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超平面／曲面のアレンジメントの組合せ論的性質

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Abstract

In this paper, we first prove that the arboricity of the dual graph of an arrangement of hyperplanes in d-dimensional Euclidian space is d. The result is extended partially to the arrangement of hypersurfaces. Also, we introduce a new measure of the complexity of the dual graph of an arrangement, and bound it by only using the property of the graph. This bound can be used in the analysis of a computationally robust algorithm for Voronoi diagrams and also to obtain another optimal randomized algorithm for finding the intersections among line segments and curves.

1 Introduction

The study of the complexity of arrangements is one of the most important subjects in computational geometry, because many geometric structures such as Voronoi diagrams are represented as structures on arrangements. The number of faces in an arrangement is a popular measure of the complexity of the arrangement. There, however, exist many cases in which it is difficult to evaluate the number of faces, for example, in the zone of the arrangement of hypersurfaces. In this paper, we study the arboricity of the dual graph of an arrangement, and introduce a new measure for the complexity of the arrangement. Since this measure is essentially based on graphs, there arise pure graph-theoretic problems related to the measure. We also obtain interesting results concerning these graph problems. The result on this new measure of the complexity of arrangements can be utilized to devise efficient algorithms for geometric problems.

A dual graph of an arrangement is the graph whose vertex set consists of cells of the arrangement and whose edge set contains an edge xy if and only if the cells x and y are adjacent in the arrangement. In Section 2, we consider the arboricity of the dual graph of an arrangement of hyperplanes. This result can be extended to the special case of the arrangement of hypersurfaces in d-dimensional Euclidian space. In Section 3, we introduce $D(G)$, which is the sum of smaller endvertex degree over edges of the undirected graph G, as a measure of the complexity of G. Chiba and Nishizeki [2] show that $D(G)$ is at most $2a(G)|E|$ where $a(G)$ is the arboricity of G (the minimum number of trees covering G). We investigate the bound for $D(G)$ in more detail, and give more refined bound for $D(G)$. We also study this bound for some special families of graphs. In Section 4, we evaluate the complexity of the arrangement by means of the measure $D(G)$ of the dual graph G of an arrangement.

This evaluation can be used in the analysis of a computationally robust divide-and-conquer algorithm for constructing the Delaunay triangulation (Oishi and Sugihara [8]). Furthermore, this can be directly used to develop an optimal randomized algorithm for constructing the intersections (or arrangements) of line segments and curves. In Section 4, we provide only an outline of this algorithm. Although such optimal (randomized) algorithms are already known to exist (Chazelle, Edelsbrunner [1], Mulmuley [6]), this indicates the usefulness of the new measure in developing new geometric algorithms.

In the following sections, hypersurfaces are considered to be topologically isomorphic to hyperplanes, that is, the word "hypersurface" means a hypersurface with no self intersections, with no holes (i.e. every topological circle in the hypersurface can be continuously transformed in the hypersurface to a point) and unbounded (i.e. not contained in a hypersphere). All graphs in this paper are considered to be undirected and may have multiple edges and loop edges, if not mentioned.

2 Arboricity of the Dual Graph of an Arrangement

Let H be a finite set of hyperplanes or a finite set of hypersurfaces in E^d, and the arrangement of H is denoted by $\mathcal{A}(H)$. For faces x of $\mathcal{A}(H)$, $\dim(x)$ denotes the dimension of x. For a face x and y of $\mathcal{A}(H)$, the relation denoted by $y \succ_i x$ means $\dim(x) = \dim(y) - i$ and the boundary of y contains x (y is incident to x), and $y \succ x$ means existence of the relation $y \succ_i x$ for some positive integer $i \leq d$. The number of k-faces of an arrangement $\mathcal{A}(H)$ is denoted by $f_k(\mathcal{A}(H))$ and will be simply denoted by $f_k(H)$. Let T be a finite set of points in E^d, and $\mathcal{A}(H)|T$, which is a subset of faces of $\mathcal{A}(H)$, is defined as follows:

$$\begin{aligned} F_d &= \{x \in \mathcal{A}(H) \mid \dim(x) = d,\ x \cap T \neq \phi\}, \\ F_i &= \{x \in \mathcal{A}(H) \mid \dim(x) = i,\ x \cap T \neq \phi\} \\ &\quad \cup \{x \in \mathcal{A}(H) \mid \dim(x) = i, \mathcal{A}(H) \cap F_{i+1} \ni y \ for \ \forall y \succ_1 x\} \ (i < d), \\ \mathcal{A}(H)|T &= \bigcup_{i=0}^{d} F_i. \end{aligned}$$

