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<th>A Short Course on $b$-Functions and Vanishing Cycles (Algebraic Geometry and Hodge Theory)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1992), 803: 48-64</td>
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<tr>
<td>Issue Date</td>
<td>1992-08</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/82897">http://hdl.handle.net/2433/82897</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
A Short Course on $b$-Functions and Vanishing Cycles

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§0. Introduction.

In this article, we use the notation appearing in [H] freely, and a $\mathcal{D}$-module means a left $\mathcal{D}$-module. Let $X$ be a complex manifold, $f$ a holomorphic function on $X$, and $\mathcal{M}$ a regular holonomic system on $X$. By Riemann-Hilbert (RH) correspondence, $\text{DR}(\mathcal{M})$ is a perverse sheaf. Hence its nearby cycle $^p\psi_f(\text{DR}(\mathcal{M}))$ and vanishing cycle $^p\phi_f(\text{DR}(\mathcal{M}))$ are perverse sheaves on $f^{-1}(0)$. If $f^{-1}(0)$ is a smooth hypersurface, again by RH correspondence there should be holonomic $\mathcal{D}_{f^{-1}(0)}$-modules $\mathcal{M}'$ and $\mathcal{M}''$ such that $^p\psi_f(\text{DR}(\mathcal{M})) = \text{DR}(\mathcal{M}')$ and $^p\phi_f(\text{DR}(\mathcal{M})) = \text{DR}(\mathcal{M}'')$. Malgrange [Ma] and Kashiwara [Kv] have given such $\mathcal{M}'$ and $\mathcal{M}''$ by using the notion of $V$-filtration. When $f^{-1}(0)$ is not smooth, the situation is reduced to the smooth case by the graph map of $f$. There are already excellent surveys [MS], [S] of this topic. This article may be considered as a very short version of [MS] or [S]. Although most proofs of assertions are omitted, those of Proposition 4.2 and 4.4 are exposed in order to convince readers that morphisms $T$, $\text{can}$, and $\text{var}$ correspond to the counterparts mentioned there.
In §1 we define $b$-functions and look at some examples. In §2 we define $V$-filtrations, which can be calculated by $b$-functions. We also look at some examples again. In §3 we state the stability under standard operations of the category of coherent $cld$-modules which admit the canonical $V$-filtrations. In §4 we define moderate nearby cycles and moderate vanishing cycles, which turn out to be quasi-isomorphic to certain graded pieces of the canonical $V$-filtration. In §5 we recall nearby cycles and vanishing cycles, and state the main theorem (Theorem 5.1).

§1. $b$-Functions.

Let $X$ be a complex manifold and $f$ a holomorphic function on it. We set $\mathcal{D}_X[s] := \mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[s]$ where $s$ is an indeterminate central element. Let $\mathcal{I}_f$ denote the left ideal of $\mathcal{D}_X[s]$ consisting of all operators $P(s, x, D)$ in $\mathcal{D}_X[s]$ such that $P(s, x, D)f(x)^s = 0$ holds for a generic $x$. A $\mathcal{D}_X[s]$-module $\mathcal{N}_f := \mathcal{D}_X[s]/\mathcal{I}_f$ has a $\mathcal{D}_X$-linear endomorphism $t$ defined by $P(s)f^s \mapsto P(s+1)f^{s+1}$. Since we have $[t, s] = t$, $\mathcal{M}_f := \mathcal{N}_f/t\mathcal{N}_f$ is a $\mathcal{D}_X[s]$-module.

**Definition 1.1 [SSM], [Be]:** The minimal polynomial $b(s)$ of the multiplication by $s$ on $\mathcal{M}_f$ is said to be the $b$-function of $f$.

**Theorem 1.2 [Be], [Bj], [Kb].** The $\mathcal{D}_X$-module $\mathcal{M}_f$ is holonomic and the
The $b$-function of $f$ locally exists.

