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A Short Course on $b$-Functions and Vanishing Cycles

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§0. Introduction.

In this article, we use the notation appearing in [H] freely, and a $\mathcal{D}$-module means a left $\mathcal{D}$-module. Let $X$ be a complex manifold, $f$ a holomorphic function on $X$, and $\mathcal{M}$ a regular holonomic system on $X$. By Riemann-Hilbert (RH) correspondence, $\text{DR}(\mathcal{M})$ is a perverse sheaf. Hence its nearby cycle $p^*\psi_f(\text{DR}(\mathcal{M}))$ and vanishing cycle $p^*\phi_f(\text{DR}(\mathcal{M}))$ are perverse sheaves on $f^{-1}(0)$. If $f^{-1}(0)$ is a smooth hypersurface, again by RH correspondence there should be holonomic $\mathcal{D}_f^{-1}(0)$-modules $\mathcal{M}'$ and $\mathcal{M}''$ such that $p^*\psi_f(\text{DR}(\mathcal{M})) = \text{DR}(\mathcal{M}')$ and $p^*\phi_f(\text{DR}(\mathcal{M})) = \text{DR}(\mathcal{M}'')$. Malgrange [Ma] and Kashiwara [Kv] have given such $\mathcal{M}'$ and $\mathcal{M}''$ by using the notion of $V$-filtration. When $f^{-1}(0)$ is not smooth, the situation is reduced to the smooth case by the graph map of $f$. There are already excellent surveys [MS], [S] of this topic. This article may be considered as a very short version of [MS] or [S]. Although most proofs of assertions are omitted, those of Proposition 4.2 and 4.4 are exposed in order to convince readers that morphisms $\text{can}$, and $\text{var}$ correspond to the counterparts mentioned there.
In §1 we define $b$-functions and look at some examples. In §2 we define $V$-filtrations, which can be calculated by $b$-functions. We also look at some examples again. In §3 we state the stability under standard operations of the category of coherent $clD$-modules which admit the canonical $V$-filtrations. In §4 we define moderate nearby cycles and moderate vanishing cycles, which turn out to be quasi-isomorphic to certain graded pieces of the canonical $V$-filtration. In §5 we recall nearby cycles and vanishing cycles, and state the main theorem (Theorem 5.1).

§1. $b$-Functions.

Let $X$ be a complex manifold and $f$ a holomorphic function on it. We set $\mathcal{D}_X[s] := \mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[s]$ where $s$ is an indeterminate central element. Let $\mathcal{I}_f$ denote the left ideal of $\mathcal{D}_X[s]$ consisting of all operators $P(s, x, D)$ in $\mathcal{D}_X[s]$ such that $P(s, x, D)f(x)^s = 0$ holds for a generic $x$. A $\mathcal{D}_X[s]$-module $\mathcal{N}_f := \mathcal{D}_X[s]/\mathcal{I}_f$ has a $\mathcal{D}_X$-linear endomorphism $t$ defined by $P(s)f^s \mapsto P(s+1)f^{s+1}$. Since we have $[t, s] = t$, $\mathcal{M}_f := \mathcal{N}_f/t\mathcal{N}_f$ is a $\mathcal{D}_X[s]$-module.

**Definition 1.1 [SSM], [Be]:** The minimal polynomial $b(s)$ of the multiplication by $s$ on $\mathcal{M}_f$ is said to be the $b$-function of $f$.

**Theorem 1.2 [Be], [Bj], [Kb].** The $\mathcal{D}_X$-module $\mathcal{M}_f$ is holonomic and the
**b-function of \( f \) locally exists.**

**Example 1.3** [Mi], [Y]: Let \( X = \mathbb{C}^n, x_1, \ldots, x_n \) a coordinate system on \( X \) and \( D_i = \frac{\partial}{\partial x_i} \) (\( 1 \leq i \leq n \)). We assume \( f \) to have an isolated singularity at the origin and \( f(0) = 0 \). We suppose that there exist \( v = \sum_{i=1}^{n} r_i x_i D_i, r \in \mathbb{Z}_{>0}, r_1, \ldots, r_n \in \mathbb{Z}_{\geq 0} \) such that \( v(f) = f \). The b-function of \( f \) at a point where \( df \) does not vanish is \( s + 1 \). Hence \( s + 1 \) is also a factor of the b-function \( b(s) \) of \( f \) at the origin. Since \( v f^* = sf^* \), \( \mathcal{M}_f \) is a singly generated \( \mathcal{D}_X \)-module. Let \( \tilde{M}_f = (s + 1) \mathcal{M}_f \) and \( \tilde{b}(s) \) denote the minimal polynomial of \( s \) on \( \tilde{M}_f \). Then we see that \( b(s) = (s + 1) \tilde{b}(s) \) and \( \mathcal{M}_f = \mathcal{D}_X/\mathcal{D}_X f_1 + \cdots + \mathcal{D}_X f_n \) where \( f_i = D_i(f) \). Let \( v^* \) be the adjoint operator of \( v \), i.e., \( v^* = - \sum_{i=1}^{n} r_i (x_i D_i + 1) \). Then we see \( \tilde{b}(s) = \) the minimal polynomial of \( s \) on \( \tilde{M}_f \) = the minimal polynomial of \( v \) on \( \tilde{M}_f \) = the minimal polynomial of \( v^* \) on \( \mathcal{O}_X/(f_1, \ldots, f_n) \). For a monomial \( x^\alpha \) where \( \alpha \) is a multi-index, we have \( v^*(x^\alpha) = - \sum_{i=1}^{n} r_i (\alpha_i + 1)x^\alpha \). We define a set \( R \) by \( R = \{ \sum_{i=1}^{n} r_i (\alpha_i + 1) \} \{ x^\alpha \}_\alpha \) is a basis for \( \mathcal{O}_X/(f_1, \ldots, f_n) \}. Then we obtain \( b(s) = (s + 1) \prod_{\beta \in R}(s + \beta) \).

