1 \beta_2 = 20$ and $m_1 \geq 4$,

$$\frac{S_2^0 / \phi_2}{S_A / (N - \nu^A(m_1m_2))} = (F^A_{2,0}, \text{ say}) \quad (5.12)$$

and (v) for $\beta_1 \beta_2 = 02$ and $m_2 \geq 4$,

$$\frac{S_0^2 / \phi_0}{S_A / (N - \nu^A(m_1m_2))} = (F^A_{0,2}, \text{ say}).$$

All of them have $F$ distributions, and the nesting procedure is continued until a significant test is obtained for each $\beta_1 \beta_2$. Note that $F^{A_{1,2}}_{\beta_1 \beta_2}$'s are central or noncentral $F$ distributions with
\[ \phi_{\beta_1 \beta_2} \text{ and } \{N - \nu^A(m_1, m_2)\} + \tau^A(a_1 a_2; \beta_1 \beta_2) \phi_{\beta_1 \beta_2} \text{ d.f., and noncentrality parameters } \lambda^{a_1 a_2}_{\beta_1 \beta_2} / \sigma^2 \text{ depending on which } a_1 a_2 \cap b_1 b_2 \text{ are true, where } \tau^A(a_1 a_2; \beta_1 \beta_2) \text{'s are some integers as above.} \]

Next consider the ANOVA and the hypothesis testing of \(2^{m_1 + m_2}\)-PBFF designs of resolution IV which satisfy Condition (B).

**Theorem 5.4.** Let \(T\) be a \(2^{m_1 + m_2}\)-PBFF design which is a PB-array with Condition (B) and \(\nu^B(m_1, m_2) < N - \nu^A(m_1, m_2)\). Then

\[
y_T^T y_T = \frac{\Sigma}{\beta_1 \beta_2} \frac{\Sigma}{a_1 a_2} \frac{\Sigma}{\beta_1 \beta_2} S_{\alpha_1 \alpha_2} + S_{e},
\]

where \(S_{e} = y_T^T y_T - y_T^T P_T y_T\).

**Theorem 5.5.** For a design \(T\) of Theorem 5.4, an unbiased estimator of \(\sigma^2\) is

\[
\hat{\sigma}^2 = S_{e} / (N - \nu^B(m_1, m_2)).
\]

**Theorem 5.6.** Let \(T\) be a design of Theorem 5.4. Then the noncentrality parameters of the quadratic forms \(y_T^T P_{\beta_1 \beta_2} y_T / \sigma^2\) for \(\beta_1 \beta_2 = 00, 10, 01, 20\) (if \(m_1 \geq 4\)) are given by

\[
\left. \frac{\lambda^{a_1 a_2}_{\beta_1 \beta_2}}{\sigma^2} \right|_{\beta_1 \beta_2 = 00, 10, 01, 20} = \frac{\Sigma}{p_1 p_2} \frac{\Sigma}{q_1 q_2} \{c_{\beta_1 \beta_2}(p_1 p_2, q_1 q_2; a_1 a_2) / \sigma^2\}
\]

\[
\times \theta^{p_1 p_2}_{\beta_1 \beta_2} (p_1 p_2, q_1 q_2) \theta_{q_1 q_2}.
\]

We now consider the hypotheses \(H_{\beta_1 \beta_2}^{1}\) against \(K_{\beta_1 \beta_2}^{1}\) for \(\beta_1 \beta_2 = 00, 10, 01, H_{\beta_1 \beta_2}^{2} \) against \(K_{\beta_1 \beta_2}^{2}\) (if \(m_1 \geq 4\)). Next if \(H_{\beta_1 \beta_2}^{1}\) (or \(H_{\beta_1 \beta_2}^{2}\)) is accepted, consider the testing hypothesis \(H_{\beta_1 \beta_2}^{3}\) (if \(m_1 = 2\)) against \(H_{\beta_1 \beta_2}^{4}\) (or \(H_{\beta_1 \beta_2}^{5}\) against \(H_{\beta_1 \beta_2}^{6}\)). If \(H_{\beta_1 \beta_2}^{4}\) is accepted, then consider \(H_{\beta_1 \beta_2}^{6}\) against \(H_{\beta_1 \beta_2}^{7}\). Third, if \(H_{\beta_1 \beta_2}^{3}\) (if \(m_1 = 3\)) (or \(H_{\beta_1 \beta_2}^{5}\) is accepted, then consider \(H_{\beta_1 \beta_2}^{6}\) against \(H_{\beta_1 \beta_2}^{7}\) (or \(H_{\beta_1 \beta_2}^{8}\) against \(H_{\beta_1 \beta_2}^{9}\)). If \(H_{\beta_1 \beta_2}^{8}\) is accepted, consider \(H_{\beta_1 \beta_2}^{9}\) against \(H_{\beta_1 \beta_2}^{10}\), and lastly if \(H_{\beta_1 \beta_2}^{10}\) is accepted, then consider \(H_{\beta_1 \beta_2}^{11}\) against \(H_{\beta_1 \beta_2}^{12}\). Note that Theorem 5.6 means that
\(a_1 a_2 b_1 b_2\) is accepted if and only if \(\chi_{a_1 a_2}^{a_1 a_2} = 0\). The test statistics, say \(F_{a_1 a_2}^{a_1 a_2}\), for the nested method are given by replacing \(S_{\beta_1 \beta_2}^A\) and \(v^A(m_1 m_2)\) of (5.1) through (5.12) with \(S_{\beta_1 \beta_2}^B\) and \(v^B(m_1 m_2)\), respectively. The \(F_{a_1 a_2}^{a_1 a_2}\)'s have \(F\) distributions similar to \(F_{\beta_1 \beta_2}^{a_1 a_2}\)’s.