**Example 1.3** [Mi], [Y]: Let $X = \mathbb{C}^n$, $x_1, \ldots, x_n$ a coordinate system on $X$ and $D_i = \frac{\partial}{\partial x_i}$ (1 ≤ $i$ ≤ $n$). We assume $f$ to have an isolated singularity at the origin and $f(0) = 0$. We suppose that there exist $v = \sum_{i=1}^{n} \frac{r_i}{r} x_i D_i, r \in \mathbb{Z}_{>0}, r_1, \ldots, r_n \in \mathbb{Z}_{\geq 0}$ such that $v(f) = f$. The $b$-function of $f$ at a point where $df$ does not vanish is $s + 1$. Hence $s + 1$ is also a factor of the $b$-function $b(s)$ of $f$ at the origin. Since $vf^* = sf^*$, $\mathcal{M}_f$ is a singly generated $\mathcal{O}_X$-module. Let $\tilde{M}_f = (s + 1)\mathcal{M}_f$ and $\tilde{b}(s)$ denote the minimal polynomial of $s$ on $\tilde{M}_f$. Then we see that $b(s) = (s + 1)\tilde{b}(s)$ and $\mathcal{M}_f = D_X/D_X f_1 + \cdots + D_X f_n$ where $f_i = D_i(f)$. Let $v^*$ be the adjoint operator of $v$, i.e., $v^* = -\sum_{i=1}^{n} \frac{r_i}{r} (x_i D_i + 1)$. Then we see $\tilde{b}(s)$ is the minimal polynomial of $s$ on $\tilde{M}_f$, the minimal polynomial of $v$ on $\tilde{M}_f$, and the minimal polynomial of $v^*$ on $\mathcal{O}_X/(f_1, \ldots, f_n)$. For a monomial $x^\alpha$ where $\alpha$ is a multi-index, we have $v^*(x^\alpha) = -\sum_{i=1}^{n} \frac{r_i}{r} (\alpha_i + 1) x^\alpha$. We define a set $R$ by $R = \{ \sum_{i=1}^{n} \frac{r_i}{r} (\alpha_i + 1) | \{x^\alpha\}_\alpha \text{ is a basis for } \mathcal{O}_X/(f_1, \ldots, f_n) \}$. Then we obtain $b(s) = (s + 1)\prod_{\beta \in R} (s + \beta)$.

**Example 1.4:** Let $X = \mathbb{C}^n$ and $f = x_1^{e_1} \cdots x_n^{e_n}$ where $e_i \in \mathbb{Z}_{\geq 0}$ (1 ≤ $i$ ≤ $n$). It is easy to check $D_1^{e_1} \cdots D_n^{e_n} f^{s+1} = \prod_{i=1}^{n} \prod_{k=1}^{e_i} (e_i s + k) f^s$. On the other hand we suppose that there exist an operator $P(s) \in D_X[s]$ and a nonzero
polynomial $b'(s) \in \mathbb{C}[s]$ such that $P(s)f^{s+1} = b'(s)f^s$. By the relative invariance under the action of $(\mathbb{C}^\times)^n$, it is easy to see that there exists $Q(s) \in \mathbb{C}[x_1D_1, \ldots, x_nD_n, s]$ such that $P(s) = Q(s)D_1^{e_1} \cdots D_n^{e_n}$. Therefore we see that the $b$-function of $f$ at the origin is $\prod_{i=1}^{n} \prod_{k=1}^{e_i}(s + \frac{k}{e_i})$.

There are many other examples of $b$-functions which can be calculated. See [Y], for instance, and [SKKO] for $b$-functions of relative invariants of prehomogeneous spaces. More generally Kashiwara has proved in [K2] that for a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ and a section $u \in \mathcal{M}$ there exists locally an operator $P(s) \in \mathcal{D}_X[s]$ and a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $P(s)f^{s+1}u = b(s)f^su$. As an application, the holonomicity of $\mathcal{H}_{[X|J^{-1}(0)]}(\mathcal{M})$ has been proved there.

§2. $V$-Filtration.

First of all we introduce the lexicographical order in $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\sqrt{-1}$. Let $Y$ be a smooth closed submanifold of $X$ of codimension one, $\mathcal{I}_Y$ the defining ideal of $Y$. For $k \in \mathbb{Z}$ we define

$$V_k \mathcal{D}_X := \{ P \in \mathcal{D}_X \mid PT^j_y \subseteq \mathcal{I}_y^{-k} \quad (\forall j \in \mathbb{Z}) \}$$

where $T^j_y = \mathcal{O}_X$ for $j \leq 0$. Then $\{ V_k \mathcal{D}_X \}_{k \in \mathbb{Z}}$ is an exhaustive increasing
filtration. Let $t$ be a local equation of $Y$ and $D_t$ a local vector field such that $[D_t, t] = 1$. We have $t \in V_1 D_X$, $D_t \in V_1 D_X$, $\text{gr}_0^Y D_X := V_0 D_X / V_1 D_X = D_Y [tD_t]$ and $V_k D_X = \{ \sum_{k \geq j+1} a_{ij}(y, D_y) t^i D^j_t \}$.

**Definition 2.1 [Kv], [Ma]:** Let $\mathcal{M}$ be a coherent $D_X$-module. An increasing filtration $\{ V_{\alpha} \mathcal{M} \}_{\alpha \in \mathbb{C}}$ satisfying the following conditions is called the canonical $V$-filtration.

1. $\mathcal{M} = \bigcup_{\alpha \in \mathbb{C}} V_{\alpha} \mathcal{M}$. Each $V_{\alpha} \mathcal{M}$ is a coherent $V_0 D_X$-submodule.

2. $(V_1 D_X)(V_{\alpha} \mathcal{M}) \subset V_{\alpha+i} \mathcal{M}$ ($\forall \alpha \in \mathbb{C}, \forall i \in \mathbb{Z}$).

3. $t(V_{\alpha} \mathcal{M}) = V_{\alpha-1} \mathcal{M}$ ($\forall \alpha < 0$).

