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HOLONOMIC $\mathcal{D}$-MODULES, PERVERSE SHEAVES AND RIEMANN-HILBERT CORRESPONDENCE

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This is a brief survey of $\mathcal{D}$-modules, Perverse Sheaves and Riemann-Hilbert correspondence. The authors hope that this becomes quick guide for readers to learn these areas.

Contents

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1. Operations on $\mathcal{D}$-modules

1.0. Definition of $\mathcal{D}$-modules.

a) Let X be a complex manifold. Then $\mathcal{D}_X$ denotes the sheaf of rings of differential operators with holomorphic coefficients on X, which is constructed as follows. First introduce the sheaf Θ_X of holomorphic vector fields on X. Then $\mathcal{D}_X$ is generated by $\mathcal{O}_X$ and Θ_X under relations

$$[\partial, f] = \partial f \quad (f \in \mathcal{O}_X, \partial \in \Theta_X). \tag{1.1}$$

b) Next we introduce, for a smooth separate scheme X of finite type over K, then sheaf $\mathcal{D}_X$ of algebraic differential operators. First we assume X to be affine. Then $\mathcal{D}(X)$ is defined as the ring generated by $\mathcal{O}_X(X)$ and $\Theta_X(X) := \mathrm{Der}_K(\mathcal{O}(X), \mathcal{O}(X))$ under the relation (1.1). Let U be open affine in X. Since we have $\Theta(U) = \mathcal{O}_X(U) \underset{\mathcal{O}_X(X)}{\otimes} \Theta_X(U)$, we deduce

$$\mathcal{D}(U) = \mathcal{O}_X(U) \underset{\mathcal{O}_X(X)}{\otimes} \mathcal{D}_X(X), \tag{1.2}$$

which entails us a unique quasi-coherent $\mathcal{O}_X$ module $\mathcal{D}_X$ on X. Now we assume to be given a general X. Then the last statement enables us to construct the sheaf $\mathcal{D}_X$ whose restriction to an open affine set U is $\mathcal{D}_U$.

c) From now on in this §1, unless otherwise stated we assume that X is a complex manifold or a smooth separate scheme of finite type over K. The ring $\mathcal{D}_X$ is naturally filtered as ring as follows. We set first $\mathcal{D}_X(0) = \mathcal{O}_X$. Next inductively we define $\mathcal{D}_X(k)$ by

$$\mathcal{D}_X(k) = \{P \in \mathcal{D}_X; [P, \mathcal{D}_X] \subset \mathcal{D}_X(k-1)\}. \tag{1.3}$$

Then we have

$$\begin{cases} \mathcal{D}_X(k) \subset \mathcal{D}_X(k+1) \\ \mathcal{D}_X(k) \cdot \mathcal{D}_X(\ell) \subset \mathcal{D}_X(k+\ell) \\ 1 \in \mathcal{D}_X(0). \end{cases} \tag{1.4}$$

For a section P of $\mathcal{D}_X$,

$$\mathrm{ord}(P) = \min\{k; P \in \mathcal{D}_X(k)\}$$

is called the order of P.

If we take a local coordinate system $x = (x_1, \cdots, x_n)$, P is written as

$$P = \sum_{|\alpha| \leqq \mathrm{ord}(P)} a_\alpha(x)\partial^\alpha. \tag{1.5}$$

Here $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\partial_n}$ with $\partial_i = \partial/\partial x_i$.

The graded ring $\mathrm{gr}\,\mathcal{D}_X$ can be identified with a sheaf of ring on the cotangent bundle T^*X. In fact we have the isomorphism

$$\mathcal{D}_X(k)/\mathcal{D}_X(k-1) \simeq \pi_*\mathcal{O}_{[T^*X]}(k)$$

with $\mathcal{O}_{[T^*X]}(k)$ being the sheaf of holomorphic functions on T^*X polynomial along fibers of $\pi : T^*X \to X$ (resp. of coordinate ring of T^*X) homogeneous in degree k in case X is a complex manifold (resp. a smooth separate scheme of finite type over K). Hence the isomorphism

$$\mathrm{gr}\,\mathcal{D}_X \simeq \pi_*(\oplus\mathcal{O}_{[T^*X]}(k)) \tag{1.6}$$

follows. The canonical morphism

$$\sigma : \mathcal{D}_X \longrightarrow \mathrm{gr}\,\mathcal{D}_X$$

is called the symbol map. For the P in (1.5), $\sigma(P)$ is written as

$$\sigma(P) = \sum_{|\alpha| = \mathrm{ord}(P)} a_\alpha(x)\xi^\alpha. \tag{1.7}$$

Here $\xi = (\xi_1, \cdots, \xi_n)$ is the associated coordinates for fibers of π.

d) Coherence of $\mathcal{D}_X$.

We quote a theorem for $\mathcal{D}_X$ of fundamental importance due to M. Kashiwara.

Theorem 1.0.0. *The sheaf $\mathcal{D}_X$ is a coherent ring.*

This theorem can be proved from the coherence of $\mathrm{gr}\,\mathcal{D}_X$.

Definition 1.0.1.

i) The category of coherent $\mathcal{D}_X$ modules is denoted by $\mathrm{Coh}(\mathcal{D}_X)$.

ii) Moreover $D_c^b(\mathcal{D}_X)$ stands for the derived category of bounded complex of $\mathcal{D}_X$ modules whose cohomologies are coherent.

We remark that if X is quasi-compact in algebraic case, we have the isomorphism

$$D^b(\mathrm{Coh}(\mathcal{D}_X)) \xrightarrow{\sim} D_c^b(\mathcal{D}_X).$$

1.1. Characteristic variety

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ module. A family $\{F_k(\mathcal{M})\}$ of $\mathcal{O}_X$ submodule of $\mathcal{M}$ is called a filtration if the following conditions i), ii), iii) are satisfied.

$$(1.8) \qquad \begin{cases} \text{i)} & F_k(\mathcal{M}) \subset F_{k+1}(\mathcal{M}), \\ \text{ii)} & \bigcup F_k(\mathcal{M}) = \mathcal{M}, \\ \text{iii)} & \mathcal{D}_X(k)F_\ell(\mathcal{M}) \subset F_{k+\ell}(\mathcal{M}). \end{cases}$$

More important is the following definition of a class of filtration.

Definition 1.1.0. Let $\{F_k(\mathcal{M})\}$ be a filtration of a coherent $\mathcal{D}_X$ module $\mathcal{M}$. Then $\{F_k(\mathcal{M})\}$ is a good filtration if i) $F_k(\mathcal{M})$ is $\mathcal{O}_X$ coherent, and ii) there exists k_o satisfying

$$\mathcal{D}_X(\ell)F_k(\mathcal{M}) = F_{k+\ell}(\mathcal{M}) \quad ({}^\forall k \geq k_0,\ {}^\forall \ell \geq 0).$$

We give several remarks concerning the above definition.

I) If $F(\mathcal{M})$ and $\tilde{F}(\mathcal{M})$ are two good filtrations for $\mathcal{M}$, then for some ℓ we have

$$\tilde{F}_{k-\ell}(\mathcal{M}) \subset F_k(\mathcal{M}) \subset \tilde{F}_{k+\ell}(\mathcal{M}).$$

II) There exists locally a good filtration for $\mathcal{M}$. In fact we take locally a resolution of $\mathcal{M}$.

$$\mathcal{D}_U^m \xrightarrow{\psi} \mathcal{M}|_U \longrightarrow 0.$$

Then it suffices to put

$$F_k(\mathcal{M}) = \psi(\mathcal{D}_U(k)^m).$$

III) In algebraic case, if X is proper over K, then there exists a global good filtration.