In other words, $\mathcal{A}(H)|T$ is recursively defined by above two conditions on the decreasing order of the dimension of its faces. The number of k-faces in $\mathcal{A}(H)|T$ is denoted by $f_k(\mathcal{A}(H)|T)$ and simply denoted by $f_k(H|T)$. For $h \in H$, the intersection of $\mathcal{A}(H)$ with h, $\{h' \cap h | h' \in \mathcal{A}(H) - h\}$ forms a $(d-1)$-dimensional arrangement in h, and this intersection arrangement is denoted by $\mathcal{A}(H/h)$. In this section, $f_k(H/h|T)$ is interpreted as $f_k(\mathcal{A}(H)|T \cap \mathcal{A}(H/h))$ instead of $f_k(\mathcal{A}(H/h)|T)$ for some $T \subseteq h$. Now we show the following lemma:

Lemma 1 *Let H be a finite set of hyperplanes in E $d, h \in H$, and T be a finite set of points. Then,*

$$
\begin{aligned}
f_d(H|T) &= f_d(H-h|T) + f_{d-1}(H/h|T), & (1)\\
f_k(H|T) &\leq f_k(H-h|T) + f_k(H/h|T) + f_{k-1}(H/h|T) \text{ for } k < d. & (2)
\end{aligned}
$$

Proof. A face f in $\mathcal{A}(H-h)|T$ is one of the following three types.

Type(A) f is not intersected by h, or f is contained in h.

Type(B) f is separated by the $(\dim(f)-1)$-face $g = f \cap h$ into two faces, and at least one of these two faces is not in $\mathcal{A}(H|T)$.

Type(C) f is separated by the $(\dim(f)-1)$-face $g = f \cap h$ into two faces, and both of these two faces are in $\mathcal{A}(H|T)$.

Every cell in $\mathcal{A}(H)|T$ is contained in some cell in $\mathcal{A}(H-h)|T$. The cells of Type(A) and Type(B) are counted once in $f_d(H|T)$, and the cells of Type(C) are counted twice in $f_d(H|T)$. For each cell f of Type(C), the $(d-1)$-face g is in $\mathcal{A}(H/h)|T$ and counted also in $f_{d-1}(H/h|T)$. Thus the equality holds in (1).

For $k < d$, every k-face in $\mathcal{A}(H)|T$ is either one of k-faces in $\mathcal{A}(H/h)|T$ or contained in some k-face in $\mathcal{A}(H-h)|T$. The k-faces of Type(A) and Type(B) are counted at most once in $f_k(H|T)$. Note that there exist the case in which the k-face f in $\mathcal{A}(H-h)|T$ is not counted in $f_k(H|T)$ illustrated in Figure 2.1. The k-faces of Type(C) are counted twice in $f_d(H|T)$, and for each k-face f of Type(C), the $(k-1)$-face g is in $\mathcal{A}(H/h)|T$ and counted also in $f_{k-1}(H/h|T)$. Thus the inequality holds in (2), and the lemma is proved. □

Using Lemma 1, we prove the following lemma.

Lemma 2 *Let H be a finite set of hyperplanes in E^d, and T be a finite set of points and $f_d(H|T) \geq 1$. Then,*

$$f_{d-1}(H|T) \leq d(f_d(H|T)-1). \tag{3}$$

Proof. We prove the inequality (3) by double induction on d and the cardinality of H. It is easy to show (3) for $H = \phi$. Thus we consider the case that $H \neq \phi$. For $h \in H$, using the inequality (2) in Lemma 1, we get