**Example 1.4:** Let \( X = \mathbb{C}^n \) and \( f = x_1^{e_1} \cdots x_n^{e_n} \) where \( e_i \in \mathbb{Z}_{\geq 0} \) (\( 1 \leq i \leq n \)). It is easy to check \( D_1^{e_1} \cdots D_n^{e_n} f^{s+1} = \prod_{i=1}^{n} \prod_{k=1}^{e_i} (e_i s + k)f^s \). On the other hand we suppose that there exist an operator \( P(s) \in \mathcal{D}_X[s] \) and a nonzero
polynomial $b'(s) \in \mathbb{C}[s]$ such that $P(s)f^{s+1} = b'(s)f^s$. By the relative invariance under the action of $(\mathbb{C}^\times)^n$, it is easy to see that there exists $Q(s) \in \mathbb{C}[x_1D_1, \ldots, x_nD_n, s]$ such that $P(s) = Q(s)D_1^{e_1} \cdots D_n^{e_n}$. Therefore we see that the $b$-function of $f$ at the origin is $\prod_{i=1}^{n} \prod_{k=1}^{e_i} (s + \frac{k}{e_i})$.

There are many other examples of $b$-functions which can be calculated. See [Y], for instance, and [SKKO] for $b$-functions of relative invariants of prehomogeneous spaces. More generally Kashiwara has proved in [K2] that for a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ and a section $u \in \mathcal{M}$ there exists locally an operator $P(s) \in \mathcal{D}_X[s]$ and a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $P(s)f^{s+1}u = b(s)f^su$. As an application, the holonomicity of $\mathcal{H}^i_{[X|J^{-1}(0)]}(\mathcal{M})$ has been proved there.

§2. $V$-Filtration.

First of all we introduce the lexicographical order in $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\sqrt{-1}$. Let $Y$ be a smooth closed submanifold of $X$ of codimension one, $I_Y$ the defining ideal of $Y$. For $k \in \mathbb{Z}$ we define

$$V_k\mathcal{D}_X := \{ P \in \mathcal{D}_X \mid PT^i \subset I_Y^{j-k} \quad (\forall j \in \mathbb{Z}) \}$$

where $I_Y^i = \mathcal{O}_X$ for $j \leq 0$. Then $\{ V_k\mathcal{D}_X \}_{k \in \mathbb{Z}}$ is an exhaustive increasing
filtration. Let \( t \) be a local equation of \( Y \) and \( D_t \) a local vector field such that \([D_t, t] = 1\). We have \( t \in V_{-1} \mathcal{D}_X, \) \( D_t \in V_1 \mathcal{D}_X, \) \( \text{gr}^V_0 \mathcal{D}_X := V_0 \mathcal{D}_X / V_{-1} \mathcal{D}_X = \mathcal{D}_Y[tD_t] \) and \( V_k \mathcal{D}_X = \{ \sum_{k \geq j - i} a_{ij}(y, D_y) t^i D^j_t \} \).

**Definition 2.1 [Kv], [Ma]:** Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X \)-module. An increasing filtration \( \{ V_\alpha \mathcal{M} \}_{\alpha \in \mathbb{C}} \) satisfying the following conditions is called the canonical \( \mathcal{V} \)-filtration.