The following proposition plays an essential role in studying characteristic varieties of coherent $\mathcal{D}_X$ modules.

Proposition 1.1.1. *Let* $0 \to \mathcal{M}_1 \to \mathcal{M} \xrightarrow{\psi} \mathcal{M}_2 \to 0$ *be an exact sequence in* $\mathrm{Coh}(\mathcal{D}_X)$, *and* $F(\mathcal{M})$ *a good filtration of* $\mathcal{M}$. *Then*

i) *the induced filtration*

$$F_k(\mathcal{M}_1) = F_k(\mathcal{M}) \cap \mathcal{M}_1$$

and the quotient filtration

$$F_k(\mathcal{M}_2) = \psi(F_k(\mathcal{M}))$$

are good, and

ii) $0 \to \mathrm{gr}(\mathcal{M}_1) \to \mathrm{gr}(\mathcal{M}) \to \mathrm{gr}(\mathcal{M}_2) \to 0$ *is exact.*

Now let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ module with a good filtration $F_k(\mathcal{M})$. Then we associate to $\mathcal{M}$ an invariant on T^*X.

Definition 1.1.2. We define the characteristic variety of $\mathcal{M}$ by

$$\mathrm{char}(\mathcal{M}) := \mathrm{supp}(\mathcal{O}_{T^*X} \underset{\pi^{-1}(\mathrm{gr}\,\mathcal{D}_X)}{\otimes} \pi^{-1}\,\mathrm{gr}(F(\mathcal{M}))).$$

We remark that the above definition is in dependent of the choice of $F(\mathcal{M})$. This fact can be shown from the remark (I) just after Definition 1.1.0.

The following proposition is used when we extend several facts about single differential equations (i.e. $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$) to those about general coherent $\mathcal{D}_X$ modules.

Proposition 1.1.3. *Let $0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0$ be an exact sequence in* $\mathrm{Coh}(\mathcal{D}_X)$. *Then*

$$\mathrm{ch}(\mathcal{M}) = \mathrm{ch}(\mathcal{M}_1) \cup \mathrm{ch}(\mathcal{M}_2).$$

This result is a simple corollary of Proposition 1.1.1.

Next we give a general remark about $\mathrm{ch}(\mathcal{M})$ by

Proposition 1.1.4. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ module. Then we have*

i) $\mathrm{ch}(\mathcal{M})$ is an analytic variety (resp. algebraic variety) in case X is analytic (resp. algebraic).

ii) $\dim \mathrm{ch}(\mathcal{M}) \geqq \dim X$.

The first part of the proposition is easy to prove. The second are results from an algebraic property of $\mathcal{D}_X$.

We give an example.

Example. Let P be a section of $\mathcal{D}_X$, and set $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$. Then

$$\mathrm{ch}(\mathcal{M}) = \{q \in T^*X; \sigma(P)(q) = 0\}.$$

Now we can give the definition of holonomic $\mathcal{D}_X$ modules.

Definition 1.1.5.

i) If $\mathrm{ch}(\mathcal{M}) = \dim X$, then $\mathcal{M}$ is holonomic.

ii) The full subcategory of $D^b(\mathcal{D}_X)$, the derived of $\mathcal{D}_X$ module, consisting of bounded complexes with holonomic cohomologies is denoted by $D^b_h(\mathcal{D}_X)$. We also denote by $Coh_h(\mathcal{D}_X)$ by the full subcategory of $Coh(\mathcal{D}_X)$ consisting of holonomic modules.

We are now in a good position to give some remarks about the characteristic variety. The cotangent bundle T^*X is endowed with the structure of homogeneous symplectic manifold. For any coherent $\mathcal{D}_X$ module $\mathcal{M}$, its characteristic variety $\mathrm{ch}(\mathcal{M})$ is involutive in the sense that for any $f, g \in \mathcal{O}_{T^*X}$ with $f|_{\mathrm{ch}(\mathcal{M})} \equiv 0$, $g|_{\mathrm{ch}(\mathcal{M})} \equiv 0$, we have

$$\{f, g\}|_{\mathrm{ch}(\mathcal{M})} \equiv 0.$$

Here $\{,\}$ denotes the Poisson bracket on T^*X. The definition for $\mathcal{M}$ to be holonomic is equivalent to the condition that $\mathrm{ch}(\mathcal{M})$ is Lagrangean manifold; i.e. $\mathrm{ch}(\mathcal{M})$ is involutive and $\omega|_{\mathrm{ch}(\mathcal{M})} = 0$ for the canonical 1-form of T^*X.

1.2. Operations of $\mathcal{D}$-modules

a) Left $\mathcal{D}_X$ modules and right $\mathcal{D}_X$-modules

We have not yet paid any attention to the difference between left $\mathcal{D}_X$-modules and right $\mathcal{D}_X$ modules. It is, however, indispensable to tell the difference to discuss several operations for $\mathcal{D}_X$ modules.

Let $\mathcal{M}$ be a $\mathcal{O}_X$ modules with an action of Θ_X; i.e. with a sheaf homomorphism

$$\Theta_X \underset{\mathcal{O}_X}{\otimes} \mathcal{M} \longrightarrow \mathcal{M} : \quad (v, m) \longmapsto v \cdot m.$$

Then $\mathcal{M}$ is given a structure of left $\mathcal{D}_X$-modules if and only if

$$\begin{cases} v(f \cdot m) & = f(vm) + v(f) \cdot m \\ f(vm) & = (fv)m \\ v_1(v_2 m) - v_2(v_1 m) & = [v_1, v_2]m \\ & (v, v_1, v_2 \in \Theta_X, f \in \mathcal{O}_X, m \in \mathcal{M}). \end{cases} \tag{1.9}$$

On the other hand, the above homomorphism entails to $\mathcal{M}$ a right $\mathcal{D}_X$ module structure if and only if

$$\begin{cases} f(vm) & = (fv)m - v(f)m \\ v(fm) & = (fv) \cdot m \\ v_1(v_2 m) - v_2(v_1 m) & = [v_1, v_2]m \\ & (v, v_1, v_2 \in \Theta_X, f \in \mathcal{O}_X, m \in \mathcal{M}). \end{cases} \tag{1.10}$$

Thus it is easy to see $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module. Moreover the sheaf of volume forms Ω_X is a right $\mathcal{D}_X$-module. In fact we define

$$v \cdot \omega = L_v \omega$$

where L_v is the Lie derivative of v.

Next we show that the category of left $\mathcal{D}_X$ modules and that of right $\mathcal{D}_X$ modules are equivalent. Given a left $\mathcal{D}_X$-module $\mathcal{M}$, then $\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \Omega_X$ becomes a right $\mathcal{D}_X$-module. In fact for $m \otimes \omega \in \mathcal{M} \otimes \Omega_X$ and $v \in \Theta_X$, an action $(m \otimes \omega)v$ can be defined by

$$(m \otimes \omega)v = -vm \otimes \omega + m \otimes \omega \cdot v.$$

On the other hand, let $\mathcal{N}$ be a right $\mathcal{D}_X$ module. Then $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega_X, \mathcal{N})$ is a left $\mathcal{D}_X$ module. In this case we define an action Θ_X on $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega_X, \mathcal{N})$ by

$$\begin{aligned} v(h)(\omega) =& h(\omega \cdot v) - h(\omega) \cdot v \\ & (h \in \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega_X, \mathcal{N}), v \in \Theta_X, \omega \in \Omega_X). \end{aligned}$$

The above two correspondences give rise to functors which are inverse to each other. Thus the category of left $\mathcal{D}_X$ modules and that of right $\mathcal{D}_X$ modules are equivalent.