4. The action of $(tD_t + 1 + \alpha)$ on $\text{gr}_\alpha^Y \mathcal{M}$ ($\forall \alpha \in \mathbb{C}$) is nilpotent where $\text{gr}_\alpha^Y \mathcal{M} = V_{\alpha} \mathcal{M} / V_{< \alpha} \mathcal{M}$ and $V_{< \alpha} \mathcal{M} = \bigcup_{\beta < \alpha} V_{\beta} \mathcal{M}$.

**Remarks 2.2:**

1. The definition of the canonical $V$-filtration does not depend on the choice of $t$ and $D_t$. The canonical $V$-filtration is unique if it exists.

2. Since the adjoint of $(tD_t + 1 + \alpha)$ is $-(tD_t - \alpha)$, the eigenvalue of $tD_t$ on $\text{gr}_\alpha^Y \mathcal{N}$ is $\alpha$ for a right $D_X$-module $\mathcal{N}$.

3. $t : \text{gr}_\alpha^Y \mathcal{M} \to \text{gr}_{\alpha-1}^Y \mathcal{M}$ and $D_t : \text{gr}_{\alpha-1}^Y \mathcal{M} \to \text{gr}_\alpha^Y \mathcal{M}$ are bijective for
\[ \alpha \neq 0. \]

**DEFINITION 2.3:** We say that a coherent \( D_X \)-module \( \mathcal{M} \) is specializable along \( Y \) and we denote \( \mathcal{M} \in B_Y \) if the following equivalent conditions are satisfied:

1. For any system of local generators \( u_1, \ldots, u_l \) of \( \mathcal{M} \) there exists a nonzero polynomial \( b(s) \in C[s] \) such that \( b(tD_t)u_i \in \sum_{j=1}^{l} (V_{-1}D_X)u_j \) \( (1 \leq \forall i \leq l) \).
2. \( \mathcal{M} \) admits the canonical \( V \)-filtration with respect to \( Y \) and there exists a finite set \( A \subset C \) such that \( \{ \alpha \in C | \text{gr}_{\alpha}^V \mathcal{M} \neq 0 \} \subset A + Z \).

Let \( \mathcal{M} \in B_Y \) and \( u \in \mathcal{M} \). Then there exists a nonzero polynomial \( b(s) \in C[s] \) such that \( b(tD_t)u \in (V_{-1}D_X)u \). The minimal polynomial among such is called the \( b \)-function of the section \( u \). The canonical \( V \)-filtration of \( \mathcal{M} \) is known to be given by \( V_{\alpha}\mathcal{M} = \{ u \in \mathcal{M} | \text{ all roots of the } b \text{-function of } u \text{ are greater than or equal to } -\alpha - 1 \} \).

**PROPOSITION 2.4.** Let \( 0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0 \) be an exact sequence of coherent \( D_X \)-modules. Then we have

1. \( \mathcal{M} \in B_Y \Leftrightarrow \mathcal{M}', \mathcal{M}'' \in B_Y \).
2. The induced sequence \( 0 \rightarrow V_{\alpha}\mathcal{M}' \rightarrow V_{\alpha}\mathcal{M} \rightarrow V_{\alpha}\mathcal{M}'' \rightarrow 0 \) is exact for
\( \forall \alpha \in C \text{ if } \mathcal{M} \in B_Y. \)

(3) The induced sequence \( 0 \to \text{gr}^Y_{\alpha} \mathcal{M}' \to \text{gr}^Y_{\alpha} \mathcal{M} \to \text{gr}^Y_{\alpha} \mathcal{M}'' \to 0 \) is exact for \( \forall \alpha \in C \text{ if } \mathcal{M} \in B_Y. \)

**Remark 2.5:** Let \( \mathcal{M} \in B_Y. \) Then \( \text{gr}^Y_{\alpha} \mathcal{M} \) is a coherent \( \text{gr}^Y_0 \mathcal{D}_X = \mathcal{D}_Y[t \mathcal{D}_t] \)-module for any \( \alpha \in C. \) Since the action of \( (t \mathcal{D}_t + 1 + \alpha) \) is nilpotent on \( \text{gr}^Y_{\alpha} \mathcal{M}, \) it is a coherent \( \mathcal{D}_Y \)-module.