1. \( \mathcal{M} = \bigcup_{\alpha \in C} V_\alpha \mathcal{M} \). Each \( V_\alpha \mathcal{M} \) is a coherent \( V_0 \mathcal{D}_X \)-submodule.
2. \( (V_i \mathcal{D}_X)(V_\alpha \mathcal{M}) \subset V_{\alpha+i} \mathcal{M} \) (\( \forall \alpha \in \mathbb{C}, \forall i \in \mathbb{Z} \)).
3. \( t(V_\alpha \mathcal{M}) = V_{\alpha-1} \mathcal{M} \) (\( \forall \alpha < 0 \)).
4. The action of \((tD_t + 1 + \alpha)\) on \( \text{gr}^V_\alpha \mathcal{M} \) (\( \forall \alpha \in \mathbb{C} \)) is nilpotent where \( \text{gr}^V_\alpha \mathcal{M} = V_\alpha \mathcal{M} / V_{<\alpha} \mathcal{M} \) and \( V_{<\alpha} \mathcal{M} = \bigcup_{\beta < \alpha} V_\beta \mathcal{M} \).

**Remarks 2.2:**
1. The definition of the canonical \( \mathcal{V} \)-filtration does not depend on the choice of \( t \) and \( D_t \). The canonical \( \mathcal{V} \)-filtration is unique if it exists.
2. Since the adjoint of \((tD_t + 1 + \alpha)\) is \(-(tD_t - \alpha)\), the eigenvalue of \( tD_t \) on \( \text{gr}^V_\alpha \mathcal{N} \) is \( \alpha \) for a right \( \mathcal{D}_X \)-module \( \mathcal{N} \).
3. \( t : \text{gr}^V_\alpha \mathcal{M} \to \text{gr}^V_{\alpha-1} \mathcal{M} \) and \( D_t : \text{gr}^V_{\alpha-1} \mathcal{M} \to \text{gr}^V_\alpha \mathcal{M} \) are bijective for
$\alpha \neq 0$.

**Definition 2.3:** We say that a coherent $D_X$-module $\mathcal{M}$ is specializable along $Y$ and we denote $\mathcal{M} \in B_Y$ if the following equivalent conditions are satisfied:

1. For any system of local generators $u_1, \ldots, u_l$ of $\mathcal{M}$ there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $b(tD_t)u_i \in \sum_{j=1}^{l}(V_{-1}D_X)u_j$ $(1 \leq \forall i \leq l)$.

2. $\mathcal{M}$ admits the canonical $V$-filtration with respect to $Y$ and there exists a finite set $A \subset \mathbb{C}$ such that $\{ \alpha \in \mathbb{C} | \text{gr}^V_{\alpha} \mathcal{M} \neq 0 \} \subset A + \mathbb{Z}$.

Let $\mathcal{M} \in B_Y$ and $u \in \mathcal{M}$. Then there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $b(tD_t)u \in (V_{-1}D_X)u$. The minimal polynomial among such is called the $b$-function of the section $u$. The canonical $V$-filtration of $\mathcal{M}$ is known to be given by $V_\alpha \mathcal{M} = \{ u \in \mathcal{M} | \text{all roots of the } b\text{-function of } u \text{ are greater than or equal to } -\alpha - 1 \}$.

**Proposition 2.4.** Let $0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0$ be an exact sequence of coherent $D_X$-modules. Then we have

1. $\mathcal{M} \in B_Y \iff \mathcal{M}', \mathcal{M}'' \in B_Y$.

2. The induced sequence $0 \rightarrow V_\alpha \mathcal{M}' \rightarrow V_\alpha \mathcal{M} \rightarrow V_\alpha \mathcal{M}'' \rightarrow 0$ is exact for
\[ \forall \alpha \in C \text{ if } \mathcal{M} \in B_Y. \]

(3) The induced sequence \[ 0 \to \text{gr}_\alpha^Y \mathcal{M}' \to \text{gr}_\alpha^Y \mathcal{M} \to \text{gr}_\alpha^Y \mathcal{M}'' \to 0 \] is exact for \( \forall \alpha \in C \text{ if } \mathcal{M} \in B_Y. \)

**Remark 2.5:** Let \( \mathcal{M} \in B_Y. \) Then \( \text{gr}_\alpha^Y \mathcal{M} \) is a coherent \( \text{gr}_0^Y \mathcal{D}_X = \mathcal{D}_Y[tD_t]- \) module for any \( \alpha \in C. \) Since the action of \((tD_t + 1 + \alpha)\) is nilpotent on \( \text{gr}_\alpha^Y \mathcal{M}, \) it is a coherent \( \mathcal{D}_Y \)-module.