b) Dual functor

Let $\mathcal{M}$ be a left $\mathcal{D}_X$ module, or more generally an element of $D^b(\mathcal{D}_X)$. We define its dual $\mathcal{M}^*$ or $D_X(\mathcal{M})$ as

$$\begin{aligned} \mathcal{M}^* = D_X(\mathcal{M}) &= \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)[n] \underset{\mathcal{O}_X}{\otimes} \Omega_X^{-1} \\ &\simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega_X, \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X))[n]. \end{aligned}$$

We first explain that it does mean in case where X is an open subset of $\mathbb{C}^n$ with coordinates z and $\mathcal{M}$ is a single equation, i.e. $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X P$. Using the resolution of $\mathcal{M}$

$$0 \longleftarrow \mathcal{M} \longleftarrow \mathcal{D}_X \xleftarrow{\cdot P} \mathcal{D}_X \longleftarrow 0 \quad \text{(exact)},$$

$\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ can be calculated as

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \simeq \mathcal{D}_X / P\mathcal{D}_X.$$

Tensored by Ω_X^{-1} over $\mathcal{O}_X$, this induces an isomorphism

$$D_X(\mathcal{M}) = \mathcal{D}_X / \mathcal{D}_X P^*$$

where P^* is the dual operator of $P = \sum_{|\alpha| \le m} a_\alpha(z) \partial^\alpha$,

$$P^* = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \partial^\alpha \cdot a_\alpha(x).$$

We remark that P^* depends on a choice of a coordinate system and a volume element.

Now we lists up several properties of the dual functor.

i) If $\mathcal{M} \in D^b_c(\mathcal{D}_X)$, then $\mathcal{M}^* \in D^b_c(\mathcal{D}_X)$.

ii) Let $\mathcal{M} \in D^b_c(\mathcal{D}_X)$. Then we have isomorphisms

$$\begin{aligned}\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{D}_X) &\simeq \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X^{-1}, \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)[n], \mathcal{D}_X)) \\ &\simeq \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X), \mathcal{D}_X) \underset{\mathcal{O}_X}{\otimes} \Omega_X[-n],\end{aligned}$$

which induce a homomorphism

$$\mathcal{M} \longrightarrow \mathcal{M}^{**}.$$

This becomes isomorphic if $\mathcal{M} \in D^b_c(\mathcal{D}_X)$.

iii) Since the category of right $\mathcal{D}_X$ modules and left $\mathcal{D}_X$ modules are equivalent as seen in a), we have an isomorphism for $\mathcal{M}, \mathcal{N} \in D^b(\mathcal{D}_X)$,

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{N}^*) \simeq \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X), \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{N}, \mathcal{D}_X)).$$

Thus we have a morphism

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \longrightarrow \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{N}^*, \mathcal{M}^*),$$

which is isomorphic if $\mathcal{M}, \mathcal{N} \in D^b_c(\mathcal{D}_X)$.

iv) For a left coherent $\mathcal{D}_X$ module $\mathcal{M}$, or more generally for an element $\mathcal{M}$ in $D^b_c(\mathcal{D}_X)$, we define its solution complex and de Rham complex respectively by

$$\begin{aligned}\mathrm{Sol}(\mathcal{M}) &:= \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X), \\ DR(\mathcal{M}) &:= \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})[\dim X].\end{aligned}$$

Once we admit the fact that $\mathcal{O}_X^* \simeq \mathcal{O}_X$, we deduce an isomorphism, for $\mathcal{M} \in D^b_c(\mathcal{D}_X)$,

$$\mathrm{Sol}(\mathcal{M})[\dim X] \simeq DR(\mathcal{M}^*).$$

We give a further explanation in §1.3 about $\mathrm{Sol}(\mathcal{M})$ and $DR(\mathcal{M})$.

c) Algebraic Local Cohomology

Let Y be a subvariety of X, and let $\mathcal{J}_Y$ denote its defining ideal. Then for a $\mathcal{D}_X$ module $\mathcal{F}$, we define algebraic local cohomology of $\mathcal{F}$ supported in Y and $X\backslash Y$ respectively by

$$\begin{aligned}\mathbb{R}\Gamma_{[Y]}(\mathcal{F}) &= \varinjlim \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X\mathcal{J}_Y^k, \mathcal{F}), \\ \mathbb{R}\Gamma_{[X|Y]}(\mathcal{F}) &= \varinjlim \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{J}_Y^k, \mathcal{F}).\end{aligned}$$

We have distinguished triangles and a morphism of triangles

$$\begin{array}{ccccccc}
\longrightarrow & \mathbb{R}\Gamma_{[Y]}\mathcal{M} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathbb{R}\Gamma_{[X\backslash Y]}\mathcal{M} & \xrightarrow{+1} \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \mathbb{R}\Gamma_{Y}\mathcal{M} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathbb{R}j_* j^{-1}\mathcal{M} & \xrightarrow{+1}
\end{array}$$

where j denotes the canonical injective $X\backslash Y \hookrightarrow X$. In case X algebraic, the morphism above is isomorphic. If Y is hypersurface defined by a function $f; Y = \{f = 0\}$, then $\mathbb{R}\Gamma_{[X|Y]}(\mathcal{M})$ can be calculated as

$$H^i\mathbb{R}\Gamma_{[X|Y]}(\mathcal{M}) = \begin{cases} 0 & (i \geq 0) \\ \mathcal{O}_f \underset{\mathcal{O}_X}{\otimes} \mathcal{M} = \mathcal{M}_f & (i = 0) \end{cases}$$

with $\mathcal{O}_f$ the localization by f.

We have the distinguished triangle of Mayer-Vietoris type for two varieties Y_1 and Y_2 in X

$$\begin{aligned} \longrightarrow \mathbb{R}\Gamma_{[Y_1 \cap Y_2]}(\mathcal{M}) &\longrightarrow \mathbb{R}\Gamma_{[Y_1]}(\mathcal{M}) \oplus \mathbb{R}\Gamma_{[Y_2]}(\mathcal{M}) \\ &\longrightarrow \mathbb{R}\Gamma_{[Y_1 \cup Y_2]}(\mathcal{M}) \xrightarrow{+1} . \end{aligned}$$

Finally we give

Example. The following plays an essential role when we define inverse image and direct image for $\mathcal{D}$-modules. Let Y be a smooth manifold. Then we have the vanishing of cohomology

$$H^i\mathbb{R}\Gamma_{[Y]}\mathcal{O}_X = 0 \quad (i \neq d = \operatorname{codim} Y).$$

The remaining cohomology group is denoted by $\mathcal{B}_{Y|X}$;

$$\mathcal{B}_{Y|X} = \mathcal{H}^d_{[Y]}(\mathcal{O}_X) = H^d\mathbb{R}\Gamma_{[Y]}(\mathcal{O}_X).$$

d) Inverse Image

Let $f : Y \to X$ be a morphism of complex manifolds or smooth separate schemes of finite type over K. We define, for a $\mathcal{D}_X$ module $\mathcal{M}$, its inverse image $\mathbb{L}f^0\mathcal{M}$ etc. For this purpose, we first give a $(\mathcal{D}_Y, f^{-1}\mathcal{D}_X)$ bi-module $\mathcal{D}_{Y\to X}$ by

$$\mathcal{D}_{Y\to X} := \mathcal{O}_Y \underset{f^{-1}\mathcal{O}_X}{\otimes} f^{-1}\mathcal{D}_X.$$

Here the $f^{-1}\mathcal{O}_X$ module structure of $\mathcal{O}_Y$ is given by

$$h \cdot g = (h \circ f) \cdot g \quad (h \in f^{-1}\mathcal{O}_X, g \in \mathcal{O}_Y).$$

In case f is given, by coordinates, as $(x' = 0, x'') \hookrightarrow (x', x'')$ with $x' = (x_1, \cdots, x_d)$. Then we have

$$\mathcal{D}_{Y \to X} \simeq \mathcal{D}_X / (x_1, \cdots, x_d)\mathcal{D}_X.$$

In case f is written as $(t, x) \mapsto x$, then

$$\mathcal{D}_{Y \to X} \simeq \frac{\mathcal{D}_Y}{\mathcal{D}_Y(D_{t_1}, \cdots, D_{t_k})}$$

with $t = (t_1, \cdots, t_k)$.