$$f_{d-1}(H|T) \leq f_{d-1}(H-h|T) + f_{d-1}(H/h|T) + f_{d-2}(H/h|T). \tag{4}$$

By the induction hypothesis on the cardinality of h, we get

$$f_{d-1}(H-h|T) \leq d(f_d(H-h|T)-1). \tag{5}$$

By the induction hypothesis on d, we get

$$f_{d-2}(H-h|T) \leq (d-1)(f_{d-1}(H/h|T)-1). \tag{6}$$

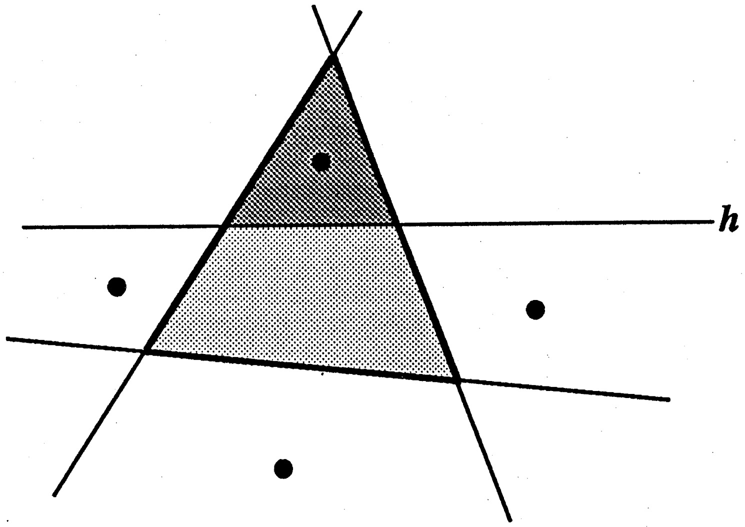

Figure 2.1. An example of $\mathcal{A}(H)$ for 2-dimensional case. The dots show the points of T and the bold lines show the faces counted in $\mathcal{A}(H-h)$ but not counted in $\mathcal{A}(H)|T$.

Using (4),(5),(6), we get

$$f_{d-1}(H|T) \leq d(f_d(H-h|T) + f_{d-1}(H/h|T) - 1) - (d-1) \tag{7}$$

Applying (1) in Lemma 1 to (7),

$$f_{d-1}(H|T) \leq d(f_d(H|T) - 1) - (d-1) \leq d(f_d(H|T) - 1) \tag{8}$$

Thus (3) is proved. □

Now we extend Lemma 2 to the arrangement of hypersurfaces. To assure the same result as in Lemma 2, we need to limit the arrangement of hypersurfaces to a special family. The cell of an arrangement of hypersurfaces is called "reducible" if every topological circle in the cell can be continuously transformed in the cell to a point. The arrangement of hypersurfaces in E^d is called "reducible" if every cell in the arrangement is reducible.

Lemma 3 *Let H be a finite set of hypersurfaces in E^d, T be a finite set of points, and $f_d(H|T) \geq 1$. If $\mathcal{A}(H)$ is reducible, then,*

$$f_{d-1}(H|T) \leq d(f_d(H|T) - 1). \tag{9}$$

Proof. To prove the inequality (9) by the same double induction as in Lemma 2, it is enough to show the following two inequalities for $h \in H$:

$$f_d(H|T) \geq f_d(H-h|T) + f_{d-1}(H/h|T), \tag{10}$$

$$f_{d-1}(H|T) \leq f_{d-1}(H-h|T) + f_{d-1}(H/h|T) + f_{d-2}(H/h|T). \tag{11}$$

Suppose a point set T' such that $T' \supseteq T$ and all points in $T' - T$ are contained in facets of $\mathcal{A}(H)$. Since $f_{d-1}(H|T) \leq f_{d-1}(H|T')$ and $f_d(H|T) = f_d(H|T')$, we can use T' instead of T. Thus, in the proof of this lemma, we can add to T new points in some facets of $\mathcal{A}(H)$. For convenience, we also use T for the point set after addition.

We need to consider the case that a cell c in $\mathcal{A}(H-h)|T$ is separated by h into more than two cells, say c_i for $i = 1, 2, ..., n$. In this case, each facet in $h \cap c$, say f_j for $j = 1, 2, ..., m$, separate c into exactly two parts. The dual graph G of these cells generated by the separation of c is described as follows:

- $V(G) = \{c_i \in \mathcal{A}(H) \cap c | i = 1, 2, ..., n\}$,
- $E(G) = \{f_j \in \mathcal{A}(H/h) \cap c | j = 1, 2, ..., m\}$,
- f_j and c_i is incident iff they are incident in $\mathcal{A}(H)$.