**Example 2.6:** Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X \)-module with \( \text{Supp}(\mathcal{M}) \subset Y, \) and \( u \in \mathcal{M}. \) Then there exists \( i \in \mathbb{Z}_{\geq 0} \) such that \( t^iu = 0. \) So we have \( \prod_{k=1}^i (tD_t + k)u = D_t^i t^iu = 0. \) Hence we obtain \( \mathcal{M} = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathcal{M}_i \) where \( \mathcal{M}_i = \{ u \in \mathcal{M} \mid (tD_t + 1 + i)u = 0 \}, \) and \( V_\alpha \mathcal{M} = \bigoplus_{i \leq \alpha} \mathcal{M}_i. \)

**Example 2.7:** Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X \)-module. We assume \( Y \) to be non-characteristic for \( \mathcal{M}, \) i.e., \( \text{Ch}(\mathcal{M}) \cap T_Y^*X \subset T_X^*X. \) Then \( \mathcal{M} \in B_Y. \) The proof could be reduced to the case of \( \mathcal{D}_X v = \mathcal{D}_X / \mathcal{D}_X P \) with \( P \in V_N \mathcal{D}_X, \) \( \bar{P} = \bar{D}_t^N \in V_N \mathcal{D}_X / V_{N-1} \mathcal{D}_X \) and \( N \in \mathbb{Z}_{>0} \) where the bar indicates the canonical image. Since \( Pv = 0, \) in \( (V_N \mathcal{D}_X)v / (V_{N-1} \mathcal{D}_X)v \) we have \( \bar{P} \bar{v} = \bar{D}_t^N \bar{v} = 0, \) i.e., \( D_t^N v \in V_{N-1} \mathcal{D}_X v. \) Hence we obtain \( \prod_{k=0}^{N-1} (tD_t - k)v = t^N D_t^N v \in V_{-1} \mathcal{D}_X v. \)

In general, any root of the \( b \)-function of any section of \( \mathcal{M} \) is a nonnegative
integer. Therefore we see by Definition 2.1 (3)

\[ V_\alpha \mathcal{M} = \begin{cases} \ t^{-[\alpha]-1} \mathcal{M} & \alpha < -1 \\ \mathcal{M} & \alpha \geq -1 \end{cases} \]

where \([\alpha] = \max \{ n \in \mathbb{Z} | n \leq \alpha \} \).

Let \( f \) be a holomorphic function, \( \mathcal{M} \) a holonomic \( \mathcal{D}_X \)-module, \( u \in \mathcal{M} \) and \( Y = f^{-1}(0) \). Let \( i_f \) denote the graph of \( f : X \to X \times \mathbb{C} \) and \( t \) the coordinate of \( \mathbb{C} \) in \( X \times \mathbb{C} \). Then we see \( \mathcal{M} \in B_Y \iff i_f^* \mathcal{M} \in B_{X \times \{0\}} \).

Furthermore there is the following correspondence under the isomorphism of \( \mathcal{D}_X[\alpha, t] \) onto \( V_0(\mathcal{D}_X[\alpha, t]) = V_0(\mathcal{D}_X[\alpha, tD]) \):

\[
\begin{align*}
    s & \mapsto -D_t t \\
    \mathcal{D}_X[s]f^* u & \mapsto V_0(\mathcal{D}_X[t, D]) u \otimes \delta(t - f) \\
    P(s)f^{s+1} u & = b(s)f^* u \mapsto P(-D_t t) tu \otimes \delta(t - f) = b(-D_t t) tu \otimes \delta(t - f).
\end{align*}
\]

By Kashiwara’s result recalled in §1, we obtain:

**Proposition 2.8.** All holonomic \( \mathcal{D}_X \)-modules belong to \( B_Y \).

**§3. Operations in \( B_Y \).**

**Proposition 3.1.** Let \( Y = \{ t = 0 \} \) and \( \mathcal{M} \in B_Y \). Then
(1) $\mathcal{M}[t^{-1}] \in B_Y$.

(2) $\mathcal{H}^i(\mathcal{M}^*) \in B_Y$ for $\forall j$. Moreover for $\forall j$ locally we have isomorphisms

$$\text{gr}^{Y}_{\alpha}(\mathcal{H}^i(\mathcal{M}^*)) \overset{\sim}{\to} \mathcal{H}^j(\text{gr}^{-1}_{\alpha}(\mathcal{M}^*)) (-1 < \alpha < 0)$$

and

$$\mathcal{H}^j(\text{gr}^{Y}_{\beta}(\mathcal{M}^*)) \overset{\sim}{\to} \mathcal{H}^i(\mathcal{M}^*) (\beta = -1, 0).$$

Under these isomorphisms the transpose $t^i: \mathcal{H}^i(\text{gr}^{-1}_{1}(\mathcal{M}^*)) \to \mathcal{H}^j(\text{gr}^{V}_{0}(\mathcal{M}^*))$ corresponds to $-D_i: \text{gr}^{-1}_{-1}(\mathcal{H}^i(\mathcal{M}^*)) \to \text{gr}^{-1}_{0}(\mathcal{H}^i(\mathcal{M}^*))$ and the transpose $t^i D_i: \mathcal{H}^i(\text{gr}^{V}_{0}(\mathcal{M}^*)) \to \mathcal{H}^j(\text{gr}^{-1}_{1}(\mathcal{M}^*))$ corresponds to $t: \text{gr}^{-1}_{0}(\mathcal{H}^i(\mathcal{M}^*)) \to \text{gr}^{-1}_{1}(\mathcal{H}^i(\mathcal{M}^*))$.