Now we define the inverse images of a $\mathcal{D}_X$-module $\mathcal{M}$

$$\begin{aligned}
\mathbb{L}f^0\mathcal{M} &= \mathcal{D}_{Y \to X} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M} = \mathcal{O}_Y \underset{f^{-1}\mathcal{O}_X}{\overset{\mathbb{L}}{\otimes}} f^{-1}\mathcal{M}, \\
\mathbb{L}f^!\mathcal{M} &= (\mathbb{L}f^0\mathcal{M})[\dim Y - \dim X], \\
\mathbb{L}f^*\mathcal{M} &= D_Y \mathbb{L}f^0 D_X[\dim X - \dim Y].
\end{aligned}$$

We explain how coherency is preserved. We associate to f two morphisms of vector bundles

$$T^*Y \xleftarrow{\rho_f} Y \times_X T^*X \xrightarrow{\varpi_f} T^*X.$$

In this situation we can formulate

Definition 1.2.1. Let $\mathcal{M}$ be a $\mathcal{D}_X$ module. Then f is non-characteristic for $\mathcal{M}$ if

$$T_Y^*X \cap \tilde{\omega}_f^{-1}(\mathrm{ch}(\mathcal{M})) \subset Y \times_X T_X^*X.$$

Here $T_Y^*X := \ker \rho_f$.

Theorem 1.2.2. *If f is non-characteristic for $\mathcal{M}$, then $\mathbb{L}f^0\mathcal{M} \simeq \mathcal{D}_{Y \to X} \underset{f^{-1}\mathcal{D}_X}{\otimes} f^{-1}\mathcal{M}$ (i.e. concentrated in degree zero) and $\mathbb{L}f^0\mathcal{M}$ is a coherent $\mathcal{D}_X$-module. Moreover*

$$\mathrm{ch}(\mathbb{L}f^0\mathcal{M}) \subset \rho_f \varpi_f^{-1}(\mathrm{ch}(\mathcal{M})).$$

This theorem can be proved by reducing it to the case where f is a embedding of codimension 1 and $\mathcal{M}$ is a single equation. Note that there is not any difficult argument to make if f is a submersion.

e) Direct image

Let $f: Y \to X$ be as in d). We define the direct image $\mathbb{R}f_*\mathcal{M}$ for a $\mathcal{D}_Y$ module $\mathcal{M}$. First we introduce a $(f^{-1}\mathcal{D}_X, \mathcal{D}_Y)$ bi-module by

$$\mathcal{D}_{X\leftarrow Y} := \mathcal{D}_{Y\to X} \underset{f^{-1}\mathcal{O}_X}{\otimes} f^{-1}\Omega_X^{-1} \underset{\mathcal{O}_Y}{\otimes} \Omega_Y.$$

Then we define, for a $\mathcal{D}_X$-module $\mathcal{M}$, its direct image by

$$\mathbb{R}f_*\mathcal{M} := \mathbb{R}f_*(\mathcal{D}_{X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{M}).$$

As for the coherency, we have

Proposition 1.2.3. *Let f be proper, and assume that $\mathcal{M}$ has a global good filtration. Then*

$$\mathbb{R}f_*\mathcal{M} \in D_c^b(\mathcal{D}_X).$$

In the analytic case, this results from Grauert's theorem on the direct image theorem for coherent sheaves.

Next we give a relative duality theorem.

Proposition 1.2.4. *Let f be proper, and $\mathcal{M} \in D_c^b(\mathcal{D}_Y)$ is assumed to have cohomologies with global good filtrations. Then*

$$\mathbb{R}f_*(D_Y\mathcal{M}) \xrightarrow{\sim} D_X(\mathbb{R}f_*\mathcal{M}).$$

1.3. Holonomic systems.

a) Examples and regular singularities of holonomic systems.

We have defined so far holonomic $\mathcal{D}_X$-modules, which are coherent $\mathcal{D}_X$-modules whose characteristic varieties are Lagrangian. Throughout this section a), X is a complex manifold or a smooth separate scheme of finite type over K.

We first see typical examples before entering into the notion of regular singularities for holonomic systems.

Example 1.3.1. The $\mathcal{D}_X$ module $\mathcal{O}_X$ is holonomic. In fact we have

$$\mathrm{ch}(\mathcal{O}_X) = T_X^*X.$$

This can be shown by taking a local coordinate system $x = (x_1, \cdots, x_n)$. Then we have the isomorphism

$$\mathcal{D}_X/\mathcal{D}_X(D_1, \cdots, D_n) \simeq \mathcal{O}_X.$$

Moreover we can construct canonically a global free resolution of $\mathcal{O}_X$ as follows. Let Θ_X denote the sheaf of holomorphic vector fields. Then we have an exact sequence

$$0 \longrightarrow \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \Lambda^n \Theta_X \longrightarrow \cdots \xrightarrow{\delta} \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \Theta_X \longrightarrow \mathcal{O}_X \longrightarrow 0$$

where the morphism δ is given by

$$\begin{aligned}
&\delta(P \otimes v_1 \wedge \cdots \wedge v_k) \\
&\sum_{i=1}^{k}(-1)^i P v_i \otimes (v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k) \\
&\quad + \sum_{1 \leq i \leq j \leq k} (-1)^{i+j} P \otimes ([v_i, v_j] \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k).
\end{aligned}$$

By means of this resolution we can calculate $DR(\mathcal{M})$ for a $\mathcal{D}_X$-module $\mathcal{M}$ by the complex

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M} \underset{\mathcal{O}_X}{\otimes} \Omega_X^1 \longrightarrow \cdots \longrightarrow \mathcal{M} \otimes \Omega_X^n \longrightarrow 0.$$

Here the derivative is given by

$$m \otimes \omega \longmapsto \nabla m \wedge \omega - (-1)^{\deg \omega} m \otimes d\omega$$

with $\nabla m = \sum_{i=1}^{n} \partial/\partial x_i \cdot m \otimes dx_i$. In fact

$$\begin{aligned}
&\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \Lambda^k \Theta_X, \mathcal{M}) \\
&\simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Lambda^k \Theta_X, \mathcal{M}) \\
&\simeq \Omega_X^k \underset{\mathcal{O}_X}{\otimes} \mathcal{M}.
\end{aligned}$$

In case $\mathcal{M} = \mathcal{O}_X$, we have

$$\mathrm{Ext}^j_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X) = \begin{cases} 0 & (j \neq n) \\ \Omega_X & (j = n). \end{cases}$$

This implies $\mathcal{O}_X^* \simeq \mathcal{O}_X$.

Example 1.3.2. Let Y be a closed variety of X. Then $H^j_{[Y]}(\mathcal{O}_X)$ is holonomic. In case Y is smooth of codimension d in X, $H^j_{[Y]}(\mathcal{O}_X) = 0$ $(j \neq d)$. We denoted by $\mathcal{B}_{Y|X}$ $H^d_{[Y]}(\mathcal{O}_X)$;

$$\mathcal{B}_{Y|X} := H^d_{[Y]}(\mathcal{O}_X).$$

If we take local coordinates $x = (x_1, \cdots, x_n)$ so that $Y = \{x_1 = \cdots = x_d = 0\}$,

$$\mathcal{B}_{Y|X} = \mathcal{D}_X/\mathcal{D}_X(x_1, \cdots, x_d, D_{d+1}, \cdots, D_n).$$

Now we give the definition of regular holonomic $\mathcal{D}_X$-modules, which is an extension of the notion of ordinary differential equations with regular singularities. First we assume that X is a complex manifold.