**Example 2.6:** Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X \)-module with \( \text{Supp}(\mathcal{M}) \subset Y, \) and \( u \in \mathcal{M}. \) Then there exists \( i \in \mathbb{Z}_{>0} \) such that \( t^i u = 0. \) So we have \( \prod_{k=1}^{i}(t \mathcal{D}_t + k)u = D_i t^i u = 0. \) Hence we obtain \( \mathcal{M} = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathcal{M}_i \) where \( \mathcal{M}_i = \{ u \in \mathcal{M} \mid (t \mathcal{D}_t + 1 + i)u = 0 \}, \) and \( V_{\alpha} \mathcal{M} = \bigoplus_{i \leq \alpha} \mathcal{M}_i. \)

**Example 2.7:** Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X \)-module. We assume \( Y \) to be non-characteristic for \( \mathcal{M}, \) i.e., \( \text{Ch}(\mathcal{M}) \cap T_{Y}^*X \subset T_{X}^*X. \) Then \( \mathcal{M} \in B_Y. \) The proof could be reduced to the case of \( \mathcal{D}_X v = \mathcal{D}_X/\mathcal{D}_X P \) with \( P \in V_N \mathcal{D}_X, \) \( \bar{P} = \bar{D}_t^N \in V_N \mathcal{D}_X/V_{N-1} \mathcal{D}_X \) and \( N \in \mathbb{Z}_{>0} \) where the bar indicates the canonical image. Since \( P v = 0, \) in \( (V_N \mathcal{D}_X)v/(V_{N-1} \mathcal{D}_X)v \) we have \( \bar{P} \bar{v} = \bar{D}_t^N \bar{v} = 0, \) i.e., \( D_t^N v \in V_{N-1} \mathcal{D}_X v. \) Hence we obtain \( \prod_{k=0}^{N-1}(t \mathcal{D}_t - k)v = t^N D_t^N v \in V_{-1} \mathcal{D}_X v. \) In general, any root of the \( b \)-function of any section of \( \mathcal{M} \) is a nonnegative
integer. Therefore we see by Definition 2.1 (3)

\[ V_{\alpha}M = \begin{cases} 
   t^{-[\alpha]-1}M & \alpha < -1 \\
   M & \alpha \geq -1
\end{cases} \]

where \([\alpha] = \max\{ n \in \mathbb{Z} | n \leq \alpha \} \).

Let \( f \) be a holomorphic function, \( M \) a holonomic \( \mathcal{D}_{X} \)-module, \( u \in \mathcal{M} \) and \( Y = f^{-1}(0) \). Let \( i_{f} \) denote the graph of \( f : X \rightarrow X \times \mathbb{C} \) and \( t \) the coordinate of \( \mathbb{C} \) in \( X \times \mathbb{C} \). Then we see \( \mathcal{M} \in B_{Y} \Leftrightarrow i_{f*}\mathcal{M} \in B_{X \times \{0\}} \).

Furthermore there is the following correspondence under the isomorphism of \( \mathcal{D}_{X}[s,t] \) onto \( V_{0}(\mathcal{D}_{X}[t,D_{t}]) = \mathcal{D}_{X}[t,tD_{t}] \):

\[ s \leftrightarrow -D_{t}t \]

\[ \mathcal{D}_{X}[s]f^{s}u \leftrightarrow V_{0}(\mathcal{D}_{X}[t,D_{t}])u \otimes \delta(t - f) \]

\[ P(s)f^{s+1}u = b(s)f^{s}u \leftrightarrow P(-D_{t}t)tu \otimes \delta(t - f) = b(-D_{t}t)u \otimes \delta(t - f). \]

By Kashiwara's result recalled in §1, we obtain:

**Proposition 2.8.** All holonomic \( \mathcal{D}_{X} \)-modules belong to \( B_{Y} \).

**§3. Operations in \( B_{Y} \).**

**Proposition 3.1.** Let \( Y = \{ t = 0 \} \) and \( \mathcal{M} \in B_{Y} \). Then
(1) $\mathcal{M}[t^{-1}] \in B_Y$.

(2) $\mathcal{H}^i(\mathcal{M}^*) \in B_Y$ for all $i$. Moreover for all locally we have isomorphisms

\[ \text{gr}_\alpha^Y(\mathcal{H}^i(\mathcal{M}^*)) \sim \mathcal{H}^i(\text{gr}_{\alpha-1}^Y(\mathcal{M})^*) \quad (-1 < \alpha < 0) \text{ and } \text{gr}_\beta^Y(\mathcal{H}^i(\mathcal{M}^*)) \sim \mathcal{H}^i(\text{gr}_{\beta}^Y(\mathcal{M})^*) \quad (\beta = -1, 0). \]

Under these isomorphisms the transpose $t: \mathcal{H}^i(\text{gr}_{-1}^Y(\mathcal{M})^*) \to \mathcal{H}^i(\text{gr}_{0}^Y(\mathcal{M})^*)$ corresponds to $-D_t: \text{gr}_{-1}^Y(\mathcal{H}^i(\mathcal{M}^*)) \to \text{gr}_{0}^Y(\mathcal{H}^i(\mathcal{M}^*))$ and the transpose $D_t: \mathcal{H}^i(\text{gr}_{0}^Y(\mathcal{M})^*) \to \mathcal{H}^i(\text{gr}_{-1}^Y(\mathcal{M})^*)$ corresponds to $t: \text{gr}_{0}^Y(\mathcal{H}^i(\mathcal{M}^*)) \to \text{gr}_{-1}^Y(\mathcal{H}^i(\mathcal{M}^*))$.