If G has a cycle, we can draw the associated topological circle in the cell c such that the intersection of each facet f_j and the topological circle is one point or empty. This topological circle can not be continuously transformed to a point, because each segment of the topological circle contained in c_i has its two endpoints on different facets and the continuous transformation can only move these endpoints in the facet the point is contained. This contradict that the arrangement is reducible. Therefore G has no cycle. Let C be the subset of $V(G)$ such that $C \in v$ iff the cell v contains a point of T. Since G has no cycle, the subgraph G' of G induced by C is a forest. Hence we get,

$$\begin{aligned} |\,\mathcal{A}(H)|T \cap c\,| &= |V(G')| = |C| \\ |\,\mathcal{A}(H/h)|T \cap c\,| &= |E(G')| \leq |C| - 1 \end{aligned}$$

Above discussion shows that the cell in $\mathcal{A}(H-h)$ which is intersected by h and counted p times in $f_d(H|T)$ has at most $p-1$ facets counted in $f_{d-1}(H/h|T)$. It is obvious that the cells not intersected by h are counted once in $f_d(H|T)$. Since every cell in $\mathcal{A}(H)|T$ is contained in some cell in $\mathcal{A}(H-h)|T$, the inequality holds in (10).

Every facet in $\mathcal{A}(H)|T$ is either one of facets in $\mathcal{A}(H/h)|T$ or contained in some facets in $\mathcal{A}(H-h)|T$. The facets in $\mathcal{A}(H-h)|T$ which are not intersected by h are counted at most once in $f_k(H|T)$. For each facets f_i in $\mathcal{A}(H) \cap \mathcal{A}(H-h)|T$ we add to T a new point in the facet. Then, for T after addition, each facet f_i in $\mathcal{A}(H)|T$ which are separated by h from the facet f in $\mathcal{A}(H)|T$ has at least one incident $(d-2)$-face g in $f \cap h$, because both two facets incident to g in f are in $\mathcal{A}(H|T)$. Hence the facets in $\mathcal{A}(H-h)|T$ which are intersected by h and counted p times in $f_d(H|T)$ has at least $p-1$ facets counted in $f_{d-1}(H/h|T)$. Therefore the inequality holeds in (11). By the same double induction using (10) and (11) as in Lemma 2, the inequality (9) is proved. □

To obtain the arboricity of the dual graph of these arrangement, we use following Nash-Williams' theorem.

Theorem 1 **(Nash-Williams [7])** *For a graph G,*

$$a(G) = \max\left\{ \left\lceil \frac{|E(S)|}{|V(S)|-1} \right\rceil \,\middle|\, S \subset G, |V(S)| \geq 2 \right\}.$$

It is easy to show that there exists the d-dimensional arrangement $\mathcal{A}(H)$ of hyperplanes H such that

$$f_{d-1}(H) > (d-1)(f_d(H)-1).$$

For example, suppose linearly independent d axes in E^d and let H be the set of enough large number of hyperplanes vertical to each one axis.

Therefore, by Theorem 1, it is shown that the arboricity of the dual graph G of d-dimensional arrangement of hypersurfaces is more than or equal to d. Now we prove the following theorem.

Theorem 2 *The arboricity of the dual graph of a reducible d-dimensional arrangement of hypersurfaces is d.*

Proof. Let H be a finite set of hypersurfaces in E^d. We consider the dual graph G of the arrangement $\mathcal{A}(H)$. For each subgraph S of G satisfying $|V(S)| \geq 2$, we take T such that T has a point in each cell which correspond to a vertex of the graph S. Then $|f_{d-1}(H|T)| = |E(S)|$, and by Lemma 3 and Theorem 1, the statement of the theorem is proved. □

The following corollary is directly obtained from this theorem.

Corollary 1 *The arboricity of the dual graph of a d-dimensional arrangement of hyperplanes is d.*

3 The sum of smaller endvertex degree over edges

For an undirected graph $G = (V, E)$ with vertex set V and edge set E, let us define $D(G)$ by

$$D(G) = \sum_{e=(u,v)\in E} \min\{\deg(u), \deg(v)\},$$

where $\deg(v)$ is the degree of a vertex v.

Theorem 3 **(Chiba, Nishizeki [2])** *For a graph G,*

$$D(G) \leq 2a(G)|E|$$

We improve this result in more detail to obtain tight bounds for the special families of graphs, and the following theorem is our result.