Definition 1.3.3. i) Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$ module. Then $\mathcal{M}$ is regular if $\mathcal{M}$ is equipped locally with a good filtration $F_k(\mathcal{M})$ satisfying

$$\begin{cases} \forall P \in \mathcal{D}_X(k) \text{ with } \sigma_k(P)|_{\mathrm{ch}(\mathcal{M})} = 0, \quad \forall \ell \\ PF_\ell(\mathcal{M}) \subset F_{k+\ell-1}(\mathcal{M}). \end{cases}$$

ii) Let $D^b_{hr}(\mathcal{D}_X)$ denote the subcategory of $D^b_h(\mathcal{D}_X)$ consisting of objects who cohomologies are regular singular, and $Coh_{hr}(\mathcal{D}_X)$ also defined in the same way.

It is remarkable that if $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module, then we can find a global good filtration satisfying the condition .

We give several examples.

Examples 1.3.4. i) Let X be $\mathbb{C}$, and P be an ordinary differential operator with regular singular points, i.e. if $z_0 \in \mathbb{C}$ is a singular point of $P(z, D_z) = a_0(z)D_z^m + \cdots + a_m(z)$, the meromorphic function $a_k(z)/a_0(z)$ has a pole of order at most k.
In this situation $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ is a regular holonomic $\mathcal{D}_X$-modules.
ii) The examples of holonomic $\mathcal{D}_X$-modules $\mathcal{O}_X$ and $\mathcal{B}_{Y|X}$ given above are regular singular.

Next we define regular holonomicity in the algebraic case. Let X be a separate scheme of finite type over K. We take a compactification of Nagata and Hironaka $\overline{X}$ of X

$$j : X \longrightarrow \overline{X}.$$

Then we have $\mathbb{R}j_*\mathcal{M} \in D^b_h(\mathcal{D}_{\overline{X}})$. Taking this into account, we give

Definition 1.3.7. Let $\mathcal{M}$ be a holonomy $\mathcal{D}_X$-module. Then $\mathcal{M}$ is regular singular if

$$(\mathbb{R}j_*\mathcal{M})_{an} = \mathcal{D}_{\overline{X}_{an}} \underset{(\mathcal{D}_{\overline{X}})}{\otimes} \mathbb{R}j_*\mathcal{M}$$

is regular singular.

b) Stability of holonomicity and regular holonomicity under operations.

Let X and Y be complex manifolds, $f : Y \to X$ be a holomorphic mapping, and Z be a closed analytic subvariety of X. We have defined several operations. We give a brief review of stability of holonomicity and regular holonomicity by the operations.

i) Dual operator

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module then $\mathcal{M}^*$ is concentrated in degree 0 and is holonomic. Moreover if $\mathcal{M}$ is regular holonomic, then so is $\mathcal{M}^*$.

ii) Algebraic local cohomology

Let $\mathcal{M}$ be a holonomic (resp. regular holonomic) $\mathcal{D}_X$-module, $\mathcal{M}$ are generally $\mathcal{M} \in D^b_h(\mathcal{D}_X)$ (resp. $D^b_{hr}(\mathcal{D}_X)$).

Then

$$\mathbb{R}\Gamma_{[Z]}(\mathcal{M}), \mathbb{R}\Gamma_{[X|Z]}(\mathcal{M}) \in D^b_h(\mathcal{D}_X) \quad (\text{resp. } D^b_{hr}(\mathcal{D}_X)).$$

iii) Inverse image

Let $\mathcal{M}$ be an object of $D^b_h(\mathcal{D}_X)$ (resp. $D^b_{hr}(\mathcal{D}_X)$). Then

$$\mathbb{L}f^!\mathcal{M}, \mathbb{L}f^*\mathcal{M} \in D^b_h(\mathcal{D}_Y) \quad (\text{resp. } D^b_{hr}(\mathcal{D}_Y)).$$

iv) Direct image

Let f be proper. Then, for $\mathcal{M} \in D^b_h(\mathcal{D}_Y)$ (resp. $D^b_{hr}(\mathcal{D}_Y)$) we have

$$\mathbb{R}f_*\mathcal{M} \in D^b_h(\mathcal{D}_X) \quad (\text{resp. } D^b_{hr}(\mathcal{D}_X)).$$

2. Constructible and Perverse Sheaves

In this section, we introduce (Whitney) stratification, constructible sheaves and perverse sheaves. We also give some results for the functorial properties of these sheaves and, in the last subsection, quick review of D-G-M modules and minimal extension of $\mathcal{D}$ modules.

2.0 Stratification

Let X be a topological space.

Definition 2.0.0. *A stratification $(X_\alpha)_\alpha$ of X is a partition $X = \sqcup X_\alpha$ satisfying*

1) the family (X_α) is a locally finite,

2) each X_α is locally closed smooth C^γ manifold ($2 \leq \gamma \leq \infty$ or $\gamma = \omega$) and

3) for each pair (α, β) s.t. $X_\alpha \cap \overline{X_\beta}$ is non empty, X_α is connected in $\overline{X_\beta}$ (i.e. $X_\alpha \prec X_\beta$).

Moreover the stratification which satisfies the Whitney conditions is said to be a Whitney stratification. For the reader's convenience, we give here the Whitney conditions. A pair $Z \prec Y$ satisfies the Whitney condition at $z \in Z$ if and only if it satisfies the following two conditions (a) and (b).

(a) for any sequence $y_n \in Y$ s.t. $y_n \to z$ and such that tangent spaces $T_{y_n}Y$ has a limit $\tau \subset T_z X$, one has $\tau \supset T_z Z$.

(b) for any sequence (z_n, y_n, c_n) in $Z \times Y \times \mathbb{R}^+$ s.t. $z_n \to z, y_n \to z, c_n(z_n - y_n) \to v$ and tangent spaces $T_{y_n}Y$ has a limit $\tau \subset T_z X$, one has $v \subset \tau$.

Definition 2.0.1. *A filtration $\{X_k\}$ of X is an increasing sequences $\phi = X_{-1} \subset X_1 \subset \cdots \subset X_n = X$ satisfying each $X_k - X_{k-1}$ is a submanifold.*

We remark that any analytic space X has a Whitney stratification $\{X_\alpha\}$ and $X_k := \bigcup_{\dim X_\alpha \leq k} X_\alpha$ gives a filtration of X.

2.1 Constructible sheaves

Let R be $\mathbb{R}$ or $\mathbb{C}$ and X an analytic space with dim $= n$. From now, we always assume this situation.

Definition 2.1.0. *F is a constructible sheaf if and only if there exists a filtration $\{X_k\}$ of X s.t. $F|_{X_k - X_{k-1}}$ is a locally constant sheaf of R module of finite rank.*

Denote by $D_c^b(X, R)$ the subcategory of bounded complexes of R modules whose cohomologies are constructible.

We give some stability theorems for the operation of constructible sheaves. Let $f : Y \to X$ be a morphism of analytic spaces.