PROPOSITION 3.2. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and $i$ the inclusion of $Y$ into $X$. Then

(1) The restriction $i^* \mathcal{M}$ is quasi-isomorphic to $0 \to \text{gr}^{-1}_{-1} \mathcal{M} \overset{D_i}{\to} \text{gr}^{-1}_{0} \mathcal{M} \to 0$ where the dot indicates the place of degree zero.

(2) For any $\alpha \in \mathbb{C}$, $\text{gr}^{V}_{\alpha} \mathcal{M}$ is a holonomic $\mathcal{D}_Y$-module.

PROOF: (1) By Remark 2.2 (3) and Proposition 3.1 (2) we have

$$i^! \mathcal{M}^* \overset{\sim}{\to} (0 \to \text{gr}^{V}_{0} (\mathcal{M}^*) \overset{i}{\to} \text{gr}^{-1}_{-1} (\mathcal{M}^*) \to 0)$$

$$\overset{\sim}{\to} (0 \to (\text{gr}^{-1}_{0} \mathcal{M})^* \overset{D_i}{\to} (\text{gr}^{-1}_{-1} \mathcal{M})^* \to 0).$$

Since $i^* \mathcal{M} = (i^! \mathcal{M}^*)^*$, we obtain the assertion.
(2) We know that

\[ \mathcal{M} : \text{holonomic } \Leftrightarrow \mathcal{H}^j(\mathcal{M}^*) = 0 \quad \text{for } j \neq 0. \]

Hence by Proposition 3.1 (2) we obtain \( \mathcal{H}^j((\text{gr}_V^\alpha \mathcal{M})^*) = 0 \) for \( j \neq 0 \). This means the holonomicity of \( \text{gr}_V^\alpha \mathcal{M} \).

**Proposition 3.3.** Let \( g : X' \to X \) be a proper morphism of smooth manifolds. We suppose that \( Y' := g^{-1}(Y) \) is a smooth hypersurface and \( \mathcal{M} \in B_Y \) has a global good filtration. Then for any \( j \), we have \( \mathcal{H}^j(\mathcal{R}g_* \mathcal{M}) \in B_Y \) and the canonical \( V \)-filtration of \( \mathcal{M} \) induces the one for \( \mathcal{H}^j(\mathcal{R}g_* \mathcal{M}) \).

§4. Moderate Nearby Cycles and Moderate Vanishing Cycles.

Let \( Y \) be a smooth hypersurface defined by \( t : X \to \mathbb{C} \). For a coherent \( \mathcal{D}_X \)-module \( \mathcal{M} \in B_Y, p \in \mathbb{Z}_{\geq 0} \) and \(-1 \leq \alpha < 0\), we define

\[ \mathcal{M}_{\alpha,p} := \bigoplus_{0 \leq k \leq p} \mathcal{M}[t^{-1}] \otimes e_{\alpha,k} \]

where \( e_{\alpha,k} = t^{\alpha+1}(\log t)^k/k! \). It is clear that for any \( \beta \in \mathbb{C} \)

\[ V_{\beta} \mathcal{M}_{\alpha,p} = \bigoplus_{0 \leq k \leq p} V_{\beta+\alpha+1}(\mathcal{M}[t^{-1}]) \otimes e_{\alpha,k}. \]
Then the monodromy $T = \exp(2\pi itD_t)$ induces a $D_Y$-automorphism on $\mathcal{M}_{\alpha,p}$ by $T(m \otimes e_{\alpha,k}) = m \otimes T(e_{\alpha,k})$, and accordingly on $\text{gr}^Y_{\beta}(\mathcal{M}_{\alpha,p})$.

**Definition 4.1:** For $-1 \leq \alpha \leq 0$, we define the moderate nearby cycle $\psi^m_{t,\alpha}(\mathcal{M})$ by

$$\psi^m_{t,\alpha}(\mathcal{M}) := \lim_{p} \psi^m_{t,\alpha,p}(\mathcal{M})$$

where $\psi^m_{t,\alpha,p}(\mathcal{M}) := i^*(\mathcal{M}_{\alpha,p})[-1]$.

By Proposition 3.2 we see

$$\psi^m_{t,\alpha,p}(\mathcal{M}) \sim \bigoplus_{0 \leq k \leq p} \text{gr}_{\alpha}^{V}(\mathcal{M}_{\alpha,p}) \otimes e_{\alpha,k}.$$

We remark that $T$ acts on $\psi^m_{t,\alpha}(\mathcal{M})$ as well.