Theorem 4 *For a planar undirected graph $G = (V, E)$, which is simple and connected, with more than 12 vertices, $D(G) \leq 6|E| - 36$. For a planar graph $G' = (V', E')$ with more than 10 vertices and without any cycle of length 3, $D(G') \leq 4|E'| - 16$. For an outerplanar graph $G'' = (V'', E'')$ with more than 5 vertices, $D(G'') \leq 4|E''| - 12$. Also, there exist graphs G, G' and G'' satisfying $D(G) = 6|E| - 36$, $D(G') = 4|E'| - 16$ and $D(G'') = 4|E''| - 12$.*

Proof of the upper bound We first prove the upper bound. Let $G = (V, E)$ be a simple planar graph. In considering the upper bound for $D(G)$, we can assume G is edge maximal, because adding edges to G does not reduce the value of $D(G)$. We denote the maximum degree of G by Δ. For each $i = 1, \ldots, \Delta$, define V_i to be the set of vertices whose degree is i, E_i to be the set of edges whose smaller endpoint degree is i, and let $\tilde{V}_i = \bigcup_{j=i}^{\Delta} V_j$, and let $\tilde{E}_i = \bigcup_{j=i}^{\Delta} E_j$. We also define the integer function $c(k)$ for every positive integer k as follows:

$$c(1) = 3,\ c(2) = 5,\ c(k) = 6 \text{ for } k \geq 3.$$

Since each subgraph H of G is also planar, we can obtain $|E(H)| \leq 3|V(H)| - c(|V(H)|)$ for all subgraph H of G by using Euler's relation. Then we have

$$\begin{aligned} D(G) &= \sum_{i=1}^{\Delta} i|E_i| = \sum_{i=1}^{\Delta}\sum_{j=i}^{\Delta} |E_i| = \sum_{i=1}^{\Delta} |\tilde{E}_i| \\ &\leq \sum_{i=1}^{\Delta} 3|\tilde{V}_i| - \sum_{i=1}^{\Delta} c(|\tilde{V}_i|) = 3\sum_{i=1}^{\Delta}\sum_{j=i}^{\Delta} |V_i| - \sum_{i=1}^{\Delta} c(|\tilde{V}_i|) \\ &= 3\sum_{i=1}^{\Delta} i|V_i| - \sum_{i=1}^{\Delta} c(|\tilde{V}_i|) = 6|E(G)| - \sum_{i=1}^{\Delta} c(|\tilde{V}_i|). \end{aligned}$$

Now we denote the second maximum degree and the third maximum degree by Δ_2 and Δ_3, respectively. Then we get

$$\begin{aligned} \sum_{i=1}^{\Delta} c(|\tilde{V}_i|) &\geq \Delta c(1) + \Delta_2\{c(2) - c(1)\} + \Delta_3\{c(3) - c(2)\} \\ &= 3\Delta + 2\Delta_2 + \Delta_3 \end{aligned}$$

Using that G is edge maximal planar graph with more than 3 vertices, we can show that $|E(G)| = 3|V(G)| - 6$ and that $\Delta \geq \Delta_2 \geq \Delta_3 \geq 3$.

Now we prove the case $|V(G)| > 14$. If $\Delta > 8$ or $\Delta_2 > 6$ or $\Delta_3 > 5$, then $3\Delta + 2\Delta_2 + \Delta_3 \geq 36$. Hence we can assume $\Delta \leq 8$ and $\Delta_2 \leq 6$ and $\Delta_3 \leq 5$. If $\Delta_3 < 5$, then $6|V(G)| - 12 = 2|E(G)| \leq \Delta_3(|V(G)| - 2) + \Delta_2 + \Delta \leq 4(|V(G)| - 2) + 14$, therefore $|V(G)| \leq 9$. As $|V(G)| > 14$, we can assume

$\Delta_3 = 5$. If $\Delta > 6$, then $3\Delta + 2\Delta_2 + \Delta_3 \geq 36$. Thus we can assume that $\Delta = 6$. And we get $6|V(G)| - 12 \leq \Delta_3(|V(G)| - 2) + \Delta_2 + \Delta = 5(|V(G)| - 2) + \Delta_2 + 6$, that is, $|V(G)| \leq 8 + \Delta_2 \leq 14$. This implies that $3\Delta + 2\Delta_2 + \Delta_3 \geq 36$ for $|V(G)| > 14$. We thus have the upper bound for $D(G)$ in the theorem. Note that this proof also indicates that the exceptional case for $|V(G)| = 14$ must consist of 2 adjacent vertices with degree 6 and other 12 vertices with degree 5. This, however, can not be planar. Also the case $|V(G)| = 13$ can not be planar, and we can obtain the bound for $|V(G)| > 12$.