Proposition 2.1.1. *For any $F \in D_c^b(Y, R)$ and $G \in D_c^b(X, R)$, we have $f_! F \in D_c^b(X, R)$ and $f^*G, f^!G \in D_c^b(Y, R)$. Moreover if f is proper on* Supp F, *$\mathbb{R}f_*F \in D_c^b(X, R)$.*

Proposition 2.1.2. *For any $F, G \in D_c^b(X, R)$, we have $\mathbb{R}\underline{\mathrm{Hom}}(F, G), F \overset{\mathbb{L}}{\otimes} G \in D_c^b(X, R)$.*

In particular, if Y is a point (i.e. $f : X \to pt$), we have $D_X := f^!R \in D_c^b(X, R)$. The dualizing complex D_X is very important and nothing but $R_X[\dim X_{top}]$ if X is smooth. Now we can define dualizing functor $\mathbb{D}_X$ as

$$\mathbb{D}_X F := \mathbb{R}\underline{\mathrm{Hom}}_R(F, D_X) \in D_c^b(X, R).$$

Proposition 2.1.3. *$\mathbb{D}_X$is involutive (i.e. $\mathbb{D}_X \circ \mathbb{D}_X = \mathrm{id}_X$) and we have $f^! = \mathbb{D}_Y \circ f^* \circ \mathbb{D}_X$ and $f_! = \mathbb{D}_X \circ \mathbb{R}f_* \circ \mathbb{D}_Y$.*

Typical examples of constructible sheaves are the solutions of holonomic $\mathcal{D}$ modules. The following well known result is due to Kashiwara.

Theorem 2.1.4. *Let X be a complex manifold and $\mathcal{M} \in D_h^b(\mathcal{D}_X)$. We have $DR(\mathcal{M})$, $Sol(\mathcal{M}) \in D_c^b(X, R)$. Moreover we have $\mathbb{D}_X(DR(\mathcal{M})) = DR(\mathcal{M}^*)$.*

2.2 Perverse sheaves.

We first recall the notions of t-structures and t-categories. Let D be a triangulated category. $D^{\geq 0}$ and $D^{\leq 0}$ are the full subcategories of D which satisfies

(1) $Hom(D^{\leq 0}, D^{\geq 1}) = 0$,

(2) $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 0} \supset D^{\geq 1}$ (here $D^{\leq n} := D^{\leq 0}[-n]$, etc.) and

(3) for any $X \in D$, there exists $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$ s.t. we have

$$A \to X \to B \overset{+1}{\to}.$$

Definition 2.2.0. *The category D equipped with $(D^{\leq 0}, D^{\geq 0})$ is said to be t-category.*

We list up some important properties of t-categories.

Lemma 2.2.1.
(1) $D^{\leq n}$ (resp. $D^{\geq n}$) $\hookrightarrow D$ has a right (left) ajoint functor denoted by $\tau_{\leq n}$ ($\tau_{\geq n}$).
(2) There exists an unique morphism $d \in Hom^1(\tau_{\geq 1}X, \tau_{\leq 0}X)$ and a triangle

$$\tau_{\leq 0}X \to X \to \tau_{\geq 1}X \xrightarrow{d} .$$

The above triangle is unique up to isomorphisms.
(3) For two integers $a \leq b$, we have

$$\tau_{\geq a} \circ \tau_{\leq b}X \simeq \tau_{\leq b} \circ \tau_{\geq a}X.$$

One of the important things is to show the perverse category is abelian. The following theorem explains why we define t-category and t-structure.

Theorem 2.2.2.
(1) The category $D^\circ := \tau_{\leq 0} \circ \tau_{\geq 0}D$ is abelian.
(2) $H^\circ := \tau_{\leq 0} \circ \tau_{\geq 0} : D \to D^\circ$ is a cohomological functor.

We apply the above argument to $D^b_c(X, R)$ and define perverse sheaves. Let $\{X_\alpha\}$ be a stratification of X and $p : \{X_\alpha\} \to \mathbb{Z}$. We set

$${}^pD^{\leq 0}_c(X, R) := \{F \in D^b_c(X, R); \quad H^k i^*_\alpha F = 0 \qquad (\forall \alpha, k > p(X_\alpha))\}$$

and

$${}^pD^{\geq 0}_c(X, R) := \{F \in D^b_c(X, R); \quad H^k i^!_\alpha F = 0 \qquad (\forall \alpha, k < p(X_\alpha))\}.$$

Here i_α is inclusion map $X_\alpha \hookrightarrow X$. The following proposition is key to define perverse sheaves.

Proposition 2.2.3. *$({}^pD^{\leq 0}_c(X, R), {}^pD^{\geq 0}_c(X, R))$ is t-structure of $D^b_c(X, R)$.*

Now we define the p-perverse category as $\mathcal{M}(p, X, R) :={}^p D^{\leq 0}_c(X, R) \cap {}^p D^{\geq 0}_c(X, R)$.

Using H°, several new functors are introduced in the perverse category. Let U be a locally closed subset of X and $j : U \hookrightarrow X$. We always assume U is reunion of strata. The functor ${}^pj_!, {}^pj^*, \cdots$ are respectively defined by $H^\circ \circ j_!, H^\circ \circ j^*, \cdots$. For $F \in \mathcal{M}(p, X, R)$, we have the morphism ${}^pj_!F \to {}^pj_*F$ associated with the canonical map $j_!F \to j_*F$.

Definition 2.2.4. *A new (and important) functor $j_{!*}$ is defined by*

$$j_{!*}F := \mathrm{Im}({}^p j_! F \to {}^p j_* F).$$

(Remark that the perverse category is abelian.)

The next proposition clarifies the meaning of this functor. Set $U_k = \bigcup_{p(X_\alpha)\leq k} X_\alpha$ and $j_k : U_{k-1} \to U_k$. For any integer l, we have a concrete description of $i : U_l \hookrightarrow X$ as:

Proposition 2.2.5. *Assume $p(X_\alpha) \geq p(X_\beta)$ for $X_\alpha \prec X_\beta$ and $\bigcup_{k\leq m} X_k = X$. Then for $F \in \mathcal{M}(p, U_l, R)$, we have*

$$i_{!*}F = \tau_{\leq m-1}(j_m)_* \circ \cdots \tau_{\leq l}(j_{l+1})_* F$$

where $\tau_{\leq k}$ is an usual wayout functor.

Proposition 2.2.6. *For any $F \in \mathcal{M}(p, U, R)$, $G := j_{!*}F$ is the unique extension of F to $\mathcal{M}(p, X, R)$ satisfying the following conditions.*

(+) $H^k i^*_\alpha G = 0$ $(k \geq p(X_\alpha))$ and $H^k i^!_\alpha G = 0$ $(k \leq p(X_\alpha))$ for any $X_\alpha \subset X - U$.

Finally we introduce a dual perversity and $\frac{1}{2}$-perverse.

Definition 2.2.7. *The dual perversity p^* of p is defined by*

$$p^*(X_\alpha) := -p(X_\alpha) - \dim X.$$

Verdier duality shows $\mathbb{D}_X(\mathcal{M}(p, X, R)) = \mathcal{M}(p^*, X, R)$. Moreover we have the following functorial properties.

$$\mathbb{D}_X \circ^p j_! =^p j_* \circ \mathbb{D}_U, \quad \mathbb{D}_U \circ^p j^! =^p j^* \circ \mathbb{D}_X \quad \text{and} \quad \mathbb{D}_X \circ j_{!*} = j_{!*} \circ \mathbb{D}_U.$$

If dimension of each stratum is even, we have $\mathbb{D}_X(\mathcal{M}(p_{\frac{1}{2}}, X, R)) = \mathcal{M}(p_{\frac{1}{2}}, X, R)$ where $p_{\frac{1}{2}}(X_\alpha) := -\frac{1}{2}\dim(X_\alpha)$ ($\frac{1}{2}$-perversity !). $\mathcal{M}(X) := \mathcal{M}(p_{\frac{1}{2}}, X, R)$ is called $\frac{1}{2}$-perverse sheaf. Since $\mathbb{D}_X$ is involution in $\mathcal{M}(X)$, we have

Theorem 2.2.8. *If X is compact, then $\mathcal{M}(X)$ is Noetherian and Artinian.*

After Kashiwara established Riemann-Hilbert correspondence, Deligne characterized the solutions of R.S holonomic modules (i.e. complexes concentrated in degree zero) as perverse sheaves.