**Proposition 3.2.** Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and $i$ the inclusion of $Y$ into $X$. Then

(1) The restriction $i^*\mathcal{M}$ is quasi-isomorphic to $0 \to \text{gr}_{-1}^Y\mathcal{M} \overset{D_t}{\to} \text{gr}_{-1}^Y\mathcal{M} \to 0$ where the dot indicates the place of degree zero.

(2) For any $\alpha \in \mathbb{C}$, $\text{gr}_\alpha^Y\mathcal{M}$ is a holonomic $\mathcal{D}_Y$-module.

**Proof:** (1) By Remark 2.2 (3) and Proposition 3.1 (2) we have

\[
i^i\mathcal{M}^* \xrightarrow{\sim} (0 \to \text{gr}_{0}^Y(\mathcal{M}^*) \overset{t}{\to} \text{gr}_{-1}^Y(\mathcal{M}^*) \to 0)
\]

\[
\sim \xrightarrow{\text{qis}} (0 \to (\text{gr}_{0}^Y\mathcal{M})^* \overset{D_t}{\to} (\text{gr}_{-1}^Y\mathcal{M})^* \to 0).
\]

Since $i^*\mathcal{M} = (i^i\mathcal{M}^*)^*$, we obtain the assertion.
(2) We know that

\[ \mathcal{M} : \text{holonomic} \Leftrightarrow \mathcal{H}^j(\mathcal{M}^*) = 0 \] 

for \( j \neq 0 \). Hence by Proposition 3.1 (2) we obtain \( \mathcal{H}^j((\text{gr}^\alpha \mathcal{M})^*) = 0 \) for \( j \neq 0 \). This means the holonomicity of \( \text{gr}^\alpha \mathcal{M} \).

**Proposition 3.3.** Let \( g : X' \to X \) be a proper morphism of smooth manifolds. We suppose that \( Y' := g^{-1}(Y) \) is a smooth hypersurface and \( \mathcal{M} \in B_Y \), has a global good filtration. Then for any \( j \), we have \( \mathcal{H}^j(\mathbb{R}g_* \mathcal{M}) \in B_Y \) and the canonical \( V \)-filtration of \( \mathcal{M} \) induces the one for \( \mathcal{H}^j(\mathbb{R}g_* \mathcal{M}) \).

§4. Moderate Nearby Cycles and Moderate Vanishing Cycles.

Let \( Y \) be a smooth hypersurface defined by \( t : X \to \mathbb{C} \). For a coherent \( \mathcal{D}_X \)-module \( \mathcal{M} \in B_Y \), \( p \in \mathbb{Z}_{\geq 0} \) and \( -1 \leq \alpha < 0 \), we define

\[ \mathcal{M}_{\alpha,p} := \bigoplus_{0 \leq k \leq p} \mathcal{M}[t^{-1}] \otimes e_{\alpha,k} \]

where \( e_{\alpha,k} = t^{\alpha+1} (\text{Log } t)^k / k! \). It is clear that for any \( \beta \in \mathbb{C} \)

\[ V_\beta \mathcal{M}_{\alpha,p} = \bigoplus_{0 \leq k \leq p} V_{\beta+\alpha+1}(\mathcal{M}[t^{-1}]) \otimes e_{\alpha,k}. \]
Then the monodromy $T = \exp(2\pi itD_t)$ induces a $D_Y$-automorphism on $\mathcal{M}_{\alpha,p}$ by $T(m \otimes e_{\alpha,k}) = m \otimes T(e_{\alpha,k})$, and accordingly on $gr_{\beta}^{V}(\mathcal{M}_{\alpha,p})$.

**Definition 4.1:** For $-1 \leq \alpha \leq 0$, we define the moderate nearby cycle $\psi_{t,\alpha}^{m}(\mathcal{M})$ by

$$\psi_{t,\alpha}^{m}(\mathcal{M}) := \lim_{p} \psi_{t,\alpha,p}^{m}(\mathcal{M})$$

where $\psi_{t,\alpha,p}^{m}(\mathcal{M}) := i^*(\mathcal{M}_{\alpha,p})[-1]$.

By Proposition 3.2 we see

$$\psi_{t,\alpha,p}^{m}(\mathcal{M}) \xrightarrow{\text{qis}} (0 \to gr_{-1}^{V}(\mathcal{M}_{\alpha,p}) \xrightarrow{D_t} gr_{0}^{V}(\mathcal{M}_{\alpha,p}) \to 0).$$

We remark that $T$ acts on $\psi_{t,\alpha}^{m}(\mathcal{M})$ as well.

**Proposition 4.2.** For $\mathcal{M} \in B_Y$ and $-1 \leq \alpha < 0$, we have a quasi-iso-

morphism $gr_{\alpha}^{V}(\mathcal{M}) \xrightarrow{\text{qis}} \psi_{t,\alpha}^{m}(\mathcal{M})$. Here the action of $T$ on $\psi_{t,\alpha}^{m}(\mathcal{M})$ corre-

sponds to that of $\exp(-2\pi itD_t)$.