We finally consider the ANOVA and the hypothesis testing of \(2^{m_1 + m_2}\)-PBFF designs satisfying Condition (C).

**Theorem 5.7.** Let T be a \(2^{m_1 + m_2}\)-PBFF design which is a PB-array with Condition (C) and \(v^C(m_1 m_2) < N v^B(m_1 m_2)\). Then we have

\[y_1 y_T = \sum_{x_1 x_2} a_1 a_2 \sum_{x_1 x_2} S_{\beta_1 \beta_2}^{a_1 a_2} + S_C^c,\]

where \(S_C^c = y_1^T P_C^c y_T\).

**Theorem 5.8.** Let T be a design of Theorem 5.7, then an unbiased estimator of \(\sigma^2\) is given by

\[\hat{\sigma}^2 = S_C^c / (N - v^C(m_1 m_2)).\]

**Theorem 5.9.** For a design T of Theorem 5.7, the noncentrality parameters of the quadratic forms \(y_1 T P_{a_1 a_2}^{a_1 a_2} y_1 / \sigma^2\) (\(\beta_1 \beta_2 = 00, 10, 01\)) are

\[\chi_{\beta_1 \beta_2}^{a_1 a_2} / \sigma^2 = \sum_{p_1 p_2} \sum_{q_1 q_2} (c_{\beta_1 \beta_2} (p_1 p_2, q_1 q_2; a_1 a_2) / \sigma^2) \times \theta_{p_1 p_2}^{c} (p_1 p_2, q_1 q_2).\]

Consider the testing hypotheses \(H_{\beta_1 \beta_2}^{11}\) against \(K_{\beta_1 \beta_2}^{11}\) for \(\beta_1 \beta_2 = 00, 10, 01\). Next if \(H_{\beta_1 \beta_2}^{11}\) (or \(H_{\beta_1 \beta_2}^{01}\)) is accepted, then consider the testing hypothesis \(H_{\beta_1 \beta_2}^{11}\) against \(H_{\beta_1 \beta_2}^{11}\) (or \(H_{\beta_1 \beta_2}^{01}\) against \(H_{\beta_1 \beta_2}^{01}\)). If \(H_{\beta_1 \beta_2}^{11}\) is accepted, then consider \(H_{\beta_1 \beta_2}^{11}\) against \(H_{\beta_1 \beta_2}^{11}\). Third if \(H_{\beta_1 \beta_2}^{11}\) (or \(H_{\beta_1 \beta_2}^{01}\)) is accepted, consider \(H_{\beta_1 \beta_2}^{11}\) against \(H_{\beta_1 \beta_2}^{11}\), and if \(H_{\beta_1 \beta_2}^{11}\) is accepted, consider \(H_{\beta_1 \beta_2}^{11}\) against \(H_{\beta_1 \beta_2}^{11}\). If \(H_{\beta_1 \beta_2}^{11}\) is ac-
cepted, consider $H_{\beta_1 \beta_2}^{a_1 a_2}$ against $H_{\beta_1 \beta_2}^{a_1 a_2}$, and lastly if $H_{\beta_1 \beta_2}^{a_1 a_2}$ is accepted, consider $H_{\beta_1 \beta_2}^{a_1 a_2}$ against $H_{\beta_1 \beta_2}^{a_1 a_2}$. Note that Theorem 5.9 implies that $a_1 a_2 b_1 b_2 \beta_1 \beta_2$ is accepted if and only if $a_1 a_2 = 0$. The test statistics, say $F_{\beta_1 \beta_2}^{a_1 a_2}$ for the nested method are given by replacing $S_{\beta_1 \beta_2}^{a_1 a_2}$ and $\nu^{a_1 a_2}(m_1 m_2)$ of (5.1) through (5.11) with $S_{\beta_1 \beta_2}^{a_1 a_2}$ and $\nu^{c}(m_1 m_2)$, respectively. The $F_{\beta_1 \beta_2}^{a_1 a_2}$'s have $F$ distributions similar to $F_{\beta_1 \beta_2}^{a_1 a_2}$'s.

References


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