Now, suppose that G does not have any cycle of length 3. We redefine the integer function $c(k)$ for every positive integer k as follows:

$$c(1) = 2,\ c(2) = 3,\ c(k) = 4 \text{ for } k \geq 3.$$

Then, Euler's relation states that, for any subgraph H of G,

$$|E(H)| \leq 2|V(H)| - c(|V(H)|).$$

Applying the above arguments to this case, we can obtain the upper bound. We will give an outline of the proof. First we get

$$D(G) \leq 4|E(G)| - \sum_{i=1}^{\Delta} c(|\tilde{V}_i|) \leq 4|E(G)| - 2\Delta - \Delta_2 - \Delta_3.$$

To show $2\Delta + \Delta_2 + \Delta_3 \geq 16$, we can assume $\Delta \leq 5$ and $\Delta_2 \leq 4$ and $\Delta_3 \leq 3$. Using the same argument, it is shown that $\Delta_3 = 3$ and $\Delta = 4$, and we get $|V(G)| \leq 6 + \Delta_2 \leq 10$. We thus have the upper bound for $D(G')$ in the theorem.

Finally, suppose that G is outerplanar. We redefine the integer function $c(k)$ for every positive integer k as follows:

$$c(1) = 2,\ c(k) = 3 \text{ for } k \geq 2.$$

Then, for any subgraph H of G,

$$|E(H)| \leq 2|V(H)| - c(|V(H)|).$$

Now we prove the case $|V(G)| > 7$. Applying the same arguments to this case, we can obtain the upper bound. We will give an outline of the proof. Here we get

$$D(G) \leq 4|E(G)| - \sum_{i=1}^{\Delta} c(|\tilde{V}_i|) \leq 4|E(G)| - 2\Delta - \Delta_2.$$

To show $2\Delta + \Delta_2 \geq 12$, we can assume $\Delta \leq 4$ and $\Delta_2 \leq 3$. The same argument shows that $\Delta_2 = 3$ and $\Delta = 4$, and we get $|V(G)| \leq 7$. The case $|V(G)| = 7$ and $|V(G)| = 6$ can not be outerplanar. We thus have the upper bound for $D(G'')$