**Proof:** Since $V_{<0}\mathcal{M} = V_{<0}(\mathcal{M}[t^{-1}])$, we have

$$gr_{-1}^{V}(\mathcal{M}_{\alpha,p}) = \bigoplus_{0 \leq k \leq p} gr_{\alpha}^{V}(\mathcal{M}[t^{-1}]) \otimes e_{\alpha,k} = \bigoplus_{0 \leq k \leq p} gr_{\alpha}^{V}(\mathcal{M}) \otimes e_{\alpha,k}.$$ 

As $\mathcal{M}_{\alpha,p} = \mathcal{M}_{\alpha,p}[t^{-1}]$, we know $\mathcal{H}^{0}(\psi_{t,\alpha,p}^{m}(\mathcal{M})) = \text{Ker}(D_t) = \text{Ker} (tD_t : gr_{-1}^{V}(\mathcal{M}_{\alpha,p}) \to gr_{-1}^{V}(\mathcal{M}_{\alpha,p}))$. Since $tD_t(m \otimes e_{\alpha,k}) = [(tD_t + \alpha + 1)m] \otimes$
\(e_{\alpha,k} + m \otimes e_{\alpha,k-1}\), we see \(\sum_{k=0}^{p} m_{k} \otimes e_{\alpha,k} \in \text{Ker}(tD_{t}) = \mathcal{H}^{0}(\psi_{t,\alpha,p}^{m}(\mathcal{M})) \iff (tD_{t} + \alpha + 1)m_{k} + m_{k+1} = 0 (0 \leq \forall k \leq p-1) \iff m_{k} = \ldots \). Hence for \(p\) such that \((tD_{t} + \alpha + 1)^{p} = 0\) on \(\text{gr}_{\alpha}^{V}(\mathcal{M})\), the morphism \(\text{gr}_{\alpha}^{V}(\mathcal{M}) \ni m_{0} \mapsto \sum_{k=0}^{p}[-(tD_{t} + \alpha + 1)]^{k}m_{0} \otimes e_{\alpha,k} \in \mathcal{H}^{0}(\psi_{t,\alpha,p}^{m}(\mathcal{M}))\) is isomorphic.

Let \(x = \sum_{k=0}^{p}[-(tD_{t} + \alpha + 1)]^{k}m_{0} \otimes e_{\alpha,k} \in \text{Ker}(tD_{t})\). Then we have

\[
0 = (tD_{t})x = \sum_{k=0}^{p}[-(tD_{t} + \alpha + 1)]^{k}(tD_{t})m_{0} \otimes e_{\alpha,k} + \sum_{k=0}^{p}[-(tD_{t} + \alpha + 1)]^{k}m_{0} \otimes (tD_{t})e_{\alpha,k},
\]

and thus \(\sum_{k=0}^{p}[-(tD_{t} + \alpha + 1)]^{k}m_{0} \otimes (2\pi itD_{t})e_{\alpha,k} = \sum_{k=0}^{p}[-(tD_{t} + \alpha + 1)]^{k}((-2\pi itD_{t})m_{0}) \otimes e_{\alpha,k}\). Hence the monodromy \(T\) corresponds to \(\exp(-2\pi itD_{t})\).

Since \(t\) induces an isomorphism \(\text{gr}_{0}^{V}(\mathcal{M}_{\alpha,p}) \simeq \text{gr}_{-1}^{V}(\mathcal{M}_{\alpha,p})\), we see \(\mathcal{H}^{1}(\psi_{t,\alpha,p}^{m}(\mathcal{M})) = \text{Coker}(D_{t}) = \text{Coker}(D_{t} : \text{gr}_{0}^{V}(\mathcal{M}_{\alpha,p}) \rightarrow \text{gr}_{0}^{V}(\mathcal{M}_{\alpha,p}))\). For \(\sum_{k=0}^{p} m_{k} \otimes e_{\alpha,k} \in \bigoplus_{0 \leq k \leq p} \text{gr}_{\alpha+1}^{V}(\mathcal{M}[t^{-1}]) \otimes e_{\alpha,k} = \text{gr}_{0}^{V}(\mathcal{M}_{\alpha,p})\), we have

\[
D_{t}t(\sum_{k=0}^{p} m_{k} \otimes e_{\alpha,k}) = \sum_{k=0}^{p}((D_{t}t + \alpha + 1)m_{k} \otimes e_{\alpha,k} + m_{k} \otimes e_{\alpha,k-1}) = \sum_{k=0}^{p} m'_{k} \otimes e_{\alpha,k}\]

where \(m'_{k} = (D_{t}t + \alpha + 1)m_{k} + m_{k+1}\). Hence for \(l\) such that \((D_{t}t + \alpha + 1)^{l} = 0\) on \(\text{gr}_{\alpha+1}^{V}(\mathcal{M}[t^{-1}])\), we have \(m \otimes e_{\alpha,k} = D_{t}t(\sum_{i=1}^{l}[-(D_{t}t + \alpha + 1)]^{i-1}m \otimes e_{\alpha,k+i})\) and thus \(\mathcal{H}^{1}(\psi_{t,\alpha}^{m}(\mathcal{M})) = 0\).