Theorem 2.2.9. *Let X be a complex manifold and $\mathcal{M} \in D^b_{hr}(\mathcal{D}_X)$. Then $\mathcal{M} \in Coh_{hr}(\mathcal{D}_X$ if and only if $DR(\mathcal{M}) \in \mathcal{M}(X)$.*

2.3 D-G-M modules and minimal extensions

Let X be a complex manifold, $j : Y \hookrightarrow X$ an irreducible complex subvariety of X, Z a subvariety with $Y_{sing} \subset Z \subsetneq Y$ and L an irreducible local system in $Y - Z$. In this subsection, we study simple objects in the both categories $\mathcal{M}(X)$ and $Coh_{hr}(X)$ and their relationship. First we give a complete description of a simple object in $\mathcal{M}(X)$.

Theorem 2.3.0. *$j_{!*}(L[\dim_{\mathbb{C}} Y])$ is a simple object of $\mathcal{M}(X)$. Conversely all simple objects of $\mathcal{M}(X)$ are described by the above form.*

A simple object of $\mathcal{M}(X)$ is called D-G-M modules and denoted by $\pi_Y(L)$. Moreover we have $\mathbb{D}_Y(\pi_Y(L)) = \pi_Y(L^*)$ where $L^* = \underline{\mathrm{Hom}}_{\mathbb{C}}(L, \mathbb{C})$. Using this sheaf, we can define intersection homologies of the middle perversity due to G-M as follows.

$$IH_k(Y, L) := H^{-k+\dim_{\mathbb{C}} Y}(Y, \pi_Y(L)).$$

Then we have generalized Poincare duality.

Theorem 2.3.1. *Assume Y is compact. Then $IH_l(Y, L) \otimes IH_{\dim_{\mathbb{R}} Y - l}(Y, L^*) \to \mathbb{C}$ is perfect.*

Next we study a simple object in the category of regular holonomic modules.

Lemma 2.3.2. *Let $\mathcal{M}$ be a simple holonomic module with $\mathrm{Supp}(\mathcal{M}) \subset Y$. Then there exists a subvariety $Z \supset Y_{sing}$ and an irreducible local system L in $Y - Z$ satisfying the following two conditions.*

(1) $\mathcal{M}|_{X-Z} = L \underset{\mathbb{C}}{\otimes} \mathcal{B}_{Y-Z|X-Z}$.

(2) $H^0_Z(\mathcal{M}) = H^0_Z(\mathcal{M}^*) = 0$.

Conversely we have

Proposition 2.3.3. *Let Z be a subvariety with $Y_{sing} \subset Z \subset Y$ and L a locally constant sheaf of finite rank in $Y - Z$. Then there exists an unique regular holonomic module $\mathcal{M}$ in X satisfying conditions (1) and (2) in Lemma 2.3.2.*

We denote such a module by $\mathcal{L}(X, Y, L)$. Therefore a simple object of $Coh_{hr}(\mathcal{D}_X)$ is described by $\mathcal{L}(X, Y, L)$.

One important fact is $\mathcal{L}(X, Y, L)$ and $\pi_Y(X)$ are connected by the DR functor.

Theorem 2.3.4. *We have $DR(\mathcal{L}(X, Y, L)) = \pi_Y(L)$.*

3. Riemann-Hilbert Correspondence

3.0. Introduction

The Riemann Hilbert correspondence between regular holonomic complexes and constructible complexes is originated in the classical problem of ordinary differential operators. Let X be a smooth curve over $\mathbb{C}$, and P be a differential operator on X with regular singular points $\{a_1, \cdots, a_N\}$. If we set $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ and $F = \underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ satisfying the condition that $F|_{\{X\backslash a_1, \cdots, a_N\}}$ is a local system of finite rank. Taking a base point $x_0 \in X\backslash\{a_1, \cdots, a_N\}$, we obtain a linear representation

$$\pi_1(X\backslash\{a_1, \cdots, a_N\}, x_0) \longrightarrow \mathrm{End}(F_{x_0})$$

called the monodoromy representation. In this situation, it was a classical Riemann-Hilbert problem to fined an operator P enjoying a given monodoromy.

Now let X be a regular holonomic complexes. Then $DR(\mathcal{M})$ is a constructible complex. This correspondence entails a generalization of Riemann-Hilbert problem to higher dimension;

$$DR : D^b_{hr}(\mathcal{D}_X) \longrightarrow D^b_c(\mathbb{C}_X),$$

where we abberaviate $D^b_c(X, \mathbb{C})$ by $D^b_c(\mathbb{C}_X)$ in this section. Given an object $F \in D^b_c(\mathbb{C}_X)$, the problem is to find $\mathcal{M} \in D^b_{hr}(\mathcal{D}_X)$ with $DR(\mathcal{M}) = F$. It is shown by Kashiwara that the DR functor gives rise to an equivalence of categories. Moreover the inverse is constructed by using tempered distribution.

3.1. The functor TH

a) Tempered distributions

Let M be a real analytic manifold, and U be an open subset of M. Then temperedness of distributions is defined by

Definition 3.1.1. Let $u \in \mathcal{D}b_M(U)$, a distribution on U, and $p \in M$. Then u is tempered at p if there exist an open neighborhood W of p and $w \in \mathcal{D}b_M(W)$ with the property $u|_{W\cap U} = w|_{W\cap U}$. Moreover u is tempered on M if so it is at any point of M.

Remark that u is tempered on M if and only if u is extended on M as distribution.

b) $\mathbb{R}$-constructible sheaves.

Let M be a real analytic manifold, and F is a sheaf on M. Then M is $\mathbb{R}$-constructible if there exists a subanalytic stratification $M = \bigcup_\alpha M_\alpha$ for which $F|_{M_\alpha}$ is locally constant of finite rank. We denote by $\mathbb{R}$-const(M) the category of $\mathbb{R}$-constructible sheaves on M. We remark that we have the equivalent of categories

$$(3.1) \qquad D^b(\mathbb{R} - \text{const}(M)) \xrightarrow{\sim} D^b_{\mathbb{R}-c}(M)$$

where $D^b_{\mathbb{R}-c}(M)$ denotes the category consisting of objects with $\mathbb{R}$-constructible cohomologies.

c) $T -$ Hom

We follows the notation in b). Let F be a $\mathbb{R}$-constructible sheaf. Then a subsheaf $T - \underline{\text{Hom}}_{\mathbb{C}_M}(F, \mathcal{D}b_M)$ of $\underline{\text{Hom}}_{\mathbb{C}_M}(F, \mathcal{D}b_M)$ is the assigning to an open subset U in M the space

$$\{\varphi \in \Gamma(U; \text{Hom}_{\mathbb{C}_M}(F, \mathcal{D}b_M); \varphi \,\text{satisfies the condition } (T)\}$$

where the condition (T) is that

$$(T)\begin{cases} \text{for any relatively compact open} \\ \text{subanalytic set } V \text{ of } U \text{ and} \\ \text{for any } s \in F(V), \varphi(s) \text{ is a} \\ \text{tempered distribution.} \end{cases}$$

The sheaf $T - \underline{\text{Hom}}(F, \mathcal{D}b_M)$ is a sheaf of $\mathcal{D}_M$ module, which is written for short as $TH_M(F)$. Now we list up the principal properties of $TH_M(F)$.

i) The sheaf $TH_M(F)$ is a soft sheaf.

ii) Let U be a subanalytic open subset in M and Ω an open subset in M. Then

$$\Gamma(\Omega; TH_M(\mathbb{C}_U)) \\ = \{u \in \Gamma(U \cap \Omega; \mathcal{D}b_M); u \text{ is tempered at any point in } \Omega\}.$$

iii) Let Z be a closed subanalytic subset in M. Then we have

$$TH_M(\mathbb{C}_Z) = \Gamma_Z(\mathcal{D}b_M).$$

Considering (3.1), we have the derived functor $\mathbb{R}TH_M(\cdot)$ of $TH_M(\cdot)$.

iv) The following theorem concerning the functorial properties of $\mathbb{R}TH_M(\cdot)$ with respect to the direct image.