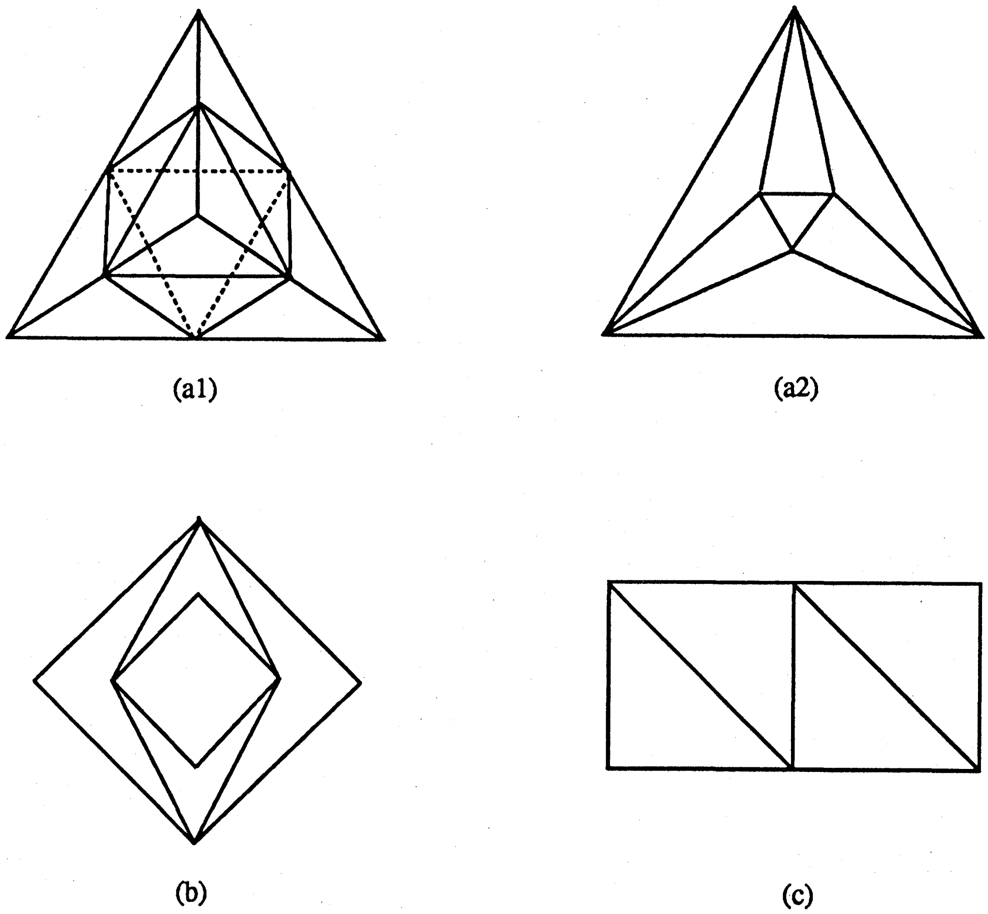

Figure 3.1. (a1) T_1, (a2) P_2, (b) S_2 and (c) R_2

in the theorem.

Proof of the lower bound We next consider lower bounds of the summation in the theorem for planar graphs. Consider a regular tetrahedron T_0 whose edges are of unit length. Each face is a regular triangle, and there are 4 faces, 6 edges and 4 vertices of degree 3. By connecting the midpoints of the edges, each triangle may be partitioned into four regular subtriangles; by repeating this process k times (denote the resultant polyhedron by T_k), each original triangle is divided into 4^k regular subtriangles with edges of length 2^{-k}. Figure 3.1(a1) depicts T_1. Then, in the interior of each original face, there are $1 + 2^{2k-1} - 3 \cdot 2^{k-1}$ vertices of degree 6. Hence, in total, among $n = 4(1 + 2^{2k-1} - 3 \cdot 2^{k-1}) + 6(2^k - 1) + 4 = 2 + 2^{2k+1}$ vertices, only 4 vertices have degree 3 and the others degree 6, and no edge connects vertices of degree 3. Therefore, for such $n = 2 + 2 \cdot 4^k$ and $m = 6 \cdot 4^k$ with $k = 2, 3, \ldots$, there exists a planar graph with n vertices and m edges for which the summation in the theorem is $6m - 36$. We can construct another series of graphs which attain the lower bound. P_1 is a triangle. P_{i+1} is constructed from P_i by adding new larger triangle surrounding P_i and connect each new vertices in the larger triangle to the corresponding two vertices on the outer triangle of P_i. Figure 3.1(a2) illustrates P_2. P_i has $n = 3i$ vertices and $m = 9i - 6$ edges. $D(P_i) = 6m - 36$ for $i \geq 3$.

To obtain a lower bound for planar graphs without cycle of length 3, construct the following series of graphs. S_0 is a square. We regard a pair of diagonal vertices in S_0 as new vertices. S_{i+1} is constructed from S_i by adding two new vertices and connect each of them with the new vertices in S_i. Figure 3.1(b) illustrates S_2. S_i has $n = 4 + 2i$ vertices and $m = 4 + 4i$ edges. $D(S_i) = 4m - 16$ for $i \geq 1$.

To obtain a lower bound for outerplanar graphs, construct the following series of graphs. R_1 is a square with a diagonal. R_{i+1} is constructed from R_i by adding two new vertices to make a copy of R_1 outside on the right edge of R_i. Figure 3.1(c) illustrates R_2. R_i has $n = 2 + 2i$ vertices and $m = 1 + 4i$ edges. $D(S_i) = 4m - 12$ for $i \geq 2$. □

4 An Optimal Randomized Algorithm for Arrangements of Curves

In this paper, we will describe an $O(N^2)$-time randomized algorithm for constructing an arrangement of N lines, without using the zone theorem for lines. This illustrates, for the problem of constructing an arrangement of N curves such that any two curves intersect at a constant number of points, how a simple incremental algorithm using a careful search technique with $O(N^2)$ randomized time complexity may be devised based on the inequality in Theorem.