**Definition 4.3**: We define the moderate vanishing cycle \(\phi_{t,0}^{m}((\mathcal{M}))\) to be the inductive limit of the mapping cone \(\phi_{t,0,p}^{m}((\mathcal{M}))\) of the natural morphism...
$i^*\mathcal{M}[-1] \rightarrow i^*\mathcal{M}_{-1,p}[-1] = \psi_{t,-1,p}^m(\mathcal{M})$, i.e.,

$$
\phi_{t,0,p}^m = (0 \rightarrow gr_{-1}^V \mathcal{M} \xrightarrow{j\oplus -D_t} gr_{-1}^V \mathcal{M}_{-1,p} \oplus gr_0^V \mathcal{M} \xrightarrow{D_t+j} gr_0^V \mathcal{M}_{-1,p} \rightarrow 0)
$$

where \( j \) is the natural morphism \( \mathcal{M} \rightarrow \mathcal{M}_{-1,p} = \bigoplus_{0 \leq k \leq p} \mathcal{M}[t^{-1}] \otimes e_{-1,k} \).

We define morphisms \( \text{can} : \psi_{t,-1}^m(\mathcal{M}) \rightarrow \phi_{t,0}^m(\mathcal{M}) \) and \( \text{var} : \phi_{t,0}^m(\mathcal{M}) \rightarrow \psi_{t,-1}^m(\mathcal{M}) \) by the morphisms \( \text{id} : \psi_{t,-1,p}^m(\mathcal{M}) \rightarrow \phi_{t,0,p}^m(\mathcal{M}) \) and \( T - \text{id} : \phi_{t,0,p}^m(\mathcal{M}) \rightarrow \psi_{t,-1,p}^m(\mathcal{M}) \) respectively.

**Proposition 4.4.** For \( \mathcal{M} \in \mathcal{B}_Y \), we have a quasi-isomorphism \( gr_0^V \mathcal{M} \xrightarrow{\sim} \phi_{t,0}^m(\mathcal{M}) \). Moreover can corresponds to \( D_t : gr_{-1}^V \mathcal{M} \rightarrow gr_0^V \mathcal{M} \) and var to \( [(\exp(-2\pi itD_t)-1)]t : gr_0^V \mathcal{M} \rightarrow gr_{-1}^V \mathcal{M} \).

**Proof:** Let \( x = \sum_{k=0}^{p} m_k \otimes e_{-1,k} + n_0 \in gr_{-1}^V \mathcal{M}_{-1,p} \oplus gr_0^V \mathcal{M} = (\bigoplus_{k=0}^{p} gr_{-1}^V \mathcal{M} \otimes e_{-1,k}) \oplus gr_0^V \mathcal{M} \). Then we can check

\[
x \in \text{Ker}(D_t + j) \iff \left\{ \begin{array}{l}
m_1 = -tD_tm_0 - tn_0 \\
m_{k+1} = -tD_tm_k \quad (k \geq 1).
\end{array} \right.
\]

Hence we obtain an isomorphism \( gr_{-1}^V \mathcal{M} \oplus gr_0^V \mathcal{M} \xrightarrow{\sim} \text{Ker}(D_t + j) \) defined by \( m_0 + n_0 \mapsto \sum m_k \otimes e_{-1,k} + n_0 \) with \((*) \). So we see \( gr_0^V \mathcal{M} \xrightarrow{\sim} \mathcal{H}^0(\phi_{t,0}^m) \). Since \( m \equiv D_tm \mod \text{Im}(j \oplus -D_t) \) for \( m \in gr_{-1}^V \mathcal{M} \), the morphism \( \text{can} \) corresponds to \( D_t : gr_{-1}^V \mathcal{M} \rightarrow gr_0^V \mathcal{M} \). The element \( \sum_{k \geq 1} (-tD_t)^{k-1} (-tn) \otimes e_{-1,k} + \)
$n \in \text{Ker}(D_t + j)$ corresponds to $n \in \text{gr}_0^V \mathcal{M}$. Since the coefficient of $(T - id)(\sum_{k \geq 1}(-tD_t)^{k-1}(-tn) \otimes e_{-1,k})$ at $e_{-1,0}$ is $\sum_{k \geq 1}(2\pi i)^k \frac{(-tD_t)^{k-1}}{k!}(-tn)$, the morphism $\text{var}$ corresponds to $\left[\frac{\exp(-2\pi itD_t)-1}{itD_t}\right]t : \text{gr}_0^V \mathcal{M} \rightarrow \text{gr}_{-1}^V \mathcal{M}$.

§5. Nearby Cycles and Vanishing Cycles.