Theorem 3.1.2. *Let $f : M \to N$ be a morphism of real analytic manifolds, and $F \in D^b_{\mathbb{R}-c}(M)$. We assume that $\overline{\operatorname{supp}\mathcal{H}^j(F)}$ is proper over N for any j. Then*

$$\begin{aligned}&\mathbb{R}f_*(\mathcal{D}_{N\leftarrow M} \overset{\mathbb{L}}{\underset{\mathcal{D}_M}{\otimes}} \mathbb{R}TH_M(P^{\cdot}))\\ &\simeq \mathbb{R}TH_N(\mathbb{R}f_*F^{\cdot}).\end{aligned}$$

The above theorem is a crucial part of the construction of Riemann-Hilbert correspondence.

3.2. Riemann-Hilbert correspondence

Let X be a complex manifold. Then we have the diagram of functors

$$\begin{tikzcd} D^b_{rh}(\mathcal{D}_X) \arrow[rr, "J(\cdot):=\mathcal{D}_X{}^{\infty}\otimes_{\mathcal{D}_X}(\cdot)"] \arrow[drr, shift right, "\Phi = Sol"'] & & D^b_h(\mathcal{D}^\infty_X) \arrow[d, shift left, "\Phi^\infty = Sol"] \\ & & D^b_c(\mathbb{C}_X)^{opp}. \arrow[ull, shift right, "\Psi"'] \arrow[u, shift left, "\Psi^\infty"] \end{tikzcd}$$

Here $\mathcal{D}^\infty_X$ is the sheaf of rings of differential operators of infinite order on X, $D^b_h(\mathcal{D}^\infty_X)$ denotes the derived category of holonomic $\mathcal{D}^\infty_X$ modules, and Ψ and Ψ^∞ are constructed as follows. For $F \in D^b_c(\mathbb{C}_X)$,

$$\begin{aligned}\Psi^\infty(F) &:= \mathbb{R}\underline{\mathrm{Hom}}_{\mathbb{C}_X}(F, \mathcal{O}_X),\\ \Psi(F) &:= \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathbb{R}TH_{X_\mathbb{R}}(F)).\end{aligned}$$

In the above situation, Kashiwara has shown

Theorem 3.2.1. *The functors J,Φ,Ψ and Ψ^∞ give rise to equivalence of categories. Moreover Φ and Ψ (resp. Φ^∞ and Ψ^∞) are inverse to each other.*

The fact that $\Psi^\infty \circ \Phi^\infty = Id$ is proved by Kashiwara-Kawai, and thus there are two facts shown by Kashiwara.

i) $\Psi(D^b_c(X)) \subset D^b_{hr}(\mathcal{D}_X)$,

ii) $\Phi \circ \Psi = Id$.

The fact ii) is relatively easy. The first one can be reduced, with the aid of Hironaka's resolution of singularities and the following Proposition 3.2.2, to the case where F is a $\mathbb{R}$-constructible sheaf and, for a normal normal crossing subvariety Y of X, $F|_Y = 0$ and $F|_{X\setminus Y}$ is a local system.

Proposition 3.2.2. *Let $f : X \to Y$ be a morphism of complex manifolds, and $F^{\cdot} \in D^b_{\mathbb{R}-c}(X_{\mathbb{R}})$ satisfying the condition $\overline{\mathrm{supp}\ \mathcal{H}^j(F^{\cdot})}$ is proper over Y.*
Then

$$\mathbb{R}f_*(\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes} \Psi_X(F^{\cdot}))[\dim X]$$
$$\simeq \Psi_Y(\mathbb{R}f_*(F^{\cdot}))[\dim Y].$$

This theorem results from Theorem 3.1.2.

We have so far studied the correspondence between regular holonomic completes and constructible complexes through the solution functor $\mathrm{Sol}_X(\cdot) = \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\cdot, \mathcal{O}_X)$. With the aid of dual operation we can translate it the one with respect to the de Rham functor $DR(\cdot) = \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{O}_X, \cdot)[\dim X]$.

Theorem 3.2.3. *We have the equivalence of categories*

$$DR : D^b_{hr}(\mathcal{D}_X) \xrightarrow{\sim} D^b_c(\mathbb{C}_X).$$

Remark that in algebraic case, we take the DR functor as

$$DR(\mathcal{M}) := \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_{X_{an}}}(\mathcal{O}_{X_{an}}, \mathcal{D}_{X_{an}} \otimes_{\mathcal{D}_X} \mathcal{M})[\dim X],$$

and Riemann-Hilbert correspondence also holds.

By Theorem 2.2.9 and Theorem 2.3.4, we also have

Theorem 3.2.4. *We have the equivalence of categories*

$$DR : Coh^b_{hr}(\mathcal{D}_X) \xrightarrow{\sim} \mathcal{M}(X).$$

Moreover simple objects of $D^b_{hr}(\mathcal{D}_X)$ are correspondent to D-G-M modules.

3.3. Several operations

We finally list up correspondence of functorial operations in both categories. Let $Y \to X$ be a morphism complex manifolds.

a) Inverse image

Let $\mathcal{M} \in D^b_{hr}(\mathcal{D}_X)$. We have

$$DR_Y(\mathbb{L}f^*\mathcal{M}) \simeq f^*DR_X(\mathcal{M}), \quad \text{and} \quad DR_Y(\mathbb{L}f^!\mathcal{M}) \simeq f^!DR_X(\mathcal{M}).$$

Remark that we have defined $\mathbb{L}f^!$ and $\mathbb{L}f^*$ as no shift are needed in the above correspondence. There are textbooks which give different definitions for the shift.

b) Direct image

We assume the same situation as a). Moreover we assume f is proper in analytic case. There always exist global good filtrations of coherent $\mathcal{D}$ modules in algebraic case, we need no assumption. Let $\mathcal{N} \in D^b_{hr}(\mathcal{D}_Y)$. we have

$$DR_X(\mathbb{R}f_*\mathcal{N}) \simeq \mathbb{R}f_*DR_Y(\mathcal{N}).$$

This is direct consequence of Theorem 3.2.2.

References

Since there are many books and papers for this area, we pick up only some of them. Therefor this is not complete references.

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We recommend textbooks [**Sc**], [**K 2**], and [**Me 1**] for readers to study the theory of $\mathcal{D}_X$ modules, and [**Bo 1**] in algebraic case. The coherency of the Ring $\mathcal{D}$ and several fundamental results have been shown in **[K 1]** , and one of the most important result that the solutions of holonomic systems are constructible is found in **[K 3]** . Regular singular holonomic $\mathcal{D}$ modules were intensively studied in **[K-K]** . For the proof of stability theorems of several operations, see **[K 1]** , **[K 2]** , **[K 3]** and **[H-Sc]** .

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