**Proposition 4.2.** For $\mathcal{M} \in B_Y$ and $-1 \leq \alpha < 0$, we have a quasi-isomorphism $\text{gr}^V_{\alpha} \mathcal{M} \sim \psi^m_{t,\alpha}(\mathcal{M})$. Here the action of $T$ on $\psi^m_{t,\alpha}(\mathcal{M})$ corresponds to that of $\exp(-2\pi itD_t)$.

**Proof:** Since $V_{<0} \mathcal{M} = V_{<0}(\mathcal{M}[t^{-1}])$, we have

$$\text{gr}^V_{-1}(\mathcal{M}_{\alpha,p}) = \bigoplus_{0 \leq k \leq p} \text{gr}_{\alpha}^{V}(\mathcal{M}[t^{-1}]) \otimes e_{\alpha,k}.$$

As $\mathcal{M}_{\alpha,p} = \mathcal{M}_{\alpha,p}[t^{-1}]$, we know $\mathcal{H}(\psi^m_{t,\alpha,p}(\mathcal{M})) = \text{Ker}(D_t) = \text{Ker}(tD_t : \text{gr}^V_{-1}(\mathcal{M}_{\alpha,p}) \to \text{gr}^V_{-1}(\mathcal{M}_{\alpha,p}))$. Since $tD_t(m \otimes e_{\alpha,k}) = [(tD_t + \alpha + 1)m] \otimes$
\(e_{\alpha,k} + m \otimes e_{\alpha,k-1}\), we see \(\sum_{k=0}^{p} m_{k} \otimes e_{\alpha,k} \in \text{Ker}(tD_{t}) = \mathcal{H}^{0}(\psi_{t,\alpha,p}^{m}(\mathcal{M})) \iff (tD_{t}+\alpha+1)m_{k}+m_{k+1} = 0 \quad (0 \leq k \leq p-1) \iff m_{k} = \left[-(tD_{t}+\alpha+1)\right]^{k}m_{0} \quad (0 \leq k \leq p)\). Hence for \(p\) such that \((tD_{t}+\alpha+1)^{p} = 0\) on \(\text{gr}_{\alpha}^{V}(\mathcal{M})\), the morphism \(\text{gr}_{\alpha}^{V}(\mathcal{M}) \ni m_{0} \mapsto \sum_{k=0}^{p} \left[-(tD_{t}+\alpha+1)\right]^{k}m_{0} \otimes e_{\alpha,k} \in \mathcal{H}^{0}(\psi_{t,\alpha,p}^{m}(\mathcal{M}))\) is isomorphic.

Let \(x = \sum_{k=0}^{p} \left[-(tD_{t}+\alpha+1)\right]^{k}m_{0} \otimes e_{\alpha,k} \in \text{Ker}(tD_{t})\). Then we have
\[
0 = (tD_{t})x = \sum_{k=0}^{p} \left[-(tD_{t}+\alpha+1)\right]^{k}(tD_{t})m_{0} \otimes e_{\alpha,k} + \sum_{k=0}^{p} \left[-(tD_{t}+\alpha+1)\right]^{k}m_{0} \otimes (tD_{t})e_{\alpha,k},
\]
and thus \(\sum_{k=0}^{p} \left[-(tD_{t}+\alpha+1)\right]^{k}m_{0} \otimes (2\pi itD_{t})e_{\alpha,k} = \sum_{k=0}^{p} \left[-(tD_{t}+\alpha+1)\right]^{k}((-2\pi itD_{t})m_{0}) \otimes e_{\alpha,k}\). Hence the monodromy \(T\) corresponds to \(\exp(-2\pi itD_{t})\).

Since \(t\) induces an isomorphism \(\text{gr}_{0}^{V}(\mathcal{M}_{\alpha,p}) \sim \text{gr}_{-1}^{V}(\mathcal{M}_{\alpha,p})\), we see \(\mathcal{H}^{1}(\psi_{t,\alpha,p}^{m}(\mathcal{M})) = \text{Coker}(D_{t}) = \text{Coker}(D_{t}t : \text{gr}_{0}^{V}(\mathcal{M}_{\alpha,p}) \rightarrow \text{gr}_{0}^{V}(\mathcal{M}_{\alpha,p}))\). For \(\sum_{k=0}^{p} m_{k} \otimes e_{\alpha,k} \in \bigoplus_{0 \leq k \leq p} \text{gr}_{\alpha+1}^{V}(\mathcal{M}[t^{-1}]) \otimes e_{\alpha,k} = \text{gr}_{0}^{V}(\mathcal{M}_{\alpha,p})\), we have
\[
D_{t}t(\sum_{k=0}^{p} m_{k} \otimes e_{\alpha,k}) = \sum_{k=0}^{p} ((D_{t}t + \alpha + 1)m_{k} \otimes e_{\alpha,k} + m_{k} \otimes e_{\alpha,k-1}) = \sum_{k=0}^{p} m'_{k} \otimes e_{\alpha,k}
\]
where \(m'_{k} = (D_{t}t + \alpha + 1)m_{k} + m_{k+1}\). Hence for \(l\) such that \((D_{t}t + \alpha + 1)^{l} = 0\) on \(\text{gr}_{\alpha+1}^{V}(\mathcal{M}[t^{-1}])\), we have \(m \otimes e_{\alpha,k} = D_{t}t(\sum_{i=1}^{l} \left[-(D_{t}t + \alpha + 1)\right]^{i-1}m \otimes e_{\alpha,k+i})\) and thus \(\mathcal{H}^{1}(\psi_{t,\alpha}^{m}(\mathcal{M})) = 0\).