Let $f$ be a nonconstant holomorphic function on $X$, $i$ the inclusion of $f^{-1}(0)$ into $X$ and $K \in D_c^b(C_X)$. Let $\tilde{C}^x$ denote the universal covering of $\mathbb{C}^x$ and $p$ the natural map $\tilde{X}^x := X \times \tilde{C}^x \rightarrow X$. Following [SGA7] we define the nearby cycle $\psi_f(K)$ by $\psi_f(K) := i^{-1}\mathcal{R}p_*p^{-1}K \in D_c^b(C_{J^{-1}(0)})$. The natural morphism $K \rightarrow \mathcal{R}p_*p^{-1}K$ induces a morphism $i^{-1}K \rightarrow \psi_f(K)$, whose mapping cone $\phi_f(K) \in D_c^b(C_{J^{-1}(0)})$ is called the vanishing cycle. By the definition of $\phi_f(K)$ we have the canonical morphism $\text{can} : \psi_f(K) \rightarrow \phi_f(K)$. Associated to the canonical generator of $\pi_1(\mathbb{C}^x)$ the monodromy automorphism $T$ acts on $\psi_f(K)$ and $\phi_f(K)$. Since $(T - id)|_{J^{-1}(0)} = 0$, $T - id$ induces the variation $\text{var} : \phi_f(K) \rightarrow \psi_f(K)$. For $\lambda \in \mathbb{C}^x$ we define a subcomplex $\psi_{f,\lambda}(K)$ of $\psi_f(K)$ by

$$\psi_{f,\lambda}(K) := \{ x \in \psi_f(K) \mid (T - \lambda id)^m x = 0 \ (m >> 0) \}.$$

Since $\psi_f(K) \in D_c^b(C_{f^{-1}(0)})$, we have a quasi-isomorphism $\bigoplus_{\lambda \in \mathbb{C}^x} \psi_{f,\lambda}(K) \xrightarrow{\sim} \psi_f(K)$. Similarly we have $\bigoplus_{\lambda \in \mathbb{C}^x} \phi_{f,\lambda}(K) \xrightarrow{\sim} \phi_f(K)$ as well. For
convenience we set $p\psi_f(K) := \psi_f(K)[-1]$ and $p\phi_f(K) := \phi_f(K)[-1]$.

Let $Y$ be a smooth hypersurface of $X$ defined by $t = 0$. When $\mathcal{M}$ is a regular holonomic $D_X$-module, we have quasi-isomorphisms

$$\text{DR}(\psi_{t, \alpha}^m(\mathcal{M})) \sim_{\text{qis}} p\psi_{t, e^{2\pi i \alpha}}(\text{DR}(\mathcal{M})) \quad (-1 \leq \alpha < 0)$$

$$\text{DR}(\phi_{t, 0}^m(\mathcal{M})) \sim_{\text{qis}} p\phi_{t, 1}(\text{DR}(\mathcal{M}))$$

(see [SGA7]). Hence we obtain:

**Theorem 5.1** [Kv], [Ma]. For a regular holonomic $D_X$-module $\mathcal{M}$, we have

$$\text{DR}(\text{gr}^{\mathcal{V}}_{\alpha} \mathcal{M}) \sim_{\text{qis}} \begin{cases} p\psi_{t, e^{2\pi i \alpha}}(\text{DR}(\mathcal{M})) & (-1 \leq \alpha < 0) \\ p\phi_{t, e^{2\pi i \alpha}}(\text{DR}(\mathcal{M})) & (-1 \leq \alpha \leq 0) \end{cases}$$

Moreover under the above quasi-isomorphisms we have the following correspondences:

$$\exp(-2\pi itD_t) \leftrightarrow T$$

$$D_t : \text{gr}^{\mathcal{V}}_{-1} \mathcal{M} \to \text{gr}^{\mathcal{V}}_{0} \mathcal{M} \leftrightarrow \text{can} : p\psi_{t, 1}(\mathcal{M}) \to p\phi_{t, 1}(\mathcal{M})$$

$$\frac{[\exp(-2\pi itD_t) - 1]}{tD_t} t : \text{gr}^{\mathcal{V}}_{0} \mathcal{M} \to \text{gr}^{\mathcal{V}}_{-1} \mathcal{M} \leftrightarrow \text{var} : p\phi_{t, 1}(\mathcal{M}) \to p\psi_{t, 1}(\mathcal{M}).$$
COROLLARY 5.2. For a regular holonomic $D_X$-module $\mathcal{M}$, we have
\[
p^\ast \psi_t(D_X(DR(\mathcal{M}))) \overset{\sim}{\longrightarrow} D_Y p^\ast \psi_t(DR(\mathcal{M}))
\]
\[
p^\ast \phi_t(D_X(DR(\mathcal{M}))) \overset{\sim}{\longrightarrow} D_Y p^\ast \phi_t(DR(\mathcal{M})).
\]

REFERENCES