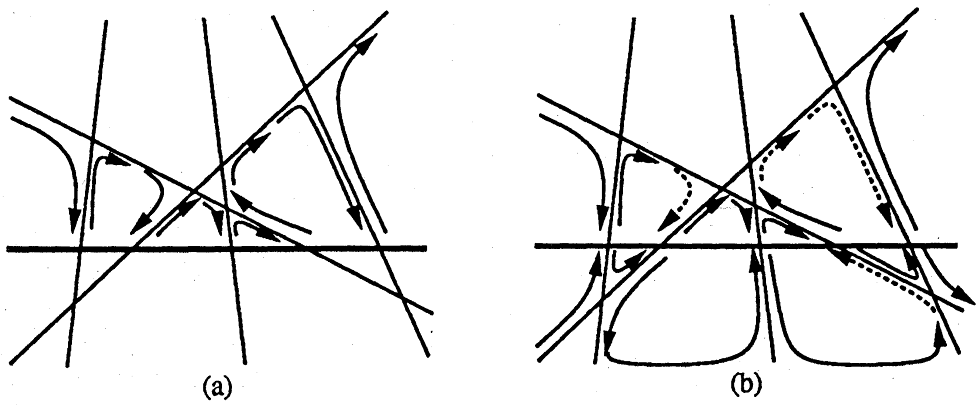

Figure 4.1. (a) A simple incremental algorithm and (b) an incremental algorithm which chooses shorter paths

An incremental algorithm for constructing the arrangement of N lines $l_1, l_2, \ldots, l_N$ works roughly as follows: at the first stage, construct a trivial arrangement of one line l_1; at the ith stage ($i = 2, 3, \ldots, N$), add line l_i to the arrangement of lines $l_1, \ldots, l_{i-1}$, which has been computed already, to obtain the arrangement of lines $l_1, \ldots, l_{i-1}, l_i$. Here, the arrangement is represented by a standard data structure for planar subdivisions.

The main step here is to add l_i to the arrangement A_{i-1} of $l_1, \ldots, l_{i-1}$. To do this, we find an edge e of the arrangement A_{i-1} that is just above l_i at $x = -\infty$, and a cell c intersecting l_i at $x = -\infty$. This can be done in linear time by finding, from among the lines $l_1, \ldots, l_{i-1}$, the line of largest slope less than that of l_i. We then traverse A_{i-1} along l_i by following edges of the cell c in clockwise order, starting with e, to find a new intersection point of l_i with an edge e' of the cell. We iterate for $e := e'$ and $c :=$cell adjacent to c at e' until a cell is found intersecting l_i at $x = +\infty$. See Figure 4.1(a).

The time complexity of adding l_i to A_{i-1} is proportional to the number of edges of cells in A_{i-1} intersecting l_i (these cells form a zone of l_i, and this number is the complexity of the zone). The well-known zone theorem for lines (e.g., [3]) states that the complexity of this zone is $O(i)$. Hence, it takes $O(i)$ time to insert l_i to A_{i-1} to construct A_i, and in total the arrangement of N lines can be constructed in $O(N^2)$ time. Note that this time complexity is worst-case optimal, since the size of a simple arrangement is $\Theta(N^2)$.

In the above algorithm, of the cells intersecting l_i, only the portion above l_i is traversed. Instead of this, we may traverse edges of the upper and lower parts of a cell intersecting l_i one by one simultaneously so that a new intersection point of l_i with the cell may be found in time proportional to the length of the shorter of the two paths (upper and lower) from the old intersection point to the new one. See Figure 4.1(b). This way of traversing cells is sometimes used in other geometric algorithms.

Now, let $e(l_j)$ be the number of edges traversed in adding l_j to the arrangement of lines $\{l_1, \dots, l_i\} - \{l_j\}$ with choosing shorter paths as above $(j = 1, \dots, i)$. Consider the dual graph of the arrangement of $l_1, \dots, l_i$ as a planar graph. This dual graph has at most i^2 edges, and does not have any cycle of length 3. Hence, applying the Theorem, it is seen that

$$\sum_{j=1}^{i} e(l_j) \leq 2 \cdot 4i^2.$$

By randomizing the order of insertion of lines in this modified incremental algorithm, the number of edges traversed in adding the ith line is at most $8i$ on the average. This implies that in total this algorithm constructs the arrangement of N lines in $O(N^2)$ average time.

This idea can be carried over for the case of the arrangement of curves, for which we need some of the techniques developed in [4, 6], say the vertical decomposition of the arrangement. Besides these applications and that of [8], Theorem could be useful in the analysis of other geometric and graph problems.

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