**Definition 4.3:** We define the moderate vanishing cycle \(\phi_{t,0}^{m}(\mathcal{M})\) to be the inductive limit of the mapping cone \(\phi_{t,0,p}^{m}(\mathcal{M})\) of the natural morphism
\[ i^* \mathcal{M}[-1] \to i^* \mathcal{M}_{-1,p}[-1] = \psi_{t,-1,p}^m(\mathcal{M}), \text{ i.e.,} \]

\[ \phi_{t,0,p}^m = (0 \to \text{gr}_{-1}^V \mathcal{M}_{-1,p} \oplus \text{gr}_0^V \mathcal{M} \xrightarrow{D_t+j} \text{gr}_0^V \mathcal{M}_{-1,p} \to 0) \]

where \( j \) is the natural morphism \( \mathcal{M} \to \mathcal{M}_{-1,p} = \bigoplus_{0 \leq k \leq p} \mathcal{M}[t^{-1}] \otimes e_{-1,k} \).

We define morphisms \( \text{can} : \psi_{t,-1}^m(\mathcal{M}) \to \phi_{t,0}^m(\mathcal{M}) \) and \( \text{var} : \phi_{t,0}^m(\mathcal{M}) \to \psi_{t,-1}^m(\mathcal{M}) \) by the morphisms \( \text{id} : \psi_{t,-1}^m(\mathcal{M}) \to \phi_{t,0}^m(\mathcal{M}) \) and \( T - \text{id} : \phi_{t,0}^m(\mathcal{M}) \to \psi_{t,-1}^m(\mathcal{M}) \) respectively.

**Proposition 4.4.** For \( \mathcal{M} \in B_Y \), we have a quasi-isomorphism \( \text{gr}_0^V \mathcal{M} \xrightarrow{\sim} \phi_{t,0}^m(\mathcal{M}) \). Moreover \( \text{can} \) corresponds to \( D_t : \text{gr}_{-1}^V \mathcal{M} \to \text{gr}_0^V \mathcal{M} \) and \( \text{var} \) to \( \frac{[\exp(-2\pi itD_{-1})]t}{tD_t} \text{gr}_0^V \mathcal{M} \to \text{gr}_{-1}^V \mathcal{M} \).

**Proof:** Let \( x = \sum_{k=0}^p m_k \otimes e_{-1,k} + n_0 \in \text{gr}_{-1}^V \mathcal{M}_{-1,p} \oplus \text{gr}_0^V \mathcal{M} = \bigoplus_{k=0}^p \text{gr}_{-1}^V \mathcal{M} \otimes e_{-1,k} \oplus \text{gr}_0^V \mathcal{M} \). Then we can check

\[ x \in \text{Ker}(D_t + j) \iff \begin{cases} m_1 = -tD_t m_0 - tn_0 \\ m_{k+1} = -tD_t m_k \quad (k \geq 1). \end{cases} \]

Hence we obtain an isomorphism \( \text{gr}_{-1}^V \mathcal{M} \oplus \text{gr}_0^V \mathcal{M} \xrightarrow{\sim} \text{Ker}(D_t + j) \) defined by \( m_0 + n_0 \mapsto \sum m_k \otimes e_{-1,k} + n_0 \) with \((*)\). So we see \( \text{gr}_0^V \mathcal{M} \xrightarrow{\sim} \mathcal{H}^0(\phi_{t,0}^m) \). Since \( m \equiv D_t m \mod \text{Im}(j \oplus -D_t) \) for \( m \in \text{gr}_{-1}^V \mathcal{M} \), the morphism \( \text{can} \) corresponds to \( D_t : \text{gr}_{-1}^V \mathcal{M} \to \text{gr}_0^V \mathcal{M} \). The element \( \sum_{k \geq 1} (-tD_t)^{k-1}(-tn) \otimes e_{-1,k} + \)


$n \in \text{Ker}(D_t + j)$ corresponds to $n \in \text{gr}_0^Y \mathcal{M}$. Since the coefficient of $(T - id)(\sum_{k \geq 1}(-tD_t)^{k-1}(-tn) \otimes e_{-1,k})$ at $e_{-1,0}$ is $\sum_{k \geq 1}(2\pi i)^k \frac{(-tD_t)^{k-1}}{k!}(-tn)$, the morphism $\text{var}$ corresponds to $\left(\frac{\exp(-2\pi itD_t) - 1}{tD_t}\right)t : \text{gr}_0^Y \mathcal{M} \to \text{gr}_{-1}^Y \mathcal{M}$.

§5. Nearby Cycles and Vanishing Cycles.

Let $f$ be a nonconstant holomorphic function on $X$, $i$ the inclusion of $f^{-1}(0)$ into $X$ and $K \in D^b_c(C_X)$. Let $\tilde{C}^x$ denote the universal covering of $\mathbb{C}^x$ and $p$ the natural map $\tilde{X}^x := X \times_{\mathbb{C}} \tilde{C}^x \to X$. Following [SGA7] we define the nearby cycle $\psi_f(K)$ by $\psi_f(K) := i^{-1}R_* p^{-1}K \in D^b_c(C_{J^{-1}(0)})$. The natural morphism $K \to R_* p^{-1}K$ induces a morphism $i^{-1}K \to \psi_f(K)$, whose mapping cone $\phi_f(K) \in D_c^b(C_{J^{-1}(0)})$ is called the vanishing cycle. By the definition of $\phi_f(K)$ we have the canonical morphism $\text{can} : \psi_f(K) \to \phi_f(K)$. Associated to the canonical generator of $\pi_1(\mathbb{C}^x)$ the monodromy automorphism $T$ acts on $\psi_f(K)$ and $\phi_f(K)$. Since $(T - id)|_{-1}K = 0$, $T - id$ induces the variation $\text{var} : \phi_f(K) \to \psi_f(K)$. For $\lambda \in \mathbb{C}^x$ we define a subcomplex $\psi_{f,\lambda}(K)$ of $\psi_f(K)$ by

$$\psi_{f,\lambda}(K) := \{ x \in \psi_f(K) | (T - \lambda id)^m x = 0 \text{ (} m > 0 \text{)} \}.$$ 

Since $\psi_f(K) \in D^b_c(C_{J^{-1}(0)})$, we have a quasi-isomorphism $\bigoplus_{\lambda \in \mathbb{C}^x} \psi_{f,\lambda}(K) \cong \psi_f(K)$. Similarly we have $\bigoplus_{\lambda \in \mathbb{C}^x} \phi_{f,\lambda}(K) \cong \phi_f(K)$ as well. For
convenience we set $p\psi_f(K) := \psi_f(K)[-1]$ and $p\phi_f(K) := \phi_f(K)[-1]$.

Let $Y$ be a smooth hypersurface of $X$ defined by $t = 0$. When $M$ is a regular holonomic $D_X$-module, we have quasi-isomorphisms

$$\text{DR}(\psi^m_{t,\alpha}(M)) \xrightarrow{\sim} p\psi_{t,e^{2\pi i\alpha}}(\text{DR}(M)) \quad (-1 \leq \alpha < 0)$$

$$\text{DR}(\phi^m_{t,0}(M)) \xrightarrow{\sim} p\phi_{t,1}(\text{DR}(M))$$

(see [SGA7]). Hence we obtain:

**Theorem 5.1** [Kv], [Ma]. For a regular holonomic $D_X$-module $M$, we have

$$\text{DR}(\text{gr}_{-1}^V M) \xrightarrow{\sim} \begin{cases} p\psi_{t,e^{2\pi i\alpha}}(\text{DR}(M)) & (-1 \leq \alpha < 0) \\ p\phi_{t,e^{2\pi i\alpha}}(\text{DR}(M)) & (-1 < \alpha \leq 0) \end{cases}$$

Moreover under the above quasi-isomorphisms we have the following correspondences:

$$\exp(-2\pi itD_t) \leftrightarrow T$$

$$D_t : \text{gr}_{-1}^V M \rightarrow \text{gr}_0^V M \leftrightarrow \text{can} : p\psi_{t,1}(M) \rightarrow p\phi_{t,1}(M)$$

$$\frac{\exp(-2\pi itD_t) - 1}{tD_t} : \text{gr}_0^V M \rightarrow \text{gr}_{-1}^V M \leftrightarrow \text{var} : p\phi_{t,1}(M) \rightarrow p\psi_{t,1}(M).$$
Corollary 5.2. For a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we have

$$p\psi_t (D_X(DR(\mathcal{M}))) \overset{\sim}{\longrightarrow} D_Y p\psi_t (DR(\mathcal{M}))$$

$$p\phi_t (D_X(DR(\mathcal{M}))) \overset{\sim}{\longrightarrow} D_Y p\phi_t (DR(\mathcal{M})).$$

References


